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FIXED POINT THEORY IN TERMS OF A METRIC AND OF AN ORDER RELATION

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Abstract. We consider a nonempty set X endowed with a metric d an order relation $\preceq$ and an operator $f : X \to X$, which satisfies two main assumptions:
(1) f is generalized monotone with respect to $\preceq$;
(2) f is a (generalized) contraction with respect to d on a certain subset Y of $X \times X$.
In the above terms, we will present conditions under which:
(i) f has a unique fixed point in X;
(ii) f is a Picard operator;
(iii) the fixed point problem for f is well-posed;
(iv) f has the Ostrowski property;
(v) f has the shadowing property;
(vi) f satisfies to some Gronwall type inequalities.
Then, we will apply these results to study some problems related to integral and differential equations. Several open questions are discussed.
Key Words and Phrases: Metric space, ordered set, ordered metric space, contraction, generalized contraction, increasing operator, decreasing operator, progressive operator, regressive operator, generalized monotone operator, fixed point, (weakly) Picard operator, stability, Gronwall lemma, open problem.
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1. Introduction

There are many fixed point theorems in lattices and in ordered sets: Zermelo (1908), Knaster (1928), Zorn (1935), Kantorovitch (1939), Bourbaki (1949), Witt (1951), Kleen (1952), Tarski (1955), Davis (1955), Abian-Brown (1961), Kolodner (1968), Bakhtin (1972), Tartar (1974), Markowsky (1976), Amann (1977), ... See for example [10], [65], [85], [75], [76], [3], [1], [2], [9], [17], [18], [21], [38], [41], [55], [94], [56], [92], ...

The metric fixed point theory is a subject with an intensive development. For basic results and problems of metric fixed point theory see [52], [85], [79], [84], [14], [16], [20], [23], [26], [27], [29], [30], [34], [37], [48], [49], [66], [67], [68], [69], [71], [77], [81], [83], ...

One of the main problem, considered by many authors in the last years, is to build a bridge between these two theories. See [53], [74], [70], [4], [5], [6], [7], [8], [11], [12], [13], [19], [21], [22], [23], [25], [28], [31], [32], [33], [34], [36], [40], [42], [43], [45], [44], [51], [57], [58], [59], [60], [61], [62], [64], [73], [87], [86], [93], [97], [98], [100], ...

The purpose of this paper is to shed more light on the bridge between these two important theories, using the following framework: let X be nonempty set endowed with a metric d, an order relation $\preceq$ and an operator $f : X \to X$, which satisfies two main assumptions:
(1) f is generalized monotone with respect to $\preceq$;
(2) f is a (generalized) contraction with respect to d on a certain subset Y of $X \times X$.

Several conclusions emerge from these assumptions. Then, we will apply these results to study various problems related to integral and differential equations. Some open questions are also discussed.

2. Preliminaries

Let X be a nonempty set and $f : X \to X$ be an operator. Then, we will denote by $f^0 := 1_X,\ f^1 := f, \ldots, f^{n+1} = f \circ f^n,\ n \in \mathbb{N}$ the iterate operators of f. By $I(f) := \{Y \subset X | f(Y) \subseteq Y\}$ we will denote the set of all nonempty invariant subsets of f and by $F_f := \{x \in X |\ x = f(x)\}$ we denote the fixed point set of f. Also, by $Graph(f) := \{(x, y) \in X \times X | f(x) = y\}$ we will denote the graph of f.

Let X be a nonempty set. Denote $s(X) := \{(x_n)_{n\in N} | x_n \in X,\ n \in N\}$.

Let $c(X) \subset s(X)$ a subset of $s(X)$ and $Lim : c(X) \to X$ an operator. By definition, the triple $(X, c(X), Lim)$ is called an L-space (Fréchet [39]) if the following conditions are satisfied:

(i) If $x_n = x,\ \forall\ n \in N$, then $(x_n)_{n\in N} \in c(X)$ and $Lim(x_n)_{n\in N} = x$.

(ii) If $(x_n)_{n\in N} \in c(X)$ and $Lim(x_n)_{n\in N} = x$, then for all subsequences, $(x_{n_i})_{i\in N}$, of $(x_n)_{n\in N}$ we have that $(x_{n_i})_{i\in N} \in c(X)$ and $Lim(x_{n_i})_{i\in N} = x$.

By definition, an element of $c(X)$ is a convergent sequence, $x := Lim(x_n)_{n\in N}$ is the limit of this sequence and we also write $x_n \to x$ as $n \to +\infty$.

In what follow we denote an L-space by $(X, \to)$. In particular, if (X, d) is a metric space, then X together with the convergence generated by d is an L-space.

We recall now the following important concepts, see [80], [78], [79], [85], [15].

Definition 2.1. Let $(X, \to)$ be an L-space. An operator $f : X \to X$ is, by definition, a Picard operator (briefly PO) if:

(i) $F_f = \{x^*\}$;

(ii) $(f^n(x))_{n\in\mathbb{N}} \to x^*$ as $n \to \infty$, for all $x \in X$.

For example, any contraction $f : X \to X$ on a complete metric space (X, d) is a PO.

Definition 2.2. Let $(X, \to)$ be an L-space. Then, $f : X \to X$ is called a weakly Picard operator (briefly WPO) if, for all $x \in X$, the sequence $(f^n(x))_{n\in N}$ converges and the limit (which may depend on x) is a fixed point of f.

For example, any continuous graphic contraction $f : X \to X$ on a complete metric space (X, d) is a WPO.

Notice that, if $f : X \to X$ is a WPO, then we define the operator $f^\infty : X \to F_f$ by

$$f^\infty(x) := \lim_{n\to\infty} f^n(x).$$

A triple $(X, \to, \preceq)$ is called an ordered L-space if $(X, \to)$ is an L-space and $\preceq$ is a partially order relation on X, which is closed with respect to $\to$, i.e., if $(x_n)_{n\in N}, (y_n)_{n\in N}$ are sequences in X such that $x_n \preceq y_n$ for every $n \in \mathbb{N}$ and $x_n \to x$, $y_n \to y$ as $n \to \infty$, then $x \preceq y$.

As usual, if $\preceq$ is a partially order relation on X, then, for $x, y \in X$, we denote $x \prec y$ if $x \preceq y$ and $x \neq y$.

The following abstract Gronwall type lemmas take place for POs and WPOs.

Lemma 2.1. *Let $(X, \to, \preceq)$ be an ordered L-space and $f : X \to X$ be an operator. We suppose:*

(a) f is a PO with respect to $\to$ (we denote by x_f^* its unique fixed point);

(b) f is increasing with respect to $\preceq$;

Then, we have:

(i) $x \in X$, $x \preceq f(x)$ implies $x \preceq x_f^*$;

(ii) $x \in X$, $x \succeq f(x)$ implies $x \succeq x_f^*$.

Lemma 2.2. *Let $(X, \to, \preceq)$ be an ordered L-space and $f : X \to X$ be an operator. We suppose:*

(a) f is a WPO with respect to $\to$;

(b) f is increasing with respect to $\preceq$;

Then, we have:

(i) the operator f^∞ is increasing;

(ii) $x \in X$, $x \preceq f(x)$ implies $x \preceq f^\infty(x)$;

(iii) $x \in X$, $x \succeq f(x)$ implies $x \succeq f^\infty(x)$.

In particular, the above results take place in ordered metric spaces $(X, d, \preceq)$, where the convergence in X is generated by the metric d.

Remark 2.1. In many papers, some authors choose to define an ordered metric space as a nonempty set endowed with a metric and a partial order. Here, we have chosen to consider the notion in the Bourbaki' spirit, i.e., with the additional assumption of the compatibility between the two structures: the metric structure and the ordered one.

Another important concept is given in the context of a metric space.

Definition 2.3. Let (X, d) be a metric space. Then, $f : X \to X$ is called a ψ-weakly Picard operator (briefly ψ-WPO) if f is a WPO, $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ is an increasing, continuous in 0 with $\psi(0) = 0$, such that the following relation holds:

$$d(x, f^\infty(x)) \leq \psi(d(x, f(x))), \text{ for all } x \in X.$$

In particular, if f is a PO and $x^* \in X$ denotes its unique fixed point, then f is said to be a ψ-Picard operator (briefly ψ-PO) if

$$d(x, x^*) \leq \psi(d(x, f(x))), \text{ for all } x \in X.$$

In both cases, if $\psi(t) := ct$, for every $t \in \mathbb{R}_+$ (for some $c > 0$), then f is called a c-WPO, respectively c-PO.

Definition 2.4. Let (X, d) be a metric space and $f : X \to X$ be an operator. Then, two elements $x, y \in X$ are called asymptotically equivalent if $d(f^n(x), f^n(y)) \to 0$ as $n \to \infty$;

Recall that $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is said to be a comparison function (see [79]) if it is increasing and $\varphi^n(t) \to 0$, as $n \to +\infty$. As a consequence, we also have $\varphi(t) < t$, for each $t > 0$, $\varphi(0) = 0$ and φ is continuous in 0. A function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is said to be a strong comparison function (see [79]) if it is increasing and $\sum_{n\geq 0} \varphi^n(t) < \infty$, for every $t > 0$. In this case, the function $s(t) := \sum_{n\geq 0} \varphi^n(t)$ $(t \in \mathbb{R}_+)$ is increasing and continuous at 0.

Definition 2.5. Let (X, d) be a metric space. An operator $f : X \to X$ is called a φ-contraction if $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is a comparison function and

$$d(f(x), f(y)) \leq \varphi(d(x, y)), \text{ for all } x, y \in X.$$

We present now some concepts from stability theory (see [72], [83], [82]).

Let (X, d) be a metric space and $f : X \to X$ be an operator. In this context we have the following notions.

Definition 2.6. (a) The fixed point problem $x = f(x)$ is well-posed if $F_f = \{x^*\}$ and for any sequence $\{u_n\}$ in X with $d(u_n, f(u_n)) \to 0$ we have that $u_n \to x^*$ as $n \to \infty$;

(b) The operator f has the Ostrowski property if $F_f = \{x^*\}$ and for any sequence $\{y_n\}$ in X with $d(y_{n+1}, f(y_n)) \to 0$ we have that $y_n \to x^*$ as $n \to \infty$;

(c) The operator f has the limit shadowing property if for any sequence $\{y_n\}$ in X with $d(y_{n+1}, f(y_n)) \to 0$, there exists $x \in X$ such that $d(y_n, f^n(x)) \to 0$ as $n \to \infty$;

(d) The operator f has the shadowing property if for any $\varepsilon > 0$ there exists $\delta > 0$ such that for each sequence $\{y_n\}$ in X with $d(y_{n+1}, f(y_n)) < \delta$ for every $n \in \mathbb{N}$, there exists $x \in X$ such that $d(y_n, f^n(x)) < \varepsilon$, for every $n \in \mathbb{N}$.

(e) The fixed point equation $x = f(x)$ is Ulam-Hyers stable if there exists $c > 0$ such that, for every $\epsilon > 0$ and any $z \in X$ with $d(z, f(z)) \leq \epsilon$, there exists $x^* \in F_f$ with $d(z, x^*) \leq c \cdot \epsilon$;

(f) The fixed point equation $x = f(x)$ is generalized Ulam-Hyers stable if there exists a function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ increasing, continuous at 0 and $\psi(0) = 0$ such that for every $\epsilon > 0$ and any $z \in X$ with $d(z, f(z)) \leq \epsilon$, there exists $x^* \in F_f$ with $d(z, x^*) \leq \psi(\epsilon)$.

For example, in the case of contraction mappings, we have the following result (see [83], [63]).

Theorem 2.1. *(Saturated principle of contraction) Let (X, d) be a complete metric space and $f : X \to X$ be an L-contraction. Then the following conclusions hold:*

(i) there exists $x^* \in X$ such that $F_f = F_{f^n} = \{x^*\}$;

(ii) f is a PO;

(iii) f is a $\frac{1}{1-L}$-PO;
(iv) the fixed point problem $x = f(x)$ is well-posed;
(v) the operator f has the Ostrowski property;
(vi) the operator f has the limit shadowing property;
(vii) the operator f has the shadowing property;
(viii) the fixed point equation $x = f(x)$ is Ulam-Hyers stable.

Finally, we notice that the aim of this paper is to present a similar result in the case of a set endowed with a metric and an ordered relation.

3. Generalized monotone operators

We need some notions from ordered set theory (see, for example, [66]). Let $(X, \preceq)$ be an ordered set and $f : X \to X$ be an operator.

By definition, f is called:
(1) increasing if $(x, y \in X, x \preceq y)$ imply $f(x) \preceq f(y)$;
(2) decreasing if $(x, y \in X, x \preceq y)$ imply $f(x) \succeq f(y)$;
(3) monotone if f is increasing or decreasing;
(4) progressive if $x \preceq f(x)$, for every $x \in X$;
(5) regressive if $x \succeq f(x)$, for every $x \in X$.

We introduce now some new classes of operators on an ordered set.

Let us consider the following sets:

$$X_{\preceq} := \{(x, y) \in X \times X | x \preceq y \text{ or } y \preceq x\} \text{ and } X_f := \{x \in X | (x, f(x)) \in X_{\preceq}\}.$$

We also define $(f \times f)(x, y) := (f(x), f(y))$, for $(x, y) \in X \times X$.

Definition 3.1. Let $(X, \preceq)$ be an ordered set and $f : X \to X$ be an operator. Then, f is called a generalized monotone operator if $(f \times f)(X_{\preceq}) \subset X_{\preceq}$.

Remark 3.1. We observe that:
(a) $F_f \subset X_f$;
(b) If f is a generalized monotone operator, then:
  (i) $f(X_f) \subset X_f$;
  (ii) for $x \in X_f$ we have $O_f(x) := \{f^n(x) | n \in \mathbb{N}\} \subset X_f$;
  (iii) if $F_f = \{x^*\}$ and $X_{x^*} := \{x \in X | (x, x^*) \in X_{\preceq}\}$, then $f(X_{x^*}) \subset X_{x^*}$.

Remark 3.2. There are two classes of operators defined in terms of X_f:
- operators f such that $X_f = X$;
- operators f such that $X_f = f(X)$.

First class includes progressive and regressive operators, while second class contains operators f which are progressive or regressive on $f(X)$.

We give now some examples of generalized monotone operators.

Example 3.1. (1) Let $(X, \preceq)$ be an ordered set and $f : X \to X$ be an operator. If f is increasing with respect to $\preceq$, then f is a generalized monotone operator.

(2) Let $(X, \preceq)$ be an ordered set and $f : X \to X$ be an operator. If f is decreasing with respect to $\preceq$, then f is a generalized monotone operator.

(3) Let $(X, \preceq)$ be a totally ordered set. Then, any operator $f : X \to X$ is a generalized monotone operator.

(4) Let $(X, \preceq)$ be an ordered set and suppose that $X = Y_1 \cup Y_2$ is a partition of X, such that $Y_1 \times Y_2 \subset X \times X \setminus X_{\preceq}$. Let $f : X \to X$ be an operator, such that $f : Y_1 \to X$ is increasing with respect to $\preceq$ and $f : Y_2 \to X$ is decreasing with respect to $\preceq$. Then, f is a generalized monotone operator.

(5) Let $(X, \preceq_X)$ and $(Y, \preceq_Y)$ be two ordered sets, $f : X \to X$ be an increasing operators and $g : Y \to Y$ be a decreasing operators. Let $Z := X \cup Y$ be the disjoint union of the sets X and Y. We consider on Z the following order relation: on X we define $\preceq_Z := \preceq_X$, on Y we define $\preceq_Z := \preceq_Y$, while if $z_1 \in X$ and $z_2 \in Y$, then z_1, z_2 are not comparable with respect to $\preceq_Z$.

In the above conditions, the operator $h : Z \to Z$ defined by

$$h(z) = \begin{cases} f(z), & \text{if } z \in X \\ g(z), & \text{if } z \in Y \end{cases}$$

is generalized monotone and $Z_{\preceq_Z} = X_{\preceq_X} \cup Y_{\preceq_Y}$, $Z_h = X_f \cup Y_g$.

We recall some classes of ordered sets which appear in some fixed point results.

Definition 3.2. A nonempty ordered set $(X, \preceq)$ is said to be directed upward if for each pair of elements $x, y \in X$ there exists $z \in X$ such that $x \preceq z$ and $y \preceq z$. Dually, $(X, \preceq)$ is said to be directed downward if for each pair of elements $x, y \in X$ there exists $z \in X$ such that $x \succeq z$ and $y \succeq z$.

We introduce now a new class of ordered sets, which generalizes the above concepts and will we used in our main theorems.

Definition 3.3. A nonempty ordered set $(X, \preceq)$ is said to be a generalized directed set if for each pair of elements $x, y \in X$ there exists $z \in X$ such that (x, z) and (y, z) are in $X_{\preceq}$.

4. Generalized contractions and generalized monotone operators on a set endowed with a metric and an order relation

The main idea of this section is given by the following result.

Lemma 4.1. *Let X be a nonempty set, d be a metric on X and $\preceq$ be an order relation on X. We consider an operator $f : X \to X$ having the generalized monotone property. We suppose:*

(i) $(X, \preceq)$ is a generalized directed set;

(ii) if $(x, y) \in X_{\preceq}$, then x and y are asymptotically equivalent;

(iii) $X_f \neq \emptyset$ and $f : X_f \to X_f$ is a WPO.

Then, $f : X \to X$ is a PO.

Proof. Let $x \in X$ be arbitrarily chosen. Let $y \in X_f$. For the pair $(x, y) \in X \times X$, by (i), there is $z \in X$ such that $(x, z), (y, z) \in X_{\preceq}$. By (ii) it follows that

$$d(f^n(x), f^n(z)) \to 0 \text{ and } d(f^n(y), f^n(z)) \to 0, \text{ and } n \to \infty.$$

By (iii) we have that $f^n(y) \to f^\infty(y) \in F_f$ as $n \to \infty$. Thus, $f^n(x) \to f^\infty(y)$ as $n \to \infty$.

Hence, if $x^* := f^\infty(y) \in F_f$, then, for each $x \in X$, we have that $f^n(x) \to x^*$ as $n \to \infty$. As a conclusion, f is a PO. □

Remark 4.1. If X is a $\vee$-semi-lattice or a $\wedge$-semi-lattice, then the assumption (i) is satisfied.

By the above result, the following question arises.

Problem 1. Which metric assumptions on f with respect to $X_\preceq$ assure that the following conditions are realized:

(i) $(x,y) \in X_\preceq \;\Rightarrow\;$ x and y are asymptotically equivalent;

(ii) $f : X_f \to X_f$ is a WPO.

In the last part of this section some results related to the above problem are given.

Theorem 4.1. *Let X be a nonempty set, d be a complete metric on X and $\preceq$ be an order relation on X. We consider an operator $f : X \to X$ having the generalized monotone property. We suppose:*

(i) $(X, \preceq)$ is a generalized directed set;

(ii) there exists $L \in]0,1[$ such that

$$d(f(x), f(y)) \le Ld(x,y), \text{ for every } x, y \in X \text{ with } x \preceq y;$$

(iii) $X_f \neq \emptyset$ and $f : X \to X$ is orbitally continuous.

Then, the following conclusions hold:

(1) $f : X \to X$ is a PO;

(2) $f : X_f \to X_f$ is a $\frac{1}{1-L}$-PO;

(3) $f : X_{x^*} \to X_{x^*}$ is L-quasicontraction;

(4) $f : X_{x^*} \to X_{x^*}$ is a $\frac{1}{1-L}$-PO;

(5) if $(y_n)_{n\in\mathbb{N}} \subset X_{x^*}$ and $d(y_n, f(y_n)) \to 0$ as $n \to \infty$, then $y_n \to x^*$ as $n \to \infty$, i.e., the fixed point problem is well-posed for $f|_{X_{x^*}}$;

(6) if $(y_n)_{n\in\mathbb{N}} \subset X_{x^*}$ and $d(y_{n+1}, f(y_n)) \to 0$ as $n \to \infty$, then $y_n \to x^*$ as $n \to \infty$, i.e., $f|_{X_{x^*}}$ has the Ostrowski property;

(7) for each $\varepsilon > 0$ there exists $\delta > 0$ such that, if $(y_n)_{n\in\mathbb{N}} \subset X_{x^*}$ and $d(y_{n+1}, f(y_n)) < \delta$ for every $n \in \mathbb{N}$, then there exists $x \in X_{x^*}$ such that $d(y_n, f^n(x)) < \varepsilon$, for every $n \in \mathbb{N}$, i.e., $f|_{X_{x^*}}$ has the shadowing property.

Proof. Notice first that, by the symmetry of the metric d, the condition (ii) is satisfied for all $(x,y) \in X_\preceq$.

(1) By (ii) it follows that $f : X_f \to X_f$ is a graphic L-contraction. Then, $f^n(x) \to f^\infty(x)$ as $n \to \infty$, for each $x \in X_f$. By the orbital continuity of f, we get that $f^\infty(x) \in F_f$, i.e., $f : X_f \to X_f$ is a WPO. By (ii) we obtain that, for each $(x,y) \in X_\preceq$, the elements x and y are asymptotically equivalent. The conclusion follows now by Lemma 4.1.

(2) By (1) we have that $F_f = \{x^*\}$. Let $x \in X_f$ be arbitrary. Since $f : X_f \to X_f$ is a graphic contraction, we have that

$$\begin{aligned} d(x,x^*) &\le d(x,f(x)) + d(f(x),f^2(x)) + \cdots + d(f^n(x),f^{n+1}(x)) + d(f^{n+1}(x),x^*) \\ &\le \frac{1}{1-L}d(x,f(x)) + d(f^{n+1}(x),x^*),\ \text{for all } n \in \mathbb{N}^* \text{ and every } x \in X_f. \end{aligned}$$

Letting $n \to \infty$, we obtain that

$$d(x,x^*) \le \frac{1}{1-L}d(x,f(x)),\ \text{for each } x \in X_f,$$

proving that $f : X_f \to X_f$ is a $\frac{1}{1-L}$-PO.

(3) Let $x \in X_{x^*}$. Then $d(f(x),x^*) = d(f(x),f(x^*)) \le Ld(x,x^*)$. Thus, $f : X_{x^*} \to X_{x^*}$ is an L-quasicontraction.

(4) By (1) we know that $f : X_{x^*} \to X_{x^*}$ is a PO. Then, for every $x \in X_{x^*}$, we have $d(x,x^*) \le d(x,f(x)) + d(f(x),x^*) \le d(x,f(x)) + Ld(x,x^*)$. Thus

$$d(x,x^*) \le \frac{1}{1-L}d(x,f(x)),\ \text{for all } x \in X_{x^*}.$$

(5) Let $(y_n)_{n\in\mathbb{N}} \subset X_{x^*}$ such that $d(y_n,f(y_n)) \to 0$ as $n \to \infty$. Then, we have

$$d(y_n,x^*) \le d(y_n,f(y_n)) + d(f(y_n),x^*) \le d(y_n,f(y_n)) + Ld(y_n,x^*).$$

Thus, we get that $d(y_n,x^*) \to 0$ as $n \to \infty$.

(6) Let $(y_n)_{n\in\mathbb{N}} \subset X_{x^*}$ such that $d(y_{n+1},f(y_n)) \to 0$ as $n \to \infty$. Then, we have

$$\begin{aligned} d(y_{n+1},x^*) &\le d(y_{n+1},f(y_n)) + d(f(y_n),x^*) \le d(y_{n+1},f(y_n)) + Ld(y_n,x^*) \\ &\le d(y_{n+1},f(y_n)) + Ld(y_n,f(y_{n-1})) + L^2d(y_{n-1},x^*) \le \cdots \\ &\le \sum_{k=0}^{n} L^{n-k}d(y_{k+1},f(y_k)) + L^{n+1}d(y_0,x^*). \end{aligned}$$

The conclusion follows by Cauchy-Toeplitz Lemma (see, for example, [85]). □

Remark 4.2. If in Theorem 4.1, instead of the assumption (i) we suppose that $(X,\preceq)$ is a lattice, then, for all $x \in X$, we have

$$d(f^n(x),f^n(x \vee x^*)) \le L^n d(x,x \vee x^*)$$

and

$$d(f^n(x \vee x^*),x^*) \le L^n d(x \vee x^*,x^*).$$

Hence, we get the following estimation:

$$d(f^n(x),x^*) \le L^n\left[d(x,x \vee x^*) + d(x \vee x^*,x^*)\right],\ \text{for every } x \in X \text{ and } n \in \mathbb{N}^*.$$

Similarly, we obtain

$$d(f^n(x),x^*) \le L^n\left[d(x,x \wedge x^*) + d(x \wedge x^*,x^*)\right],\ \text{for every } x \in X \text{ and } n \in \mathbb{N}^*.$$

Another fixed point result of this type uses the notion of φ-contraction.

Theorem 4.2. *Let X be a nonempty set, d be a complete metric on X and $\preceq$ be an order relation on X. We consider an operator $f : X \to X$ having the generalized monotone property. We suppose:*

(i) $(X, \preceq)$ is a generalized directed set;

(ii) there exists a strong comparison function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$d(f(x), f(y)) \leq \varphi(d(x, y)), \text{ for every } x, y \in X \text{ with } x \preceq y;$$

(iii) $X_f \neq \emptyset$ and $f : X \to X$ is orbitally continuous.

Then, the following conclusions hold:

(1) $f : X \to X$ is a PO;

(2) $f : X_f \to X_f$ is a s-PO, where $s(t) := \sum_{n \geq 0} \varphi^n(t)$, for $t \in \mathbb{R}_+$;

(3) $f : X_{x^*} \to X_{x^*}$ is φ-quasicontraction;

(4) if, additionally, $t - \varphi(t) \to \infty$ as $t \to \infty$, then $f : X_{x^*} \to X_{x^*}$ is a η_φ-PO, where $\eta_\varphi(u) = \sup\{t \in \mathbb{R}_+ | t - \varphi(t) \leq u\}$;

(5) if, additionally, $t - \varphi(t) \to \infty$ as $t \to \infty$ and $(y_n)_{n \in \mathbb{N}} \subset X_{x^*}$ is such that $d(y_n, f(y_n)) \to 0$ as $n \to \infty$, then $y_n \to x^*$ as $n \to \infty$;

(6) if, additionally, φ is a subadditive function and $(y_n)_{n \in \mathbb{N}} \subset X_{x^*}$ is such that $d(y_{n+1}, f(y_n)) \to 0$ as $n \to \infty$, then $y_n \to x^*$ as $n \to \infty$.

Proof. (1) By (ii) it follows that $f : X_f \to X_f$ is a graphic φ-contraction, i.e.,

$$d(f(x), f^2(x)) \leq \varphi(d(x, f(x))), \text{ for every } x \in X_f.$$

Then, we obtain

$$d(f^n(x), f^{n+1}(x)) \leq \varphi^n(d(x, f(x))) \to 0 \text{ as } n \to \infty, \text{ for every } x \in X_f.$$

By the strong comparison assumption on φ, we obtain that, for each $x \in X_f$, the sequence $(f^n(x))_{n \in \mathbb{N}}$ is Cauchy. Thus, $(f^n(x))_{n \in \mathbb{N}}$ is convergent in X, for each $x \in X_f$. Using the orbital continuity of $f_{|X_f}$, we have that $f^n(x) \to f^\infty(x) \in F_f$ as $n \to \infty$, for each $x \in X_f$. Thus, $f : X_f \to X_f$ is a WPO. By (ii) we obtain that, for each $(x, y) \in X_\preceq$, the elements x and y are asymptotically equivalent. The conclusion follows now by Lemma 4.1.

(2) By (1) we have that $F_f = \{x^*\}$. Let $x \in X_f$ be arbitrary. Since $f : X_f \to X_f$ is a graphic φ-contraction, we have, for every $x \in X_f$, that

$$d(x, x^*) \leq d(x, f(x)) + d(f(x), f^2(x)) + \cdots + d(f^n(x), f^{n+1}(x)) + d(f^{n+1}(x), x^*)$$
$$\leq d(x, f(x)) + \varphi(d(x, f(x))) + \cdots + \varphi^n(d(x, f(x))) + d(f^{n+1}(x), x^*), \text{ for all } n \in \mathbb{N}^*.$$

Letting $n \to \infty$, we obtain that

$$d(x, x^*) \leq \sum_{n \geq 0} \varphi^n(d(x, f(x))) = s(d(x, f(x))), \text{ for each } x \in X_f,$$

proving that $f : X_f \to X_f$ is a s-PO.

(3) Let $x \in X_{x^*}$. Then $d(f(x), x^*) = d(f(x), f(x^*)) \leq \varphi(d(x, x^*))$.
Thus, $f : X_{x^*} \to X_{x^*}$ is an φ-quasicontraction.

(4) By (1) we know that $f : X_{x^*} \to X_{x^*}$ is a PO. Then, for every $x \in X_{x^*}$, we have

$$d(x, x^*) \leq d(x, f(x)) + d(f(x), x^*) \leq d(x, f(x)) + \varphi(d(x, x^*)).$$

Thus

$$d(x,x^*) \le \eta_\varphi(d(x,f(x))), \text{ for all } x \in X_{x^*}.$$

(5) Let $(y_n)_{n\in\mathbb{N}} \subset X_{x^*}$ such that $d(y_n,f(y_n)) \to 0$ as $n \to \infty$. Then, we have

$$d(y_n,x^*) \le d(y_n,f(y_n)) + d(f(y_n),x^*) \le d(y_n,f(y_n)) + \varphi(d(y_n,x^*)).$$

Thus, we get that $d(y_n,x^*) \le \eta_\varphi(d(y_n,f(y_n))) \to 0$ as $n \to \infty$.

(6) Let $(y_n)_{n\in\mathbb{N}} \subset X_{x^*}$ such that $d(y_{n+1},f(y_n)) \to 0$ as $n \to \infty$. Then, we have

$$\begin{aligned} d(y_{n+1},x^*) &\le d(y_{n+1},f(y_n)) + d(f(y_n),x^*) \le d(y_{n+1},f(y_n)) + \varphi(d(y_n,x^*)) \\ &\le d(y_{n+1},f(y_n)) + \varphi(d(y_n,f(y_{n-1}))) + \varphi^2(d(y_{n-1},x^*)) \le \cdots \\ &\le \sum_{k=0}^{n} \varphi^{n-k}(d(y_{k+1},f(y_k))) + \varphi^{n+1}(d(y_0,x^*)). \end{aligned}$$

The conclusion follows by generalized Cauchy-Toeplitz Lemma (see, for example, [65]). □

We have the following general remark.

Remark 4.3. Let $(X,\to)$ be an L-space and $f : X \to X$ be an operator. Then, the following statements are equivalent:

(1) f is a PO;
(2) there exists $p \in \mathbb{N}^*$ such that f^p is a PO;
(3) for every $p \in \mathbb{N}^*$ we have that f^p is a PO.

By the above remark, we have the following theorem concerning the fixed points of an operator having the contraction property for one of its iterates.

Theorem 4.3. *Let X be a nonempty set, d be a complete metric on X and $\preceq$ be an order relation on X. We consider an operator $f : X \to X$ having the generalized monotone property. We suppose:*

(i) $(X,\preceq)$ is a generalized directed set;
(ii) there exist $p \in \mathbb{N}^*$ and $k \in]0,1[$ such that

$$d(f^p(x),f^p(y)) \le kd(x,y), \text{ for every } x,y \in X \text{ with } x \preceq y;$$

as $n \to \infty$;
(iii) $X_{f^p} \neq \emptyset$ and $f^p : X \to X$ is orbitally continuous.
Then, $f : X \to X$ is a PO.

Proof. Notice first that f^p satisfies all the assumptions of Theorem 4.1.
Thus, $f^p : X \to X$ is a PO. Now, the conclusion follows by Remark 4.3. □

A fixed point result for the case of Kannan operators is the following.

Theorem 4.4. *Let X be a nonempty set, d be a complete metric on X and $\preceq$ be an order relation on X. We consider an operator $f : X \to X$ having the generalized monotone property. We suppose:*

(i) $(X,\preceq)$ is a generalized directed set;
(ii) there exists $k \in]0,\frac{1}{2}[$ such that

$$d(f(x),f(y)) \le k\left(d(x,f(x)) + d(y,f(y))\right), \text{ for every } x,y \in X \text{ with } x \preceq y;$$

(iii) $X_f \neq \emptyset$ *and* $f : X \to X$ *is orbitally continuous.*
Then, $f : X_f \to X_f$ *is a WPO.*

Proof. By the Kannan type condition (ii), we obtain that the operator $f : X_f \to X_f$ is a graphic $\frac{k}{1-k}$-contraction. This (together with the orbital continuity of $f_{|X_f}$) implies that $f : X_f \to X_f$ is a WPO and we have $f^n(x) \to f^\infty(x) \in F_f$ as $n \to \infty$, for each $x \in X_f$. □

Remark 4.4. In [74], Ran and Reurings use the following relaxation of the contraction condition:

(A) there exists $k \in]0, 1[$ such that

$$d(f(x), f(y)) \le kd(x, y), \text{ for every } (x, y) \in X \times X \text{ with } x \preceq y.$$

On the other hand, in [70], the authors consider with the following assumption:

(B) there exists $k \in]0, 1[$ such that

$$d(f(x), f(y)) \le kd(x, y), \text{ for every } (x, y) \in X_\preceq.$$

It is easy to see that, because of the symmetry of the metric d and of the contraction condition, we have that (A) ⇔ (B). This remark also applies for the φ-contraction condition, for Kannan's condition, for Chatterjea's condition (see [24]) or for Ćirić's condition (see [27]). It is also worth to note that, in the absence of the symmetry of d (for example, the case of quasi-metric spaces) or in the case of non-symmetrical contraction type conditions, the above equivalence does not hold (for example, the case of almost contractions in the sense of Berinde, see [14]). In this situations, condition (B) is more restrictive.

5. Generalized contractions and increasing operators on ordered metric spaces

By the above results we obtain some fixed point theorems for increasing operators in complete metric spaces.

Theorem 5.1. *Let* $(X, d, \preceq)$ *be a complete ordered metric space and* $f : X \to X$ *be an increasing operator with respect to* $\preceq$. *We suppose:*

(i) $(X, \preceq)$ *is a generalized directed set;*

(ii) there exists $L \in]0, 1[$ *such that*

$$d(f(x), f(y)) \le Ld(x, y), \text{ for every } x, y \in X \text{ with } x \preceq y;$$

(iii) $X_f \neq \emptyset$ *and* $f : X \to X$ *is orbitally continuous.*

Then, the following conclusions hold:

(1) $f : X \to X$ *is a PO (we denote by* x^* *its unique fixed point);*

(2) $f : X_f \to X_f$ *is a* $\frac{1}{1-L}$*-PO.*

(3) $(x, x^*) \in X_\preceq$, *for every* $x \in X_f$*;*

(4) if $(y_n)_{n\in\mathbb{N}} \subset X_{x^*}$ *and* $d(y_n, f(y_n)) \to 0$ *as* $n \to \infty$, *then* $y_n \to x^*$ *as* $n \to \infty$*;*

(5) if $(y_n)_{n\in\mathbb{N}} \subset X_{x^*}$ *and* $d(y_{n+1}, f(y_n)) \to 0$ *as* $n \to \infty$, *then* $y_n \to x^*$ *as* $n \to \infty$*;*

(6) for each $\varepsilon > 0$ *there exists* $\delta > 0$ *such that, if* $(y_n)_{n\in\mathbb{N}} \subset X_{x^*}$ *and* $d(y_{n+1}, f(y_n)) < \delta$, *for every* $n \in \mathbb{N}$, *then there exists* $x \in X_{x^*}$ *such that* $d(y_n, f^n(x)) < \varepsilon$, *for every* $n \in \mathbb{N}$*;*

(7) if $x \in X$, then:
(a) $x \preceq f(x)$ implies $x \preceq x^*$;
(b) $x \succeq f(x)$ implies $x \succeq x^*$;
(8) if $x, y \in X$ such that $x \prec y, x \preceq f(x), y \succeq f(y)$, then $x \preceq x^* \preceq y$.

Proof. The first two conclusions and (5)-(6) follow by Theorem 4.1. Conclusions (3) and (7) follow by Lemma 2.1, while conclusion (8) is a consequence of (7). □

6. Generalized contractions and decreasing operators on ordered metric spaces

In some papers, after presenting fixed point results for increasing operators, some authors (see, for example, [5], [92]) notice that similar results for decreasing operators. It is clear that this is not the case here. In general, a decreasing operator from a complete lattice to itself has no fixed points. A good remark is the following (see Amann [10]): Let $(X, \preceq)$ be an ordered set and $f : X \to X$ be a decreasing operator. Then, the operator f^2 is increasing. Let us suppose that X has a minimum element τ (i.e., $\tau \preceq x$, for all $x \in X$). Then $\tau \preceq f(x) \preceq f(\tau)$. Thus, $f(X) \subset [\tau, f(\tau)]$. Hence $F_f \subset [\tau, f(\tau)]$. If $[\tau, f(\tau)]$ is a complete ordered set, then $F_{f^2} \neq \emptyset$. If $F_{f^2} = \{x^*\}$, then $F_f = \{x^*\}$.

For the fixed point theory of decreasing operators, see [1], [2], [5], [10], ... By the results given in Section 4, we also obtain some fixed point theorems for decreasing operators in complete metric spaces.

Theorem 6.1. *Let $(X, d, \preceq)$ be a complete ordered metric space and $f : X \to X$ be an decreasing operator with respect to $\preceq$. We suppose:*
(i) $(X, \preceq)$ is a generalized directed set;
(ii) there exists $L \in]0, 1[$ such that

$$d(f(x), f(y)) \leq Ld(x, y), \text{ for every } x, y \in X \text{ with } x \preceq y;$$

(iii) $X_{f^2} \neq \emptyset$ and $f^2 : X \to X$ is orbitally continuous.
Then, $f : X \to X$ is a PO.

Proof. Since f is decreasing, we get that $f^2 : X \to X$ is an increasing operator with respect to $\preceq$. By (ii) we get that f^2 is a L^2-contraction on comparable elements, i.e.,

$$d(f^2(x), f^2(y)) \leq L^2 d(x, y), \text{ for every } x, y \in X \text{ with } x \preceq y.$$

By Theorem 5.1 we get that f^2 is a PO. The conclusion follows by Remark 4.3. □

7. Applications in the theory of differential and integral equations in Banach spaces

7.1. **Banach lattices of continuous Banach lattice-valued functions.** Let us consider the Banach lattice $(\mathbb{B}, +, \mathbb{R}, \|\cdot\|, \preceq)$ (see [89], [35]) and denote

$$X := C([a, b], \mathbb{B}) := \{x : [a, b] \to \mathbb{B} | x \text{ is continuous }\}.$$

We consider on X the following norms:

$$\|x\|_\infty := \max_{t \in [a,b]} \|x(t)\|, \ \|x\|_\tau := \max_{t \in [a,b]} \left(\|x(t)\| e^{-\tau(t-a)}\right) (\tau > 0)$$

and the standard order relation

$$x \leq_C y \Leftrightarrow x(t) \preceq y(t), t \in [a,b].$$

We get the following Banach lattices $(X, +, \mathbb{R}, \|\cdot\|_\infty, \leq_C)$ and $(X, +, \mathbb{R}, \|\cdot\|_\tau, \leq_C)$.

By Theorem 5.1, we have the following result.

Theorem 7.1. *Let us consider the Banach lattice $(X, +, \mathbb{R}, \|\cdot\|_\infty, \leq_C)$ and $T : X \to X$ be an operator. We suppose:*
(i) T is increasing;
(ii) there exists $L \in [0, 1[$ such that

$$\|T(x) - T(y)\|_\infty \leq L\|x - y\|_\infty, \text{ for every } x, y \in X \text{ with } x \leq_C y;$$

(iii) $X_T \neq \emptyset$.
Then, the following conclusions hold:
(1) $F_T = F_{T^n} = \{x^*\}$, for every $n \geq 2$ and T is a PO;
(2) $T : X_T \to X_T$ is a $\frac{1}{1-L}$-PO;
(3) $(x, x^*) \in X_{\leq_C}$, for every $x \in X_T$, i.e., $T : X_{x^*} \to X_{x^*}$ is a quasicontraction;
(4) the fixed point problem $x = T(x)$ (where $T : X_{x^*} \to X_{x^*}$) is well-posed;
(5) the operator $T : X_{x^*} \to X_{x^*}$ has the Ostrowski property;
(6) the fixed point equation $x = T(x)$ (where $T : X_{x^*} \to X_{x^*}$) is Ulam-Hyers stable;
(7) for all $x \in X$ the following implications hold:
(a) $x \leq_C T(x) \Rightarrow x \leq x^*$;
(b) $T(x) \leq_C x \Rightarrow x^* \leq x$.

Remark 7.1. A similar result (with the assumption $L > 0$) holds for the Banach lattice $(X, +, \mathbb{R}, \|\cdot\|_\tau, \leq_C)$. We denote this result by Theorem 8'.

Remark 7.2. In Theorem 7.1 we can consider, instead of the Banach lattice $\mathbb{B}$, the following particular Banach lattices $(\mathbb{R}^m, +, \mathbb{R}, \|\cdot\|_\infty, \leq)$ and $(m(\mathbb{R}), +, \mathbb{R}, \|\cdot\|_\infty, \leq)$, where we denote $m(\mathbb{R}) := \{(x_n)_{n\in\mathbb{N}} \subset \mathbb{R} | (x_n)_{n\in\mathbb{N}}$ is a bounded sequence $\}$.

7.2. **Volterra integral equations and Cauchy problems.** We consider the following Volterra type integral equation in Banach spaces

$$x(t) = \int_a^t K(t, s, x(s))ds + g(t), t \in [a, b], \tag{7.1}$$

where $K \in C([a, b] \times [a, b] \times \mathbb{B}, \mathbb{B})$, $g \in C([a, b], \mathbb{B})$ and $\mathbb{B}$ is a Banach lattice.

Then, by Theorem 8', we have the following result.

Theorem 7.2. *Let us consider the equation (7.1). We suppose:*
(i) $K \in C([a, b] \times [a, b] \times \mathbb{B}, \mathbb{B})$, $g \in C([a, b], \mathbb{B})$;
(ii) there exists $L > 0$ such that, for every $t, s \in [a, b]$, we have

$$\|K(t, s, u) - K(t, s, v)\| \leq L\|u - v\|, \text{ for every } u, v \in \mathbb{B} \text{ with } u \preceq v;$$

(iii) $K(t, s, \cdot)$ is increasing, for every $t, s \in [a, b]$.
Then, the following conclusions hold:
(1) there exists a unique solution $x^* \in C([a, b], \mathbb{B})$ of equation (7.1);

(2) the sequence $\{x_n\}_{n\in\mathbb{N}}$ *from* $C([a,b],\mathbb{B})$ *defined by*

$$x_0 \in C([a,b],\mathbb{B}),\ x_{n+1}(t) = \int_a^t K(t,s,x_n(s))ds + g(t), t \in [a,b]$$

converges, for each $x_0 \in C([a,b],\mathbb{B})$*, to* x^**;*

(3) if $x \in C([a,b],\mathbb{B})$ *is a lower solution of (7.1), then* $x \preceq x^*$*;*

(4) if $x \in C([a,b],\mathbb{B})$ *is an upper solution of (7.1), then* $x^* \preceq x$*.*

Proof. Let $X := C([a,b],\mathbb{B})$ and the Banach lattice $(X,+,\mathbb{R},\|\cdot\|_\tau,\leq_C)$. Define $T : X \to X$ by $Tx(t) := \int_a^t K(t,s,x(s))ds + g(t), t \in [a,b]$. The conclusion follows by Theorem 8' applied for T. $\square$

By the above theorem, we can get an existence and uniqueness results for a Cauchy problem in Banach spaces.

We consider the following Cauchy problem

$$\begin{cases} x'(t) = f(t,x(t)), t \in [a,b] \\ x(a) = x^0, \end{cases} \tag{7.2}$$

where $f \in C([a,b]\times\mathbb{B},\mathbb{B})$, $x^0 \in \mathbb{B}$ and $\mathbb{B}$ is a Banach lattice. We are looking for solutions $x \in C^1([a,b],\mathbb{B})$ of this problem.

Notice first that problem (7.2) is equivalent to the following integral equation of Volterra type

$$x(t) = \int_a^t f(s,x(s))ds + x^0, t \in [a,b]. \tag{7.3}$$

Any solution $x \in C([a,b],\mathbb{B})$ of (7.3) is a solution of (7.2) and vice-versa. Thus, by Theorem 7.2, we get the following existence and uniqueness result.

Theorem 7.3. *Let us consider the Cauchy problem (7.2). We suppose:*

(i) $f \in C([a,b]\times\mathbb{B},\mathbb{B})$ *and* $x^0 \in \mathbb{B}$*;*

(ii) there exists $L > 0$ *such that, for every* $s \in [a,b]$*, we have*

$$\|f(s,u) - f(s,v)\| \leq L\|u - v\|,\ \textit{for every } u,v \in \mathbb{B} \textit{ with } u \preceq v;$$

(iii) $f(s,\cdot)$ *is increasing, for every* $s \in [a,b]$*.*

Then, the following conclusions hold:

(1) there exists a unique solution x^* *of the Cauchy problem (7.2);*

(2) the sequence $\{x_n\}_{n\in\mathbb{N}}$ *in* $C([a,b],\mathbb{B})$*, defined by*

$$x_0 \in C([a,b],\mathbb{B}),\ x_{n+1}(t) = \int_a^t f(s,x_n(s))ds + x^0, t \in [a,b]$$

converges, for each $x_0 \in C([a,b],\mathbb{B})$*, to* x^**;*

(3) if $x \in C([a,b],\mathbb{B})$ *is a lower solution of (7.2), then* $x \preceq x^*$*;*

(4) if $x \in C([a,b],\mathbb{B})$ *is an upper solution of (7.2), then* $x^* \preceq x$*.*

7.3. **Fredholm integral equations and bilocal problems.** We consider the following Fredholm type integral equation in Banach spaces

$$x(t) = \int_a^b K(t, s, x(s))ds + g(t), t \in [a, b], \tag{7.4}$$

where $K \in C([a, b] \times [a, b] \times \mathbb{B}, \mathbb{B})$, $g \in C([a, b], \mathbb{B})$ and $\mathbb{B}$ is a Banach lattice.

Then, by Theorem 7.1, we have the following result.

Theorem 7.4. *Let us consider the equation (7.4). We suppose:*
(i) $K \in C([a, b] \times [a, b] \times \mathbb{B}, \mathbb{B})$, $g \in C([a, b], \mathbb{B})$;
(ii) there exists $L \in [0, 1[$ *such that* $L(b-a) < 1$ *and, for every* $t, s \in [a, b]$, *we have*

$$\|K(t, s, u) - K(t, s, v)\| \leq L\|u - v\|, \text{ for every } u, v \in \mathbb{B} \text{ with } u \preceq v;$$

(iii) $K(t, s, \cdot)$ *is increasing, for every* $t, s \in [a, b]$.
Then, the following conclusions hold:
(1) there exists a unique solution $x^* \in C([a, b], \mathbb{B})$ *of equation (7.4);*
(2) the sequence $\{x_n\}_{n \in \mathbb{N}}$ *from* $C([a, b], \mathbb{B})$, *defined by*

$$x_0 \in C([a, b], \mathbb{B}), \ x_{n+1}(t) = \int_a^b K(t, s, x_n(s))ds + g(t), t \in [a, b]$$

converges, for each $x_0 \in C([a, b], \mathbb{B})$, *to* x^*;
(3) if $x \in C([a, b], \mathbb{B})$ *is a lower solution of (7.4), then* $x \preceq x^*$;
(4) if $x \in C([a, b], \mathbb{B})$ *is an upper solution of (7.4), then* $x^* \preceq x$.

Proof. Let $X := C([a, b], \mathbb{B})$ and the Banach lattice $(X, +, \mathbb{R}, \|\cdot\|_\infty, \leq_C)$. Define $T : X \to X$ by $Tx(t) := \int_a^b K(t, s, x(s))ds + g(t), t \in [a, b]$. The conclusion follows by Theorem 7.1 applied for the operator T. □

By the above theorem, we can get an existence and uniqueness results for a bilocal problem in Banach spaces.

We consider the following bilocal problem

$$\begin{cases} -x''(t) = f(t, x(t)), t \in [a, b] \\ x(a) = x(b) = \Theta, \end{cases} \tag{7.5}$$

where $f \in C([a, b] \times \mathbb{B}, \mathbb{B})$ and $\Theta \in \mathbb{B}$ is the null element of the Banach lattice $\mathbb{B}$. We are looking for solutions $x \in C^2([a, b], \mathbb{B})$ of this problem.

Notice first that problem (7.5) is equivalent to the following Fredholm type integral equation

$$x(t) = \int_a^b G(s, t)f(s, x(s))ds, t \in [a, b], \tag{7.6}$$

where G is the Green function corresponding to the above problem. Notice that any solution $x \in C([a, b], \mathbb{B})$ of (7.6) is a solution of (7.5) and vice-versa. Hence, by Theorem 7.4, we get the following existence and uniqueness result.

Theorem 7.5. *Let us consider the bilocal problem (7.5). We suppose:*
(i) $f \in C([a, b] \times \mathbb{B}, \mathbb{B})$;

(ii) there exists $L \in [0,1[$ such that, for every $s \in [a,b]$, we have

$$\|f(s,u) - f(s,v)\| \leq L\|u - v\|, \text{ for every } u, v \in \mathbb{B} \text{ with } u \preceq v;$$

(iii) $L\|G(t,s)\|(b-a) < 1$;

(iii) $f(s,\cdot)$ is increasing, for every $s \in [a,b]$.

Then, the following conclusions hold:

(1) there exists a unique solution x^* of the bilocal problem (7.5);

(2) the sequence $\{x_n\}_{n\in\mathbb{N}}$ in $C([a,b],\mathbb{B})$, defined by

$$x_0 \in C([a,b],\mathbb{B}),\ x_{n+1}(t) = \int_a^b G(s,t)f(s,x_n(s))ds, t \in [a,b]$$

converges, for each $x_0 \in C([a,b],\mathbb{B})$, to x^*;

(3) if $x \in C([a,b],\mathbb{B})$ is a lower solution of (7.5), then $x \preceq x^*$;

(4) if $x \in C([a,b],\mathbb{B})$ is an upper solution of (7.5), then $x^* \preceq x$.

8. Open questions

The above considerations give rise to some open questions.

8.1. **Fixed point theory for Y-contractions.** Let (X,d) be a metric space, $f : X \to X$ be an operator and $Y \subset X \times X$ a nonempty set. By definition, f is called a Y-contraction if there exists $L \in [0,1[$ such that

$$d(f(x),f(y)) \leq Ld(x,y), \text{ for every } (x,y) \in Y.$$

The problem is to give conditions on X, Y and f, which imply similar conclusions to those from the Saturated Principle of Contraction.

In this paper, we studied the problem for the case when X is endowed with a metric d and an ordered relation $\preceq$. In this case, $Y := X_{\preceq}$. Following [81] (see also [85], page 282) we present other examples of Y-contractions.

(1) If (X,d) is a metric space and $f : X \to X$ is a graphic contraction, then f is a $Graph(f)$-contraction;

(2) (Weingram (1969)) Contractions outside a compact set are Y-contractions. For example, let us consider $X := \mathbb{R}^m$ endowed with one of the classical metric d, $Y := \mathbb{R}^m \times \mathbb{R}^m \setminus (Z \times Z)$, where $Z \subset \mathbb{R}^m$ is a nonempty compact set.

(3) (Kirk-Srivasan-Veeramani (2003)) Let (X,d) be a metric space, $f : X \to X$ be an operator and A_i (for $i \in \{1,2,\cdots,p\}$) be nonempty subsets of X such that $X = \bigcup_{i=1}^{p} A_i$ and $f(A_i) \subset A_{i+1}$, for each for $i \in \{1,2,\cdots,p\}$, where $A_{p+1} = A_1$. Suppose that f is a cyclic L-contraction, i.e., $L \in [0,1[$ and

$$d(f(x),f(y)) \leq Ld(x,y), \text{ for every } x \in A_i, \text{ and } y \in A_{i+1}, \text{ where } i \in \{1,2,\cdots,p\}.$$

Then, f is a $\bigcup_{i=1}^{p}(A_i \times A_{i+1})$-contraction.

(4) (Suzuki (2008), see [91]) Let (X, d) be a metric space and $f : X \to X$ be an operator for which there exists $L \in [0, 1[$ such that

$$x, y \in X,\ \frac{1}{2}d(x, f(x)) \leq d(x, y) \Rightarrow d(f(x), f(y)) \leq Ld(x, y).$$

Then f is an Y-contraction with $Y := \{(x, y) \in X \times X | \frac{1}{2}d(x, f(x)) \leq d(x, y)\}$.

(5) (Jachymski (2008), see [50]) Let (X, d) be a metric space, G be a directed graph such that the set of its vertiges $V(G)$ coincides with X and the set of all edges $E(G)$ contains all loops. Let $f : X \to X$ be an operator such that f preserves the edges of G and there exists $L \in [0, 1[$ such that

$$d(f(x), f(y)) \leq Ld(x, y), \text{ for every } (x, y) \in E(G).$$

Then, f is an Y-contraction with $Y := \{(x, y) \in X \times X | (x, y) \in E(G)\}$.

We notice that the results of this paper can be extended to Y-contractions provided Y satisfies the following conditions:

(i) $(x, x) \in Y$, for every $x \in X$;
(ii) $(x, y) \in Y \Rightarrow (y, x) \in Y$;
(iii) for each $(x, y) \in Y$ there exists $z \in X$ such that $(x, z), (y, z) \in Y$.

8.2. **Y-contractions with the shadowing property.** Notice that $X_{\preceq}$-contractions do not have, in general, the shadowing property. The problem is in which conditions an Y-contraction has the (limit) shadowing property.

8.3. **The case of dislocated metric spaces.** Another open question is to extend the results of this paper to various generalized metric spaces (see [85], [16], [30], [37], [39], [49], [52], [79], ...) For example, consider the case of dislocated metric spaces. Let X be a nonempty set. By definition, a functional $d : X \times X \to \mathbb{R}_+$ is a dislocated metric if the following axioms hold:

(a) $d(x, y) = 0 \Rightarrow x = y$;
(b) $d(x, y) = d(y, x)$, for every $x, y \in X$;
(c) $d(x, y) \leq d(x, z) + d(z, y)$, for every $x, y, z \in X$.

The dislocated metrics and the stronger notion of partial metrics have applications to logic programming and theoretical computer science, see, for example, [49] and the references therein. See also [16], [30], [85].

References

[1] A. Abian, *A fixed point theorem for image-intersecting mapping*, Proc. Amer. Math. Soc., **73**(1979), 300-302.

[2] A. Abian, *A fixed point theorem for nonincreasing mappings*, Boll. Un. Mat. Ital., **4**(1969), 200-201.

[3] S. Abian, A.B. Brown, *A theorem on partially ordered sets, with applications to fixed point theorems*, Canad. J. Math., bf 13(1961), 78-82.

[4] M. Abtahi, Z. Kadelburg, S. Radenović, *Fixed points and coupled fixed points in partially ordered ν-generalized metric spaces*, Applied General Topology, 19(2018), 189-201.

[5] R.P. Agarwal, M.A. El-Gebeily, D. O'Regan, *Generalized contractions in partially ordered metric spaces*, Applicable Anal., **87**(2008), 109-116.

[6] R.P. Agarwal, E. Karapinar, A.-F. Roldan-Lopez-de-Hierro, *Some remarks on "Multidimensional fixed point theorems for isotone mappings in partially ordered metric spaces"*, Fixed Point Theory Appl., **2014** 2014:245, 13 pp.

[7] A. Alam, M. Imdad, *Monotone generalized contractions in ordered metric spaces*, Bull. Korean Math. Soc., **53**(2016), 61-81.

[8] A. Alam, Q.H. Khan, M. Imdad, *Discussion on some recent order-theoretic metrical coincidence theorems involving nonlinear contractions*, J. Function Spaces, **2016** Article ID 6275367, 11 pp.

[9] H. Amann, *Fixed point equations and nonlinear eigenvalue problems in ordered Banach spaces*, SIAM Review, **18**(1976), 620-708.

[10] H. Amann, *Order Structures and Fixed Points*, Atti del Seminario di Analisi Funzionale e Applicazioni, University of Calabria (1977) 1–51.

[11] A. Amini-Harandi, H. Emami, *A fixed point theorem for contraction type maps in partially ordered metric spaces and application to ordinary differential equations*, Nonlinear Analysis: Theory, Methods & Appl., 72(2010), 2238-2242.

[12] H. Argoubi, B. Samet, C. Vetro, *Nonlinear contractions involving simulation functions in a metric space with a partial order*, J. Nonlinear Sci. Appl., **8**(2015), 1082-1094.

[13] H. Ben-El-Mechaiekh, *The Ran-Reurings fixed point theorem without partial order: A simple proof*, J. Fixed Point Theory Appl., **16**(2014), 373-383.

[14] V. Berinde, *Approximating fixed points of weak contractions using the Picard iteration*, Nonlinear Anal. Forum, **9**(2004), 43-53.

[15] V. Berinde, A. Petruşel, I.A. Rus, M.A. Şerban, *The retraction-displacement condition in the theory of fixed point equation with a convergent iterative algorithm*, In: Mathematical Analysis, Approximation Theory and Their Applications (Th. Rassias, V. Gupta-Eds.) Springer Optimization and Its Applications, vol. 111. Springer, Cham, 75-106, 2016.

[16] M.M. Bonsangue, F. van Breugel, J.J.M.M. Rutten, *Generalized metric spaces: completation, topology and power domains, via the Yoneda embedding*, Theoretical Computer Sci., **193**(1998), 1-51.

[17] A. Brøndsted, *Fixed points and partial orders*, Proc. Amer. Math. Soc., **60**(1976), 365-366.

[18] A. Brøndsted, *On a lemma of Bishop and Phelps*, Pacific J. Math., **55**(1974), 335-341.

[19] J. Caballero, J. Harjani, K. Sadarangani, *Contractive-like mapping principles in ordered metric spaces and application to ordinary differential equations*, Fixed Point Theory Appl., **2010** 2010:916064, 14 pp.

[20] J. Caristi, *Fixed point theorems for mappings satisfying inwardness conditions*, Trans. Amer. Math. Soc., **215**(1976), 241-251.

[21] S. Carl, S. Heikkilä, *Fixed Point Theory in Ordered Sets and Applications*, Springer Berlin, 2011.

[22] S. Chandok, M.S. Khan, T.D. Narang, *Fixed point theorems in partially ordered metric spaces for generalized contractions*, Azerbaijan J. Math., **5**(2015), 89-96.

[23] B.S. Choudhury, P. Maity, *Weak contractions on chains in a generalized metric space with a partial order*, Afrika Math., **25**(2014), 745-756.

[24] S.K. Chatterjea, *Fixed-point theorems*, C.R. Acad. Bulgare Sci. **25**(1972) 727-730.

[25] C.-M. Chen, J.-C. Lee, C.-H. Chen, *Fixed point theory of weak contractions in partially ordered metric spaces*, J. Applied Math., **2013**, Article ID 302438, 7 pp.

[26] M. M. Choban, V. Berinde, *Multiple fixed point theorems for contractive and Meir-Keeler type mappings defined on partially ordered spaces with a distance*, Applied General Topology, bf 18(2017), 317-330.

[27] Lj.B. Ćirić, *A generalization of Banach's contraction principle*, Proc. Amer. Math. Soc., **45**(1974), 267-273.

[28] Lj.B. Ćirić, N. Cakić, M. Rajović, and J. S. Ume, *Monotone generalized nonlinear contractions in partially ordered metric spaces*, Fixed Point Theory Appl., **2008**(2008), article ID 131294, 11 pp.

[29] K. Deimling, *Nonlinear Functional Analysis*, Springer-Verlag, Berlin, 1980.

[30] M.M. Deza, F. Deza, *Encyclopedia of Distances*, Springer Berlin, 2009.

[31] B.C. Dhage, *Basic hybrid fixed point theorems for contractive mappings in partially ordered metric spaces*, Thai J. Math., **15**(2017), 721-732.

[32] M. Dinarvand, *Fixed points for generalized contractions via rational expressions in partially ordered b-metric spaces and applications to integral equations*, Afrika Math., **29**(2018), 175-193.
[33] D. Doric, Z. Kadelburg, S. Radenovic, P. Kumam, *A note on fixed point results without monotone property in partially ordered metric space*, Revista de la Real Academia de Ciencias Exactas Fis. Natur. Serie A-Matematicas, **108**(2014), 503-510.
[34] J. Eisenfeld, V. Lakshmikantham, *Remarks on nonlinear contraction and comparison principle in abstract cones*, J. Math. Anal Appl., **61**(1977), 116-121.
[35] Z. Ercan, A.W. Wickstead, *Banach lattices of continuous Banach lattice-valued functions*, J. Math. Anal. Appl., **198**(1996), 121-136.
[36] Y.-P. Fang, Z. Han, *Fixed points for mixed increasing operators in ordered Banach spaces with an application*, Indian J. Pure Applied Math., **33**(2002), 1217-1226.
[37] B. Flagg, R. Kopperman, *Continuity spaces: reconciling domains and metric spaces*, Theoretical Computer Sci., **177**(1997), 111-138.
[38] T.N. Fomenko, D.A. Podoprikhin, *Fixed points and coincidences of mappings of partially ordered sets*, J. Fixed Point Theory Appl., bf 18(2016), 823-842.
[39] M. Fréchet, *Les espaces abstraits*, Gauthier-Villars, Paris, 1928.
[40] M. A. Gordji, M. Ramezani, F. Sajadian, Y. J. Cho, C. Park, *A new type fixed point theorem for a contraction on partially ordered generalized complete metric spaces with applications*, Fixed Point Theory Appl., **2014** 2014:15, 9 pp.
[41] D. Guo, Y.J. Cho, J. Zhu, *Partial Ordering Methods in Nonlinear Problems*, Nova Science Publishers Inc., Hauppauge, 2004.
[42] J. Harjani, K. Sadarangani, *Fixed point theorems for weakly contractive mappings in partially ordered sets*, Nonlinear Anal. Theory, Methods & Appl., **71**(2009), 3403-3410.
[43] J. Harjani, F. Sabetghadam, K. Sadarangani, *Generalized contractions in partially ordered metric spaces and applications to ordinary differential equations*, Nonlinear Anal. Theory, Methods & Appl., **72**(2010), 1188-1197.
[44] J. Harjani, K. Sadarangani, *Fixed point theorems for monotone generalized contractions in partially ordered metric spaces and applications to integral equations*, J. Convex Anal., **19**(2012), 853-864.
[45] J. Harjani, K. Sadarangani, *Fixed point theorems for cyclic weak contractions in partially ordered sets endowed with a complete metric*, Carpathian J. Math., **29**(2013), 179-186.
[46] S. Heikkilä, V. Lakshmikantham, *Monotone Iterative Techniques for Discontinuous Nonlinear Differential Equations*, CRC Press, 1994.
[47] G. Herzog, *On ordinary differential equations with quasimonotone increasing right-hand side*, Archiv der Mathematik, **70**(1998), 142-146.
[48] T.L. Hicks and B.E. Rhoades, *A Banach type fixed point theorem*, Math. Japonica, **24**(1979), 327-330.
[49] P. Hitzler, *Generalized Metric and Topology in Logic Programming Semantics*, Ph.D. Dissertation, National Univ. Ireland, 2001.
[50] J. Jachymski, *The contraction principle for mappings on a metric space with a graph*, Proc. Amer. Math. Soc., **136**(2008), 1359-1373.
[51] M. Jleli, B. Samet, *Best proximity point results for MK-proximal contractions on ordered sets*, J. Fixed Point Theory Appl., **17**(2015), 439-452.
[52] W.A. Kirk, B. Sims (editors), *Handbook of Metric Fixed Point Theory*, Kluwer Acad. Publ., Dordrecht, 2001.
[53] M.A. Krasnoselskii, *Positive Solutions of Operator Equations*, Noordhoff, Leyden, 1964.
[54] M.A. Krasnoselskii, *Geometrical Methods in Nonlinear Analysis*, Springer, Berlin, 1984.
[55] M. Marinacci, L. Montrucchio, Unique Tarski Fixed Points, Working Papers 604(2017), IGIER (Innocenzo Gasparini Institute for Economic Research), Bocconi University, Italy.
[56] G. Markowsky, *Chain-complete posets and directed sets with applications*, Algebra Univ., **6**(1976), 53-68.
[57] Z. Mustafa, E. Karapinar, H. Aydi, *A discussion on generalized almost contractions via rational expressions in partially ordered metric spaces*, J. Ineq. Appl., **2014** 2014:219.

[58] H.K. Nashine, I. Altun, *Fixed point theorems for generalized weakly contractive condition in ordered metric spaces*, Fixed Point Theory Appl., **2011** 2011:132367, 20 pp.
[59] J. J. Nieto, A. Ouahab, R. Rodriguez-Lopez, *Random Fixed point theorem in partial order metric space*, Fixed Point Theory Appl., **2016** 2016:98, 19 pp.
[60] J.J. Nieto, R. Rodriguez-Lopez, *Contractive mapping theorems in partially ordered sets and applications to ordinary differential equations*, Order, **22**(2005), 223-239.
[61] J.J. Nieto, R. Rodriguez-Lopez, *Existence and uniqueness of fixed point in partially ordered sets and applications to ordinary differential equations*, Acta Math. Sinica (English Series), **23**(2007), 2205-2212.
[62] J.J. Nieto, R.L. Pouso, R. Rodriguez-Lopez, *Fixed point theorems in ordered abstract spaces*, Proc. Amer. Math. Soc., **135**(2007), 2505-2517.
[63] J. Ombach, *The simplest shadowing*, Ann. Polon. Math., **58**(1993), 253-258.
[64] D. O'Regan and A. Petruşel, *Fixed point theorems for generalized contractions in ordered metric spaces*, J. Math. Anal. Appl., **341**(2008), 1241-1252.
[65] M. Păcurar, I.A. Rus, *Fixed point theory for cyclic φ-contractions*, Nonlinear Anal., **72**(2010), 1181–1187
[66] M. Păcurar, I.A. Rus, *Some remarks on the notations and terminology in the ordered set theory*, Creat. Math. Inform., **27**(2018), 191-195.
[67] A. Petruşel, *Fixed points Vs. coupled fixed points*, J. Fixed Point Theory Appl., (2018) 20:150. https://doi.org/10.1007/s11784-018-0630-6
[68] A. Petruşel, G. Petruşel, Y.-B. Xiao, J.-C. Yao, Fixed point theorems for generalized contractions with applications to coupled fixed point theory, J. Nonlinear Convex Anal., **19** (2018), 71-88.
[69] A. Petruşel, G. Petruşel, *Coupled fixed points and coupled coincidence points via fixed point theory*, in: Mathematical Analysis and Applications: Selected Topics (M. Ruzhansky, H. Dutta, R.P. Agarwal - Eds.), Wiley, 2018, 661-708.
[70] A. Petruşel, I.A. Rus, *Fixed point theorems in ordered L-spaces*, Proc. Amer. Math. Soc., **134** (2005), 411-418.
[71] A. Petruşel, I.A. Rus, *Graphic contraction principle and applications*, in: Mathematical Analysis and Applications (Th. Rassias, P. Pardalos - Eds.), Springer, Berlin 2019, to appear.
[72] S.Yu. Pilyugin, *Shadowing in Dynamical Systems*, Springer Berlin, 1999.
[73] S. Radenović, Z. Kadelburg, *Generalized weak contractions in partially ordered metric spaces*, Comput Math. Appl., **60**(2010), 1776-1783.
[74] A.C.M. Ran and M.C.B. Reurings, *A fixed point theorem in partially ordered sets and some applications to matrix equations*, Proc. Amer. Math. Soc., **132**(2004), 1435-1443.
[75] I. Rival, *The problem of fixed points in ordered sets*, Ann. Discrete Math., **8**(1980), 283-292.
[76] I.A. Rus, *Fixed point theory in algebraic structures*, Babeş-Bolyai Univ. Cluj, 1971 (in Romanian).
[77] I.A. Rus, *Some equivalent conditions in the metrical fixed point theory*, Mathematica, **23**(1981), 213-218.
[78] I.A. Rus, Weakly Picard mappings, *Comment. Math. Univ. Carol.*, **34**(1993), 769-773.
[79] I.A. Rus, *Generalized Contractions and Applications*, Cluj Univ. Press, 2001.
[80] I.A. Rus, *Picard operators and applications*, Sci. Math. Japon., **58**(2003), 191-219.
[81] I.A. Rus, *Cyclic representation and fixed points*, Annals of the Tiberiu Popoviciu Seminar, **3**(2005), 171-178.
[82] I.A. Rus, *Results and problems in Ulam stability of operatorial equations and inclusions*, in: (Th. Rassias-ed.) Handbook of Functional Equations: Stability Theory, Springer Berlin 2014, 323-352.
[83] I.A. Rus, *Some variants of contraction principle, generalizations and applications*, Stud. Univ. Babeş-Bolyai Math., **61**(2016) 343–358.
[84] I.A. Rus, M.-A. Şerban, *Basic problems of the metric fixed point theory and the relevance of a metric fixed point theorem*, Carpathian J. Math., 29(2013), 239-258.
[85] I.A. Rus, A. Petruşel, G. Petruşel, *Fixed Point Theory*, Cluj University Press Cluj-Napoca, 2008.

[86] R. Saadati, S.M. Vaezpour, *Monotone generalized weak contractions in partially ordered metric spaces*, Fixed Point Theory, **11**(2010), 375-382.
[87] B. Samet, *Ran-Reurings fixed point theorem is an immediate consequence of the Banach contraction principle*, J. Nonlinear Sci. Appl., **9**(2016), 873-875.
[88] A.M. Samoilenko, Yu.V. Teplinskii, *Countable Systems of Differential Equations*, W. De Gruyter, 2003.
[89] H.H. Schaefer, *Banach Lattices and Positive Operators*, Springer Berlin, 1974.
[90] G. Song, *Iterative technique for a integral equation in Banach space and applications*, Indian J. Pure Applied Math., **32**(2001), 355-360.
[91] T. Suzuki, *A generalized Banach contraction principle that characterizes metric completeness*, Proc. Amer. Math. Soc., **136**(2008) 1861–1869.
[92] M.R. Tasković, *Partially ordered sets and some fixed point theorems*, Publ. Inst. Math. (Beograd), **27**(1980), 241-247.
[93] M. Turinici, *Contraction maps in ordered metrical structures*, In: Mathematics without Boundaries (P. Pardalos, Th. Rassias - Eds.), Springer, New York, 2014, 533–575.
[94] M. Turinici, *Fixed point results on abstract ordered sets*, Matematiche, **49**(1994), 25-34.
[95] M. Turinici, *Selected Topics in Metrical Fixed Point Theory*, Ed. PIM, Iaşi, 2014.
[96] , M. Turinici, *Abstract comparison principles and multivariable Gronwall-Bellman inequalities*, J. Math. Anal. Appl., **117**(1986), 100-127.
[97] N. Van Luong, N. Xuan Thuan, *Fixed point theorem for generalized weak contractions satisfying rational expressions in ordered metric spaces*, Fixed Point Theory Appl., **2011** 2011:46, 10 pp.
[98] H. Viet Long, N. Thi Kim Son, R. Rodriguez-López, *Some generalizations of fixed point theorems in partially ordered metric spaces and applications to partial differential equations with uncertainty*, Vietnam J. Math., **46**(2018), 531-555.
[99] W. Walter, *Ordinary differential inequalities in ordered Banach spaces*, J. Diff. Eq., **9**(1971), 253-263.
[100] J. Zhang, R. P. Agarwal, N. Jiang, *N fixed point theorems and N best proximity point theorems for generalized contraction in partially metric spaces*, J. Fixed Point Theory Appl., (2018) 20: 18. https://doi.org/10.1007/s11784-018-0505-x.
[101] A.D. Ziebur, *Uniqueness and the convergence of successive approximations. II.*, Proc. Amer. Math. Soc., **16**(1965), 335-340.

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