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NIELSEN THEORY ON INFRA-NILMANIFOLDS MODELED ON THE GROUP OF UNI-TRIANGULAR MATRICES

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Abstract. Let Nil_m be the group of $m\times m$ uni-triangular matrices. Then it is a connected and simply connected $(m-1)$-step nilpotent Lie group. Using the averaging formulas, we compute the spectra of the Lefschetz, Nielsen and Reidemeister (coincidence) numbers of maps on infra-nilmanifolds modeled on Nil_m. As a byproduct, we prove that the Bieberbach groups of Nil_m $(m \geq 4)$ with Γ_m as its nil-radical satisfy the R_∞ property.
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1. Introduction

Let Nil_m be the group of uni-triangular (upper-triangular unipotent) matrices of size m, i.e, Nil_m consists of all $m \times m$ upper triangular real matrices with all the diagonal entries 1. Then it is an $(m-1)$-step nilpotent Lie group, diffeomorphic to $\mathbb{R}^{\frac{1}{2}m(m-1)}$. We note that Nil_2 is the abelian group $\mathbb{R}$, and Nil_3 is the Heisenberg group. We will suppress m whenever no confusion is likely.

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Let $\Gamma_m \subset \mathrm{Nil}_m$ be the subgroup consisting of all matrices with integer entries. Then Γ_m is a lattice of Nil_m. It is known that the group of automorphisms of Nil_m is

$$\mathrm{Aut}(\mathrm{Nil}_m) = \begin{cases} \mathrm{GL}(2,\mathbb{R}) & \text{if } m=2 \\ \mathrm{Nil}_3/\mathcal{Z}(\mathrm{Nil}_3) \rtimes \mathrm{GL}(2,\mathbb{R}) & \text{if } m=3 \\ \mathcal{I} \rtimes ((\mathbb{R}^*)^{m-1} \rtimes \mathbb{Z}_2) & \text{if } m \geq 4, \end{cases}$$

where $\mathcal{I}$ is a connected and simply connected nilpotent Lie group. Hence a maximal compact subgroup K of $\mathrm{Aut}(\Gamma_m)$ and of $\mathrm{Aut}(\mathrm{Nil}_m)$ is ([3])

$$K = \begin{cases} \mathrm{O}(2) & \text{if } m=2,3 \\ (\mathbb{Z}_2)^{m-1} \rtimes \mathbb{Z}_2 & \text{if } m \geq 4 \end{cases}$$

where $(\mathbb{Z}_2)^{m-1} \rtimes \mathbb{Z}_2 \subset (\mathbb{R}^*)^{m-1} \rtimes \mathbb{Z}_2 \subset \mathrm{GL}(m-1,\mathbb{Z})$.

The quotient $\Gamma_m\backslash\mathrm{Nil}_m$ is a *nilmanifold*, and a finite quotient of $\Gamma_m\backslash\mathrm{Nil}_m$ is an *infra-nilmanifold*.

It is the purpose of this work to study the Nielsen (coincidence) theory for all continuous maps of infra-nilmanifolds M that are covered essentially by the nilmanifold $\Gamma_m\backslash\mathrm{Nil}_m$ for every $m \geq 3$. We will determine the spectra of the fundamental invariants $L(f), N(f)$ and $R(f)$ of the Nielsen theory where $L(f), N(f)$ and $R(f)$ are the Lefschetz, the Nielsen and the Reidemeister numbers of f using the averaging formulas. We will also determine the spectra of the Lefschetz, the Nielsen and the Reidemeister coincidence invariants. That is, we will determine

$$\begin{aligned} \mathfrak{L}(M) &= \{L(f) \mid f \text{ is a self-map of } M\}, \\ \mathfrak{L}_{\mathfrak{h}}(M) &= \{L(f) \mid f \text{ is a self-homeomorphism of } M\}, \\ \mathfrak{LC}(M) &= \{L(f,g) \mid f,g \text{ are self-maps of } M\}. \end{aligned}$$

Similarly, we will also determine

$$\mathfrak{N}(M),\ \mathfrak{N}_{\mathfrak{h}}(M),\ \mathfrak{NC}(M),$$

and

$$\mathfrak{R}(M),\ \mathfrak{R}_{\mathfrak{h}}(M),\ \mathfrak{RC}(M).$$

2. Infra-nilmanifolds modeled on Nil_m

Let $m \geq 3$ and let M be an infra-nilmanifold that is covered essentially by the nilmanifold $\Gamma_m\backslash\mathrm{Nil}_m$. Then $M = \Pi\backslash\mathrm{Nil}_m$ where Π is a Bieberbach group of Nil_m having Γ_m as its nil-radical. This means that Π is a torsion-free group which fits in the following commutative diagram

$$\begin{tikzcd} 1 \arrow[r] & \mathrm{Nil}_m \arrow[r] & \mathrm{Nil}_m \rtimes K \arrow[r] & K \arrow[r] & 1 \\ 1 \arrow[r] & \Gamma_m \arrow[r] \arrow[u, "\cup"'] & \Pi \arrow[r] \arrow[u, "\cup"'] & \Phi \arrow[r] \arrow[u, "\cup"'] & 1 \end{tikzcd}$$

where K is a maximal compact subgroup of $\mathrm{Aut}(\mathrm{Nil}_m)$ and Φ is a finite group, called the *holonomy group* of Π. Recall that if $m=3$, then we can choose $K = O(2)$; if

$m \geq 4$ then we can choose

$$K = {\mathbb{Z}_2}^{m-1} \rtimes \mathbb{Z}_2 \subset \mathrm{GL}(m-1, \mathbb{Z}).$$

We have a complete classification of all Bieberbach groups Π of Nil_m ($m \geq 3$) with Γ_m as the discrete nil-radical.

Theorem 2.1 ([4])**.** *Let* $\Pi = \langle \Gamma, \alpha \rangle$ *where*

$$\alpha = (a, A) = \left(\begin{pmatrix} 1 & 0 & \frac{7}{24} \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} \right) \in \mathrm{Nil}_3 \rtimes \mathrm{GL}(2, \mathbb{Z}).$$

Then Π *is the only Bieberbach group of* Nil_3 *with nontrivial holonomy group and with* Γ_3 *as the discrete nil-radical.*

Theorem 2.2 ([3, Theorem 5.1])**.** *For odd* $m \geq 4$*, there is no infra-nilmanifold which is essentially covered by* $\Gamma_m \backslash \mathrm{Nil}_m$*.*

For $m = 2n \geq 4$*, there is a unique infra-nilmanifold which is essentially covered by the nilmanifold* $\Gamma_m \backslash \mathrm{Nil}_m$*. This manifold has the covering group* $\mathbb{Z}_2$ *generated by* $\alpha = (a, J) \in \mathrm{Nil} \rtimes K$*, where* $a = \mathcal{Z}[\frac{1}{2}]$ *and*

$$J = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & 0 \\ \vdots & \vdots & ⋰ & \vdots & \vdots \\ 0 & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{pmatrix} \in \mathrm{GL}(m-1, \mathbb{Z}).$$

The Nielsen theory concerns with the following basic invariants: the Lefschetz (coincidence) numbers, the Nielsen (coincidence) numbers and the Reidemeister (coincidence) numbers. In the following sections, we shall compute those basis invariants for all maps on the infra-nilmanifolds $\Pi \backslash \mathrm{Nil}_m$ where Π is a Bieberbach group of Nil_m ($m \geq 3$) with Γ_m as the discrete nil-radical.

Let $f : \Pi \backslash G \to \Pi \backslash G$ be a continuous self-map of an infra-nilmanifold $\Pi \backslash G$. Then f induces a homomorphism $\phi : \Pi \to \Pi$. Due to [9, Theorem 1.1], there exists an affine map $(d, D) \in \mathrm{Aff}(G)$ of G such that

$$\phi(\alpha) \circ (d, D) = (d, D) \circ \alpha, \quad \forall \alpha \in \Pi \subset \mathrm{Aff}(G). \tag{2.1}$$

Consequently, the affine map $(d, D) : G \to G$ restricts to a self-map of $\Pi \backslash G$ which is homotopic to f. We say that the affine map (d, D) is an *affine homotopy lift* of f.

For the computation of the basic invariants of the Nielsen theory, we will use the following averaging formulas:

Theorem 2.3. [Averaging formulas: ([10], [8], [7], [6], [5])] *Let* f *and* g *be continuous maps on an* (*orientable*) *infra-nilmanifold* $\Pi \backslash G$ *with holonomy group* Φ*.*

Let f and g have affine homotopy lifts (d, D) and (e, E) respectively. Then we have:

$$L(f,g) = \frac{1}{\#\Phi} \sum_{A\in\Phi} \det(E_* - A_* D_*),$$

$$N(f,g) = \frac{1}{\#\Phi} \sum_{A\in\Phi} |\det(E_* - A_* D_*)|,$$

and

$$R(f,g) = \frac{1}{\#\Phi} \sum_{A\in\Phi} \sigma\left(\det(E_* - A_* D_*)\right),$$

where D_*, E_* and A_* are the matrices of the differentials of the Lie group endomorphisms D, E and A with respect to the same linear basis of the Lie algebra of G, and where $\sigma : \mathbb{R} \to \mathbb{R} \cup \{\infty\}$ is defined by

$$\sigma(0) = \infty \textit{ and } \sigma(x) = |x| \textit{ for } x \neq 0.$$

3. The theoretical idea behind the computation

In order to use the averaging formulas for the computation of all possible Lefschetz, Nielsen and Reidemeister numbers of self-maps on infra-nilmanifolds $\Pi\backslash\mathrm{Nil}_m$, we have to know what the possibilities are for such self-maps. By [9, Theorem 1.1], we know that every such a map is homotopic to a map that is induced by an affine map on the Lie group. Because the Lefschetz numbers, the Nielsen numbers and the Reidemeister numbers are homotopy type invariants, we know that it suffices to find all possible affine maps that induce a self-map on the infra-nilmanifold.

For the practical approach, let (d, D) be any affine map of Nil_m. Then the map (d, D) induces a self-map on the infra-nilmanifold $\Pi\backslash\mathrm{Nil}_m$ if and only if there exists an endomorphism $\phi : \Pi \to \Pi$ satisfying the equation (2.1):

$$\phi(\alpha) \circ (d, D) = (d, D) \circ \alpha, \quad \forall \alpha \in \Pi \subset \mathrm{Aff}(\mathrm{Nil}_m).$$

When $\Pi = \Gamma_m$, the case is much simpler. In this case, (2.1) yields

$$\phi(\alpha) = (\mu(d) \circ D)(\alpha) \quad \forall \alpha \in \Gamma_m$$

where $\mu(d)$ is the conjugation by d, $x \mapsto dxd^{-1}$. Thus we obtain the following commutative diagram

$$\begin{array}{ccc} \mathrm{Nil}_m & \xrightarrow{\mu(d)\circ D} & \mathrm{Nil}_m \\ \uparrow \cup & & \uparrow \cup \\ \Gamma_m & \xrightarrow{\phi} & \Gamma_m \end{array}$$

That is, $\mu(d) \circ D$ is the extension of the endomorphism ϕ of the lattice Γ_m to the Lie group Nil_m. Furthermore, (d, D) and $\mu(d) \circ D$ induce self-maps on $\Gamma_m\backslash\mathrm{Nil}_m$, homotopic to each other. Therefore, *when $\Pi = \Gamma_m$ it suffices to understand the set of all endomorphisms of the lattice* Γ_m, $\mathrm{Endo}(\Gamma_m)$.

Now we will consider the case where $\Pi \neq \Gamma_m$. Let Φ be the nontrivial holonomy group of Π generated by an element $A \in \mathrm{GL}(m-1, \mathbb{Z})$. By Theorems 2.1 and 2.2, A is of order 3 or 2 depending on $m = 3$ or $m > 3$ respectively.

Let $m = 3$. Consider the equation (2.1) with specific $\alpha = (a, A)$. Since $\phi(\alpha) \in \Pi$ and A is of order 3, $\phi(\alpha)$ is of the form

$$\gamma,\ \gamma\alpha = (\gamma a, A),\ \gamma\alpha^2 = (\gamma a A(a), A^2),$$

where $\gamma \in \Gamma_m$. By substitution into (2.1), we have one of the following

$$DA = D, DA = AD \text{ or } DA = A^{-1}D. \tag{3.1}$$

Here, $A \in \mathrm{GL}(2, \mathbb{Z})$ and $A \in \mathrm{GL}(2, \mathbb{Z})$ can be regarded as an element of $A \in \mathrm{Aut}(\mathrm{Nil}_3)$ and of $A_* \in \mathrm{Aut}(\mathfrak{nil}_3)$

Let $m > 3$. Since $\phi(\alpha) \in \Pi$, $\phi(\alpha)$ is of the form

$$\gamma,\ \gamma\alpha = (\gamma a, A)$$

where $\gamma \in \Gamma_m$. By (2.1),

$$(d, D)(a, A) = (\gamma, I)(d, D) \ \text{ or } \ (d, D)(a, A) = (\gamma, I)(a, A)(d, D).$$

This implies that either

$$DA = D \text{ or } DA = AD. \tag{3.2}$$

Since $m > 3$, A is diagonal or anti-diagonal in $\mathrm{GL}(m-1, \mathbb{Z})$ (see [2, Lemma 3.9]) and it can be regarded as an element of $A \in \mathrm{Aut}(\mathrm{Nil}_m)$ and of $A_* \in \mathrm{Aut}(\mathfrak{nil}_m)$.

If $\Gamma_m (\subset \Pi)$ is ϕ-invariant, then the equation (2.1) induces that

$$\phi(\gamma) = \mu(d) \circ D(\gamma),\ \forall \gamma \in \Gamma_m.$$

Hence we need to find all endomorphisms D of Λ_m satisfying (3.1) when $m = 3$ and (3.2) when $m > 3$. This is the case when ϕ is an automorphism because Γ_m is a characteristic subgroup of Π.

However, Γ_m is not necessarily a fully invariant subgroup of Π. By [10, Lemma 3.1], there exists a fully invariant subgroup $\Lambda_m \subset \Gamma_m$ of Π which is of finite index. For all $\lambda \in \Lambda_m$, the equation (2.1) gives

$$\phi(\lambda) = (\mu(d) \circ D)(\lambda).$$

Thus we obtain the following commutative diagram

$$\begin{array}{ccc} \mathrm{Nil}_m & \xrightarrow{\mu(d)\circ D} & \mathrm{Nil}_m \\ \Big\uparrow \cup & & \Big\uparrow \cup \\ \Lambda_m & \xrightarrow{\phi|_\Lambda} & \Lambda_m \end{array}$$

The Lie group endomorphism $\mu(d) \circ D$ is our linearization of f.

Consequently, *when* $\Pi \neq \Gamma_m$ *we need to find a fully invariant subgroup* $\Lambda_m \subset \Gamma_m$ *of* Nil_m *and then find all endomorphisms* D *of* Λ_m, $D \in \mathrm{Endo}(\Lambda_m)$, *satisfying* (3.1) *when* $m = 3$ *and* (3.2) *when* $m > 3$ *by regarding* A *as an element of* $A \in \mathrm{Aut}(\Lambda_m)$.

4. Endomorphisms of Γ_3

Let ϕ be an endomorphism of Γ_m. Then it can be regarded as a Lie group endomorphism of Nil_m. That is, $\mathrm{Endo}(\Gamma_m) \subset \mathrm{Endo}(\mathrm{Nil}_m)$. Because of the following commutative diagram

$$\begin{tikzcd} \mathrm{Nil}_m \arrow[r, "\phi"] \arrow[d, "\log"] & \mathrm{Nil}_m \\ \mathfrak{nil}_m \arrow[r, "\mathrm{d}\phi"] & \mathfrak{nil}_m \arrow[u, "\exp"] \end{tikzcd}$$

we can identify ϕ with its differential $\mathrm{d}\phi$, so we have

$$\mathrm{Endo}(\Gamma_m) \subset \mathrm{Endo}(\mathrm{Nil}_m) = \mathrm{Endo}(\mathfrak{nil}_m).$$

First we consider the case $m = 3$. The Lie algebra $\mathfrak{nil}_3$ of Nil_3 is

$$\mathfrak{nil}_3 = \left\{ \begin{pmatrix} 0 & x_1 & x_3 \\ 0 & 0 & x_2 \\ 0 & 0 & 0 \end{pmatrix} \mid x_1, x_2, x_3 \in \mathbb{R} \right\}.$$

This algebra is linearly generated by

$$\mathbf{e}_{1,2} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ \mathbf{e}_{2,3} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \ \mathbf{e}_{1,3} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

They satisfy the only nontrivial Lie bracket

$$[\mathbf{e}_{1,2}, \mathbf{e}_{2,3}] = \mathbf{e}_{1,3}.$$

A Lie algebra endomorphism of $\mathfrak{nil}_3$ is a linear transformation of the linear space $\mathfrak{nil}_3$ preserving all Lie brackets among the linear basis $\{\mathbf{e}_{1,2}, \mathbf{e}_{2,3}, \mathbf{e}_{1,3}\}$, and vice versa. It is easy to see that the set of all Lie algebra endomorphisms of $\mathfrak{nil}_3$ is the following set of 3×3 matrices

$$\mathrm{Endo}(\mathfrak{nil}_3) = \left\{ \left(\begin{array}{cc:c} a & b & 0 \\ c & d & 0 \\ \hdashline u & v & ad - bc \end{array}\right) \mid a, b, c, d, u, v \in \mathbb{R} \right\}.$$

Let $\phi \in \mathrm{Endo}(\Gamma_3)$ with

$$\phi = \mathrm{d}\phi = \left(\begin{array}{cc:c} a & b & 0 \\ c & d & 0 \\ \hdashline u & v & ad - bc \end{array}\right).$$

Write

$$E_{1,2} = \exp \mathbf{e}_{1,2}, \ E_{2,3} = \exp \mathbf{e}_{2,3}, \ E_{1,3} = \exp \mathbf{e}_{1,3}.$$

Then

$$\begin{aligned}\phi(E_{1,2}) &= \exp\circ\phi\circ\log(E_{1,2})\\ &= \exp(a\mathbf{e}_{1,2}+c\mathbf{e}_{2,3}+u\mathbf{e}_{1,3})\\ &= \exp\begin{pmatrix}0&a&u\\0&0&c\\0&0&0\end{pmatrix} = \begin{pmatrix}1&a&u+\frac{ac}{2}\\0&1&c\\0&0&1\end{pmatrix}\\ &= E_{1,2}^{a}E_{2,3}^{c}E_{1,3}^{u-\frac{ac}{2}}.\end{aligned}$$

Similarly, we have

$$\phi(E_{2,3}) = E_{1,2}^{b}E_{2,3}^{d}E_{1,3}^{v-\frac{bd}{2}},\ \phi(E_{1,3}) = E_{1,3}^{ad-bc}.$$

Since Γ_3 is generated by $E_{1,2}, E_{2,3}$ and $E_{1,3}$, we have

$$\mathrm{Endo}(\Gamma_3) = \left\{ \left(\begin{array}{cc:c} a & b & 0\\ c & d & 0\\ \hdashline u & v & ad-bc\end{array}\right) \mid a,b,c,d\in\mathbb{Z},\ u,v\in\frac{1}{2}\mathbb{Z}\right\}.$$

5. Nielsen theory on infra-nilmanifolds modeled on Nil_3

Let $M=\Gamma_3\backslash\mathrm{Nil}_3$ be the standard nilmanifold. Then by Theorem 2.3

$$\begin{aligned}L(f,g) &= \det(\psi-\phi),\\ N(f,g) &= |\det(\psi-\phi)|,\\ R(f,g) &= \sigma(\det(\psi-\phi)).\end{aligned}$$

Here, ϕ and $\psi\in\mathrm{Endo}(\Gamma_3)\subset\mathrm{Aut}(\mathrm{Nil}_3)=\mathrm{Aut}(\mathfrak{nil}_3)$ induce maps on M which are homotopic to f and g, respectively.

Example 5.1. Let $\phi\in\mathrm{Aut}(\mathfrak{nil}_3)$ be given by

$$\phi = \left(\begin{array}{cc:c} -1 & 1 & 0\\ -1 & 2 & 0\\ \hdashline -\frac{1}{2} & 0 & -1\end{array}\right).$$

Then it can be seen that $\phi\in\mathrm{Aut}(\Gamma_3)$ and hence ϕ induces a homeomorphism f of $\Gamma_3\backslash\mathrm{Nil}_3$ whose linearization is ϕ. Therefore the Lefschetz number, the Nielsen number and the Reidemeister number of f are

$$\begin{aligned}L(f) &= \det(I_3-\phi) = -2,\\ N(f) &= |\det(I_3-\phi)| = |-2| = 2,\\ R(f) &= \sigma(\det(I_3-\phi)) = \sigma(-2) = |-2| = 2.\end{aligned}$$

In the following we will determine the possible values of the Lefschetz numbers, the Nielsen numbers and the Reidemeister numbers for all homeomorphisms f of the standard nilmanifold $\Gamma_3\backslash\mathrm{Nil}_3$.

Theorem 5.2. *We have*

$$\begin{aligned}\mathfrak{L}_{\mathfrak{h}}(\Gamma_3\backslash\mathrm{Nil}_3) &= 2\mathbb{Z},\\ \mathfrak{N}_{\mathfrak{h}}(\Gamma_3\backslash\mathrm{Nil}_3) &= 2\mathbb{N}\cup\{0\},\\ \mathfrak{R}_{\mathfrak{h}}(\Gamma_3\backslash\mathrm{Nil}_3) &= 2\mathbb{N}\cup\{\infty\}.\end{aligned}$$

Proof. Let f be a homeomorphism of the nilmanifold $\Gamma_3\backslash\mathrm{Nil}_3$. Then our linearization $F\in\mathrm{Aut}(\Gamma_3)$ of f is of the form

$$F=\left(\begin{array}{c:c}\bar F & 0\\ \hdashline * & \det(\bar F)\end{array}\right)$$

with $\bar F\in\mathrm{GL}(2,\mathbb{Z})$ and $\det(\bar F)=\pm1$.

Remark that

$$\begin{aligned}\det(I_3-F) &= \det(I_2-\bar F)\cdot(1-\det(\bar F))\\ &= \left(1-\mathrm{tr}\,(\bar F)+\det(\bar F)\right)\left(1-\det(\bar F)\right).\end{aligned}$$

Consider any $\bar F\in\mathrm{GL}(2,\mathbb{Z})$ with $\det(\bar F)=1$, for example we can choose $\bar F=I_2$. Then it is obvious that $\det(I_3-F)=0$, hence $L(f)=0$. Next we consider $\bar F\in\mathrm{GL}(2,\mathbb{Z})$ so that $\det(\bar F)=-1$. Then $\det(I_3-F)=-2\,\mathrm{tr}\,(\bar F)$. If we choose $\bar F$ to be

$$\bar F=\begin{pmatrix}1+n & -n\\ 1 & -1\end{pmatrix}$$

then $L(f)=\det(I_3-F)=-2n$. This finishes the proof. $\square$

When f is a homotopically periodic map of $\Gamma_3\backslash\mathrm{Nil}_3$, we can show that f always has the Nielsen number $N(f)=0$.

Theorem 5.3. *For any homotopically periodic map f of $\Gamma_3\backslash\mathrm{Nil}_3$, the Lefschetz number, the Nielsen number and the Reidemeister number are*

$$L(f)=0,\ \ N(f)=0,\ \ R(f)=\infty.$$

Proof. Just like before, we may assume that a linearization of f is

$$F=\left(\begin{array}{c:c}\bar F & 0\\ \hdashline * & \det(\bar F)\end{array}\right)\in\mathrm{Aut}(\Gamma_3)$$

so that

$$L(f)=\det(I_3-F)=\left(1-\mathrm{tr}\,(\bar F)+\det(\bar F)\right)\left(1-\det(\bar F)\right).$$

It suffices to show that if F is of finite order then $\det(I_3-F)=0$. If $\det(\bar F)=1$ then it is clear that $\det(I_3-F)=0$. On the other hand, consider $\det(\bar F)=-1$. Then the trace of $\bar F$ is 0. For, first recall from [11, p. 180] that every element of finite order in $\mathrm{GL}(2,\mathbb{Z})$ is conjugate to one of the following matrices

$$\pm I_2,\ \begin{pmatrix}0&1\\1&0\end{pmatrix},\ \begin{pmatrix}1&0\\0&-1\end{pmatrix},\ \begin{pmatrix}0&-1\\1&-1\end{pmatrix},\ \begin{pmatrix}0&1\\-1&0\end{pmatrix},\ \begin{pmatrix}0&1\\-1&1\end{pmatrix}.$$

Since $\det(\bar{F}) = -1$, $\bar{F}$ is conjugate to $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ or $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, each of which has trace 0. Thus, we still have $\det(I_3 - F) = -2\operatorname{tr}(\bar{F}) = 0$. □

Now we consider arbitrary self-maps of the standard nilmanifold $\Gamma_3\backslash\mathrm{Nil}_3$.

Theorem 5.4. *We have*

$$\begin{aligned} \mathfrak{L}(\Gamma_3\backslash\mathrm{Nil}_3) &= \mathbb{Z}, \\ \mathfrak{N}(\Gamma_3\backslash\mathrm{Nil}_3) &= \mathbb{N} \cup \{0\}, \\ \mathfrak{R}(\Gamma_3\backslash\mathrm{Nil}_3) &= \mathbb{N} \cup \{\infty\}. \end{aligned}$$

Proof. Note that

$$F = \left(\begin{array}{cc:c} 1 & -1 & 0 \\ 1 & k-1 & 0 \\ \hdashline 0 & 0 & k \end{array}\right) \in \mathrm{Endo}(\Gamma_3).$$

Let f be a self-map of $\Gamma_3\backslash\mathrm{Nil}_3$ whose linearization is F. Then

$$L(f) = \det(I_3 - F) = 1 - k.$$

This finishes the proof. □

Because $\mathfrak{L}(\Gamma_3\backslash\mathrm{Nil}_3) \subset \mathfrak{LC}(\Gamma_3\backslash\mathrm{Nil}_3)$, immediately we have

Corollary 5.5. *We have*

$$\begin{aligned} \mathfrak{LC}(\Gamma_3\backslash\mathrm{Nil}_3) &= \mathbb{Z}, \\ \mathfrak{NC}(\Gamma_3\backslash\mathrm{Nil}_3) &= \mathbb{N} \cup \{0\}, \\ \mathfrak{RC}(\Gamma_3\backslash\mathrm{Nil}_3) &= \mathbb{N} \cup \{\infty\}. \end{aligned}$$

By Theorem 2.1, there are only two infra-nilmanifolds M which are essentially covered by $\Gamma_3\backslash\mathrm{Nil}_3$, one the nilmanifold $\Gamma_3\backslash\mathrm{Nil}_3$ itself and the other $\Pi\backslash\mathrm{Nil}_3$ whose fundamental group is $\Pi = \langle\Gamma_3, \alpha\rangle$ where

$$\alpha = (a, A) = \left(\begin{pmatrix} 1 & 0 & \frac{7}{24} \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} \right) \in \mathrm{Nil}_3 \rtimes \mathrm{GL}(2, \mathbb{Z}).$$

In the remaining of this section we shall consider the infra-nilmanifold $M = \Pi\backslash\mathrm{Nil}_3$. It is known that $\mathrm{Aut}(\mathrm{Nil}_3) = \mathrm{Nil}_3/\mathcal{Z}(\mathrm{Nil}_3) \rtimes \mathrm{GL}(2, \mathbb{R})$ and every element $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}(2, \mathbb{R})$ acts on Nil_3 as a Lie group automorphism as follows:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} : \begin{pmatrix} 1 & x_1 & x_3 \\ 0 & 1 & x_2 \\ 0 & 0 & 1 \end{pmatrix} \mapsto \begin{pmatrix} 1 & ax_1 + bx_2 & x_3' \\ 0 & 1 & cx_1 + dx_2 \\ 0 & 0 & 1 \end{pmatrix}$$

where

$$x_3' = \frac{1}{2}\left(ax_1(cx_1 + 2dx_2) + x_2(bdx_2 - 2x_1)\right) + (ad - bc)x_3.$$

Consequently, $\mathrm{GL}(2,\mathbb{R})$ can be regarded as a subgroup of $\mathrm{Aut}(\mathfrak{nil}_3)=\mathrm{GL}(3,\mathbb{R})$ by

$$\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in \mathrm{GL}(2,\mathbb{R})\hookrightarrow \left(\begin{array}{cc:c} a & b & 0\\ c & d & 0\\ \hdashline 0 & 0 & ad-bc\end{array}\right)\in \mathrm{GL}(3,\mathbb{R}).$$

In particular,

$$A=\begin{pmatrix} 0 & -1\\ 1 & -1\end{pmatrix}\in \mathrm{GL}(2,\mathbb{Z})\hookrightarrow A_*=\left(\begin{array}{cc:c} 0 & -1 & 0\\ 1 & -1 & 0\\ \hdashline 0 & 0 & 1\end{array}\right)\in \mathrm{GL}(3,\mathbb{R})$$

Theorem 5.6. *Let $M=\varPi\backslash\mathrm{Nil}_3$. Then*

$$\mathfrak{L}_{\mathfrak{h}}(\varPi\backslash\mathrm{Nil}_3)=\{0\},\ \mathfrak{N}_{\mathfrak{h}}(\varPi\backslash\mathrm{Nil}_3)=\{0\},\ \mathfrak{R}_{\mathfrak{h}}(\varPi\backslash\mathrm{Nil}_3)=\{\infty\}.$$

Proof. Let f be a homeomorphism of $M=\varPi\backslash\mathrm{Nil}_3$ with linearization

$$F=\left(\begin{array}{c:c} \bar F & 0\\ \hdashline * & \det(\bar F)\end{array}\right)\in \mathrm{Aut}(\Gamma_3).$$

By (3.1) together with the fact that F is invertible, $\bar F\in \mathrm{GL}(2,\mathbb{Z})$ satisfies that

$$\bar F A = A\bar F \text{ or } \bar F A = A^{-1}\bar F.$$

If $\det(\bar F)=1$, then $\det(I-F)=\det(I-\bar F)\cdot(1-\det(\bar F))=0$; hence

$$\begin{aligned} L(f)&=\frac13\big(\det(I-F)+\det(I-AF)+\det(I-A^2F)\big)\\ &=\frac13(0+0+0)=0.\end{aligned}$$

If $\det(\bar F)=-1$, there is no solution for $\bar F A=A\bar F$. If $\det(\bar F)=-1$ and $\bar F A=A^{-1}\bar F$, then $\bar F$ is one of the following:

$$\bar F=\pm\begin{pmatrix} 0 & 1\\ 1 & 0\end{pmatrix},\ \pm\begin{pmatrix} 1 & -1\\ 0 & -1\end{pmatrix},\ \pm\begin{pmatrix} 1 & 0\\ 1 & -1\end{pmatrix}.$$

Therefore, the possible linearizations of f are

$$F=\begin{pmatrix} 0 & \pm1 & 0\\ \pm1 & 0 & 0\\ 0 & 0 & -1\end{pmatrix},\ \begin{pmatrix} \pm1 & \mp1 & 0\\ 0 & \mp1 & 0\\ 0 & 0 & -1\end{pmatrix},\ \begin{pmatrix} \pm1 & 0 & 0\\ \pm1 & \mp1 & 0\\ 0 & 0 & -1\end{pmatrix}.$$

Clearly, for each of these F, we have $\det(I-F)=0$. Consequently,

$$L(f)=0+0+0=0,\ N(f)=0+0+0=0 \text{ and } R(f)=\infty. \qquad\square$$

Recalling that $\varPi=\langle\Gamma_3,\alpha\rangle$ with

$$\alpha=(a,A)=\left(\begin{pmatrix} 1 & 0 & \frac{7}{24}\\ 0 & 1 & \frac12\\ 0 & 0 & 1\end{pmatrix},\begin{pmatrix} 0 & -1\\ 1 & -1\end{pmatrix}\right),$$

it is easy to see that

$$\begin{aligned}
&\alpha(E_{1,2}, I)\alpha^{-1} = (E_{2,3}, I),\\
&\alpha(E_{2,3}, I)\alpha^{-1} = (E_{1,2}, I)^{-1}(E_{2,3}, I)^{-1},\\
&\alpha(E_{1,3}, I)\alpha^{-1} = (E_{1,3}, I).
\end{aligned}$$

We will find a fully invariant subgroup $\Lambda_3 \subset \Gamma_3$ of Π which is of finite index. Every element β of Π is one of the following forms

$$\begin{aligned}
&(E_{1,2}, I)^{n_1}(E_{2,3}, I)^{n_2}(E_{1,3}, I)^{n_3} = (E_{1,2}^{n_1}E_{2,3}^{n_2}E_{1,3}^{n_3}, I),\\
&(E_{1,2}, I)^{n_1}(E_{2,3}, I)^{n_2}(E_{1,3}, I)^{n_3}\alpha = (E_{1,2}^{n_1}E_{2,3}^{n_2}E_{1,3}^{n_3}, I)\alpha,\\
&(E_{1,2}, I)^{n_1}(E_{2,3}, I)^{n_2}(E_{1,3}, I)^{n_3}\alpha^{-1} = (E_{1,2}^{n_1}E_{2,3}^{n_2}E_{1,3}^{n_3}, I)\alpha^{-1}.
\end{aligned}$$

Hence β^3 is of the form

$$\begin{aligned}
&(E_{1,2}^{n_1}E_{2,3}^{n_2}E_{1,3}^{n_3}, I)^3 = (E_{1,2}^{3n_1}E_{2,3}^{3n_2}E_{1,3}^{3(n_3-n_1n_2)}, I),\\
&((E_{1,2}^{n_1}E_{2,3}^{n_2}E_{1,3}^{n_3}, I)\alpha)^3 = (E_{1,3}^{\frac{1}{2}(2+n_1+n_1^2+n_2+2n_1n_2+n_2^2+6n_3)}, I),\\
&((E_{1,2}^{n_1}E_{2,3}^{n_2}E_{1,3}^{n_3}, I)\alpha^{-1})^3 = (E_{2,3}^{\frac{1}{2}(-2+n_1-n_1^2+n_+4n_1n_2-n_2^2+6n_3)}, I).
\end{aligned}$$

By the proof of [10, Lemma 3.1], we can choose Λ_3 as the subgroup of Π generated by the set of all elements β^3 where $\beta \in \Pi$. Then we see that

$$\Lambda_3 = \langle E_{1,2}^3, E_{2,3}^3, E_{1,3}\rangle.$$

Now we compute

$$\mathrm{Endo}(\Lambda_3) = \left\{ \left(\begin{array}{cc:c} a & b & 0\\ c & d & 0\\ \hdashline u & v & ad-bc \end{array}\right) \mid a, b, c, d \in \mathbb{Z}, u, v \in \frac{1}{2}\mathbb{Z} \right\} \subset \mathrm{Endo}(\mathfrak{nil}_3).$$

with respect to the linear basis $\{3\mathbf{e}_{1,2}, 3\mathbf{e}_{2,3}, \mathbf{e}_{1,3}\}$ of $\mathfrak{nil}_3$. It is also easy to compute that

$$A = \begin{pmatrix} 0 & -1\\ 1 & -1 \end{pmatrix} \in \mathrm{GL}(2, \mathbb{Z}) \hookrightarrow A_* = \left(\begin{array}{cc:c} 0 & -1 & 0\\ 1 & -1 & 0\\ \hdashline 0 & 0 & 1 \end{array}\right) \in \mathrm{Aut}(\mathfrak{nil}_3)$$

with respect to the linear basis $\{3\mathbf{e}_{1,2}, 3\mathbf{e}_{2,3}, \mathbf{e}_{1,3}\}$ of $\mathfrak{nil}_3$. (See the last paragraph of Section 3.)

Theorem 5.7. *Let $M = \Pi\backslash\mathrm{Nil}_3$. Then*

$$\begin{aligned}
&\mathfrak{L}(\Pi\backslash\mathrm{Nil}_3) = \left\{1 - (a^2+ab+b^2)^2 \mid a, b \in \mathbb{Z}\right\},\\
&\mathfrak{N}(\Pi\backslash\mathrm{Nil}_3) = \left\{|1 - (a^2+ab+b^2)^2| \mid a, b \in \mathbb{Z}\right\},\\
&\mathfrak{R}(\Pi\backslash\mathrm{Nil}_3) = \left\{ |1 - (a^2+ab+b^2)^2| \;\middle|\; \begin{array}{l} a, b \in \mathbb{Z},\\ (a,b) \neq \pm(1,0), \pm(0,1),\\ \qquad \pm(1,1), \pm(1,-1) \end{array} \right\} \cup \{\infty\}.
\end{aligned}$$

Proof. Let f be a self-map of $M = \mathit{\Pi}\backslash\mathrm{Nil}_3$ with linearization

$$F = \left(\begin{array}{c:c} \bar{F} & 0 \\ \hdashline * & \det(\bar{F}) \end{array}\right) = \left(\begin{array}{cc:c} a & b & 0 \\ c & d & 0 \\ \hdashline u & v & ad-bc \end{array}\right) \in \mathrm{Endo}(\Lambda_3).$$

Then F must satisfy (3.1):

$$FA = F, FA = AF \text{ or } FA = A^{-1}F.$$

Thus $\bar{F}$ satisfies

$$\bar{F}A = \bar{F}, \bar{F}A = A\bar{F} \text{ or } \bar{F}A = A^{-1}\bar{F}.$$

If $\bar{F}A = \bar{F}$ then $\bar{F} = 0$, hence $L(f) = \frac{1}{3}(0+0+0) = 0$. If $\bar{F}A = A\bar{F}$ or $\bar{F}A = A^{-2}\bar{F}$ then $\bar{F}$ is respectively of the form

$$\bar{F} = \begin{pmatrix} a & b \\ -b & a+b \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} b & a \\ a+b & -b \end{pmatrix} \in \mathrm{M}(2,\mathbb{Z}).$$

In the first case, we have

$$\begin{aligned} L(f) = \frac{1}{3}\big(&-(a^2+ab+b^2-1)((a-1)^2+(a-1)b+b^2) \\ &- (a^2+ab+b^2-1)((a+1)^2+ab-b+b^2) \\ &-(a^2+ab+b^2-1)(a^2+a(b+1)+(b+1)^2)\big) \\ = -(a^2&+ab+b^2)^2+1. \end{aligned}$$

In the second case, we have

$$\begin{aligned} L(f) = \frac{1}{3}\big(&-((a^2+ab+b^2)^2-1) - ((a^2+ab+b^2)^2-1) \\ &-((a^2+ab+b^2)^2-1)\big) \\ = -(a^2&+ab+b^2)^2+1. \end{aligned}$$

Hence

$$\begin{aligned} \mathfrak{L}(\mathit{\Pi}\backslash\mathrm{Nil}_3) &= \{0\} \bigcup \left\{1-(a^2+ab+b^2)^2 \mid a,b \in \mathbb{Z}\right\} \\ &= \left\{1-(a^2+ab+b^2)^2 \mid a,b \in \mathbb{Z}\right\}. \end{aligned}$$

Remark that $a^2+ab+b^2 = (a+\frac{b}{2})^2+\frac{3}{4}b^2 \geq 0$. If $a^2+ab+b^2 = 0$ then $b = 0$ and $a = 0$. In this case, $N(f) = R(f) = 1$. Consequently, we may assume that $a^2+ab+b^2 \geq 1$. Similarly, $(a-1)^2+(a-1)b+b^2 \geq 0$ and $a^2+a(b+1)+(b+1)^2 \geq 0$. Furthermore, $(a+1)^2+ab-b+b^2 = ((a+1)+\frac{b-1}{2})^2+\frac{3}{4}(b-1)^2 \geq 0$. These imply that each term in the above expressions for $L(f)$ are nonnegative. Consequently, in either case of $\bar{F}$, we have

$$N(f) = \begin{cases} 1-(a^2+ab+b^2)^2 = 1 & \text{if } a=b=0 \\ (a^2+ab+b^2)^2-1 & \text{otherwise,} \end{cases}$$

which proves the remaining assertions for $\mathfrak{N}(\mathit{\Pi}\backslash\mathrm{Nil}_3)$ and $\mathfrak{R}(\mathit{\Pi}\backslash\mathrm{Nil}_3)$. $\square$

Note that the infra-nilmanifold $M = \Pi\backslash\mathrm{Nil}_3$ is orientable because the holonomy group $\Phi = \langle A\rangle$ preserves the orientation of the standard nilmanifold $\Gamma_3\backslash\mathrm{Nil}_3$, or equivalently because $\det(A) = 1 > 0$. Hence the coincidence invariants for the Nielsen theory are defined.

Theorem 5.8. *Let $M = \Pi\backslash\mathrm{Nil}_3$. Then*

$$\begin{aligned}
\mathfrak{LC}(\Pi\backslash\mathrm{Nil}_3) &= \left\{m^2 - n^2 \mid m, n \textit{ are of the form } a^2 + ab + b^2\right\},\\
\mathfrak{NC}(\Pi\backslash\mathrm{Nil}_3) &= \left\{|m^2 - n^2| \mid m, n \textit{ are of the form } a^2 + ab + b^2\right\},\\
\mathfrak{RC}(\Pi\backslash\mathrm{Nil}_3) &= \left\{|m^2 - n^2| \neq 0 \mid m, n \textit{ are of the form } a^2 + ab + b^2\right\} \bigcup \{\infty\}.
\end{aligned}$$

Proof. Let f and g be self-maps of $M = \Pi\backslash\mathrm{Nil}_3$ with respective linearizations

$$D = \left(\begin{array}{c:c} \bar{D} & 0 \\ \hdashline * & \det(\bar{D}) \end{array}\right) = \left(\begin{array}{cc:c} a & b & 0 \\ c & d & 0 \\ \hdashline u & v & ad - bc \end{array}\right) \in \mathrm{Endo}(\Lambda_3),$$

and

$$E = \left(\begin{array}{c:c} \bar{E} & 0 \\ \hdashline * & \det(\bar{E}) \end{array}\right) = \left(\begin{array}{cc:c} k & \ell & 0 \\ m & n & 0 \\ \hdashline u' & v' & kn - \ell m \end{array}\right) \in \mathrm{Endo}(\Lambda_3).$$

By the proof of Theorem 5.7, $\bar{D}$ and $\bar{E}$ are respectively one of the following:

$$\begin{aligned}
\bar{D} &= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} a & b \\ -b & a+b \end{pmatrix} \text{ or } \begin{pmatrix} b & a \\ a+b & -b \end{pmatrix} \in \mathrm{M}(2, \mathbb{Z}),\\
\bar{E} &= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} k & \ell \\ -\ell & k+\ell \end{pmatrix} \text{ or } \begin{pmatrix} \ell & k \\ k+\ell & -\ell \end{pmatrix} \in \mathrm{M}(2, \mathbb{Z}).
\end{aligned}$$

For each pair of $(\bar{D}, \bar{E})$, we compute the Lefschetz coincidence number

$$L(f, g) = \frac{1}{3}\left(\det(E - D) + \det(E - AD) + \det(E - A^2 D)\right).$$

- For $(\bar{D}, \bar{E}) = (0, 0)$, we have

$$L(f, g) = N(f, g) = 0, \ R(f, g) = \infty.$$

- For $(\bar{D}, \bar{E}) = \left(\begin{pmatrix} a & b \\ -b & a+b \end{pmatrix}, 0\right)$, we have

$$\begin{aligned}
L(f, g) &= \frac{1}{3}\left(\det(0 - D) + \det(0 - AD) + \det(0 - A^2 D)\right)\\
&= \frac{1}{3}\left(-(a^2 + ab + b^2)^2 - (a^2 + ab + b^2)^2 - (a^2 + ab + b^2)^2\right)\\
&= -(a^2 + ab + b^2)^2,\\
N(f, g) &= (a^2 + ab + b^2)^2.
\end{aligned}$$

• For $(\bar{D},\bar{E})=\left(\begin{pmatrix} b & a\\ a+b & -b\end{pmatrix},0\right)$, we have

$$\begin{aligned}L(f,g)&=\frac{1}{3}\left((a^2+ab+b^2)^2+(a^2+ab+b^2)^2+(a^2+ab+b^2)^2\right)\\&=(a^2+ab+b^2)^2=N(f,g).\end{aligned}$$

• For

$$(\bar{D},\bar{E})=\left(\begin{pmatrix} a & b\\ -b & a+b\end{pmatrix},\begin{pmatrix} k & \ell\\ -\ell & k+\ell\end{pmatrix}\right)$$

or

$$(\bar{D},\bar{E})=\left(\begin{pmatrix} b & a\\ a+b & -b\end{pmatrix},\begin{pmatrix} \ell & k\\ k+\ell & -\ell\end{pmatrix}\right),$$

we have

$$\begin{aligned}L(f,g)=&\frac{(k^2+k\ell+\ell^2)-(a^2+ab+b^2)}{3}\\&\times\Big\{\left((a-k)^2+(a-k)(b-\ell)+(b-\ell)^2\right)\\&\quad+\left(\left((a+\ell)+\frac{b+k}{2}\right)^2+\frac{3}{4}(b-k)^2\right)\\&\quad+\left(\frac{3}{4}(a-\ell)^2+\left(\frac{a+\ell}{2}+(b+k)\right)^2\right)\Big\}\\=&(k^2+k\ell+\ell^2)^2-(a^2+ab+b^2)^2.\end{aligned}$$

• For $(\bar{D},\bar{E})=\left(\begin{pmatrix} a & b\\ -b & a+b\end{pmatrix},\begin{pmatrix} \ell & k\\ k+\ell & -\ell\end{pmatrix}\right)$, we have

$$\begin{aligned}L(f,g)=&\frac{1}{3}\Big(\left((k^2+k\ell+\ell^2)^2-(a^2+ab+b^2)^2\right)\\&+\left((k^2+k\ell+\ell^2)^2-(a^2+ab+b^2)^2\right)\\&+\left((k^2+k\ell+\ell^2)^2-(a^2+ab+b^2)^2\right)\Big)\\=&(k^2+k\ell+\ell^2)^2-(a^2+ab+b^2)^2.\end{aligned}$$

This completes the proof. □

6. Homeomorphisms of infra-nilmanifolds modeled on Nil_m

Let $m\geq 4$. The Lie algebra $\mathfrak{nil}_m$ of Nil_m is generated by ([3, Lemma 3.1])

$$\mathcal{L}_1=\{\mathbf{e}_{1,2},\ \mathbf{e}_{2,3},\ \cdots,\ \mathbf{e}_{m-1,m}\}.$$

Moreover, $\mathcal{L}_1$ forms a linear basis of the vector space $\mathfrak{nil}_m/[\mathfrak{nil}_m,\mathfrak{nil}_m]$. By [3, Proposition 3.2], this gives a natural homomorphism

$$\pi:\mathrm{Aut}(\mathfrak{nil}_m)\to\mathrm{Aut}(\mathfrak{nil}/\mathfrak{nil}^2)=\mathrm{GL}(m-1,\mathbb{R})$$

whose image is isomorphic to $(\mathbb{R}^*)^{m-1} \rtimes \mathbb{Z}_2$ where

$$(\mathbb{R}^*)^{m-1} \cong \left\{ \begin{pmatrix} r_1 & 0 & \cdots & 0 \\ 0 & r_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & r_{m-1} \end{pmatrix} \mid r_i \in \mathbb{R}^* \right\}$$

and

$$\mathbb{Z}_2 \text{ is generated by } \begin{pmatrix} 0 & \cdots & 0 & 1 \\ 0 & \cdots & 1 & 0 \\ \vdots & \iddots & \vdots & \vdots \\ 1 & \cdots & 0 & 0 \end{pmatrix}.$$

Hence it follows that $\mathrm{Aut}(\mathfrak{nil}_m) \cong \ker(\pi) \rtimes ((\mathbb{R}^*)^{m-1} \rtimes \mathbb{Z}_2)$ and

$$K := \left((\mathbb{R}^*)^{m-1} \rtimes \mathbb{Z}_2\right) \bigcap \mathrm{GL}(m-1, \mathbb{Z}) = (\mathbb{Z}_2)^{m-1} \rtimes \mathbb{Z}_2$$

is a maximal compact subgroup of $\mathrm{Aut}(\mathrm{Nil}_m) = \mathrm{Aut}(\mathfrak{nil}_m)$.

Let Γ_m be the lattice of Nil_m with integer entries. By the unique extension property, we have

$$\mathrm{Aut}(\Gamma_m) \subset \mathrm{Aut}(\mathrm{Nil}_m) = \mathrm{Aut}(\mathfrak{nil}_m)$$

and we have the following commutative diagram

$$\begin{tikzcd} \mathrm{Aut}(\mathfrak{nil}_m) \arrow[r, "\pi"] & (\mathbb{R}^*)^{m-1} \rtimes \mathbb{Z}_2 \arrow[r] & 1 \\ \mathrm{Aut}(\Gamma_m) \arrow[u, "\cup"'] \arrow[r, "\pi"] & K = (\mathbb{Z}_2)^{m-1} \rtimes \mathbb{Z}_2 \arrow[u, "\cup"'] \arrow[r] & 1 \end{tikzcd}$$

Let $\phi \in \mathrm{Aut}(\Gamma_m)$. With respect to the linear generators

$$\mathcal{L} := \{\mathbf{e}_{1,2}, \cdots, \mathbf{e}_{m-1,m}; \mathbf{e}_{1,3}, \cdots, \mathbf{e}_{m-2,m}; \cdots; \mathbf{e}_{1,m-1}, \mathbf{e}_{2,m}; \mathbf{e}_{1,m}\}$$

of $\mathfrak{nil}_m$, $\phi \in \mathrm{Aut}(\mathfrak{nil}_m)$ can be expressed as a lower triangular block matrix

$$\phi = \begin{pmatrix} F_1 & 0 & \cdots & 0 \\ * & F_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ * & * & \cdots & F_{m-1} \end{pmatrix}.$$

Note also that $\pi(\phi) = F_1 \in K$. Hence F_1 is either diagonal or anti-diagonal.

Now we are ready to state and prove our main result of this section. When $m \geq 4$, all homeomorphisms f of M has the Nielsen number $N(f) = 0$.

Theorem 6.1. *Let $m \geq 4$ and let M be an infra-nilmanifold which is essentially covered by the standard nilmanifold $\Gamma_m \backslash \mathrm{Nil}_m$. Then*

$$\mathfrak{L}_{\mathfrak{h}}(M) = \{0\}, \ \mathfrak{N}_{\mathfrak{h}}(f) = \{0\}, \ \mathfrak{R}_{\mathfrak{h}}(f) = \{\infty\}.$$

In particular, the Bieberbach groups of Nil_m having Γ_m as its nil-radical have the R_∞-property.

Proof. First assume that $M = \Gamma_m \backslash \mathrm{Nil}_m$ is the standard nilmanifold. Let f be a self-homeomorphism of M with linearization

$$F = \begin{pmatrix} F_1 & 0 & \cdots & 0 \\ * & F_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ * & * & \cdots & F_{m-1} \end{pmatrix} \in \mathrm{Aut}(\Gamma_m).$$

Then

$$L(f) = \det(I_d - F) = \prod_{i=1}^{m-1} \det(I_{m-i} - F_i)$$

where $d = \frac{1}{2}m(m-1)$.

Now we assert that $\det(I_d - F) = 0$. Indeed we will show that the product of the first two terms is zero;

$$\det(I_{m-1} - F_1)\det(I_{m-2} - F_2) = 0.$$

Consider first the case where F_1 is diagonal

$$F_1 = \mathrm{diag}(\epsilon_1, \epsilon_2, \cdots, \epsilon_{m-1}).$$

If some $\epsilon_j = 1$ then $\det(I_{m-1} - F_1) = 0$ and so $\det(I_d - F) = 0$. Hence we shall consider the case where all $\epsilon_j = -1$. In this case we can easily understand the diagonal blocks of F. Indeed, the second block is

$$F_2 = \mathrm{diag}(\epsilon_1\epsilon_2, \epsilon_2\epsilon_3, \cdots, \epsilon_{m-2}\epsilon_{m-1}).$$

This follows from the fact that $F \in \mathrm{Aut}(\mathfrak{nil}_m)$ preserves the identities

$$\mathbf{e}_{p,p+2} = [\mathbf{e}_{p,p+1}, \mathbf{e}_{p+1,p+2}], \quad \forall\, p \text{ with } 1 \le p \le m-2.$$

Because all $\epsilon_j = -1$, we have $F_2 = I_{m-2}$, hence

$$\det(I - F) = \cdots \det(I - F_2) \cdots = 0.$$

In order to study $\det(I - B)$ for an anti-diagonal B, let

$$B = \mathrm{adiag}(\delta_1, \delta_2, \cdots, \delta_k).$$

Then

$$\det(I_k - B) = \begin{cases} \displaystyle\prod_{i=1}^{n-1} (1 - \delta_i\delta_{k-i})(1 - \delta_n) & \text{when } k = 2n-1 \\ \displaystyle\prod_{i=1}^{n} (1 - \delta_i\delta_{k-i}) & \text{when } k = 2n. \end{cases}$$

Consider now the case where F_1 is anti-diagonal

$$F_1 = \mathrm{adiag}(\epsilon_1, \epsilon_2, \cdots, \epsilon_{m-1}).$$

Then

$$\begin{aligned} F_1 &= \mathrm{adiag}(\epsilon_1, \epsilon_2, \cdots, \epsilon_{m-1}), \\ F_2 &= -\,\mathrm{adiag}(\epsilon_1\epsilon_2, \epsilon_2\epsilon_3, \cdots, \epsilon_{m-2}\epsilon_{m-1}). \end{aligned}$$

If there exists i such that $\epsilon_i \cdot \epsilon_{(m-1)-i} = +1$, then

$$\det(I_{m-1} - F_1) = \cdots (1 - \epsilon_i \cdot \epsilon_{(m-1)-i}) \cdots = 0,$$

and we have $\det(I_d - F) = 0$.

Now suppose $\epsilon_i \cdot \epsilon_{(m-1)-i} = -1$ for all i. Then, among the factors of $\det(I_{m-2} - F_2)$, we have (since $m \geq 4$)

$$\begin{aligned}(1 - \delta_1 \delta_{m-2}) &= (1 - (\epsilon_1 \epsilon_2)(\epsilon_{m-2} \epsilon_{m-1})) \\ &= (1 - (\epsilon_1 \epsilon_{m-1})(\epsilon_2 \epsilon_{m-2})) \\ &= (1 - (-1)(-1)) = 0.\end{aligned}$$

Thus, again we have $\det(I_d - F) = 0$.

Consequently, $\mathfrak{L}(\Gamma_m \backslash \mathrm{Nil}_m) = \{0\}$, $\mathfrak{N}(\Gamma_m \backslash \mathrm{Nil}_m) = \{0\}$ and $\mathfrak{R}(\Gamma_m \backslash \mathrm{Nil}_m) = \{\infty\}$. In particular, Γ_m has the R_∞-property

Now we assume that M is an infra-nilmanifold which is essentially covered by the standard nilmanifold $\Gamma_m \backslash \mathrm{Nil}_m$ and which has nontrivial holonomy group. By Theorem 2.2, we must have $m = 2n \geq 4$ and $M = \mathit{\Pi} \backslash \mathrm{Nil}_m$ is double covered by the standard nilmanifold $\Gamma_m \backslash \mathrm{Nil}_m$.

Let f be a self-homeomorphism of $M = \mathit{\Pi} \backslash \mathrm{Nil}_m$. Since Γ_m is a characteristic subgroup of $\mathit{\Pi}$, f is always lifted a homeomorphism $\bar{f}$ of the nilmanifold $\Gamma_m \backslash \mathrm{Nil}_m$ so that the following diagram is commutative

$$\begin{array}{ccc} \Gamma_m \backslash \mathrm{Nil}_m & \xrightarrow{\bar{f}} & \Gamma_m \backslash \mathrm{Nil}_m \\ \big\downarrow & & \big\downarrow \\ M & \xrightarrow{f} & M \end{array}$$

Because the projection $\Gamma_m \backslash \mathrm{Nil}_m \to M$ is a double covering projection, there are exactly two liftings of f, one $\bar{f}$ and the other $\bar{g}$, both of them are homeomorphisms of the standard nilmanifold $\Gamma_m \backslash \mathrm{Nil}_m$. So, $L(\bar{f}) = L(\bar{g}) = 0$. By the averaging formula, we have

$$L(f) = \frac{1}{2} \left(L(\bar{f}) + L(\bar{g}) \right) = 0 + 0 = 0.$$

Similarly, $N(f) = 0$ and $R(f) = \infty$ for all self-homeomorphisms f of $M = \Gamma_m \backslash \mathrm{Nil}_m$. By definition, $\mathit{\Pi}$ has the R_∞-property. $\square$

Example 6.2. There are two Bieberbach groups of Nil_4 with nontrivial holonomy groups Φ with Γ_4 as the discrete nil-radical. It is generated by Γ_4 together with an element $\alpha = (a, A)$ where

$$a = \mathcal{Z}[\tfrac{1}{2}] = \begin{pmatrix} 1 & 0 & 0 & \frac{1}{2} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in \mathrm{Nil}_4$$

and A is one of the following

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}.$$

Hence, both Bieberbach groups are extensions of Γ_4 by $\mathbb{Z}_2$.

Therefore, there are only two infra-nilmanifolds which are essentially covered by the nilmanifold $\Gamma_4\backslash\mathrm{Nil}_4$.

We take $\Pi = \langle \Gamma, \alpha = (a, A)\rangle$, where

$$a = \mathcal{Z}[\tfrac{1}{2}] = \begin{pmatrix} 1 & 0 & 0 & \frac{1}{2} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in \mathrm{Nil}_4, \quad A = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}.$$

The normalizer N of Π in $\mathrm{Nil}_4 \rtimes K$ is

$$N = \Gamma_4 \cdot \mathcal{Z}(\mathrm{Nil}_4) \times \left((\mathbb{Z}_2)^2 \oplus (\mathbb{Z}_2)^2\right),$$

where $(\mathbb{Z}_2)^2 \oplus (\mathbb{Z}_2)^2$ is generated by

$$\begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & a_1 \\ 0 & a_2 & 0 \\ a_1 & 0 & 0 \end{pmatrix}, \quad \text{with } a_i = \pm 1,$$

where every element having order 2. Therefore, the group of isometries of our space $\Pi\backslash\mathrm{Nil}_4$ is

$$\mathrm{Isom}(\Pi\backslash\mathrm{Nil}_4) = N/\Gamma_4 = S^1 \rtimes (\mathbb{Z}_2)^2 \oplus (\mathbb{Z}_2)^2.$$

As an example, let $f : \Pi\backslash\mathrm{Nil}_4 \to \Pi\backslash\mathrm{Nil}_4$ be a map induced by $\alpha = (e, B)$, where

$$B = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Then f is an isometry of period 2. Of course, $L(f) = N(f) = 0$. But here is a geometric reasoning.

We will calculate the fixed point set of f explicitly. With

$$\mathbf{x} = \begin{pmatrix} 1 & x_1 & x_4 & x_6 \\ 0 & 1 & x_2 & x_5 \\ 0 & 0 & 1 & x_3 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

we solve the equation

$$B \cdot \mathbf{x} = \mathbf{a} \cdot (A \cdot \mathbf{x})$$

to get

$$x_1 = 0,\ x_2 = 0,\ x_3 = 0,\ x_6 = -\tfrac{1}{4}.$$

Consequently, the fixed points of f on the universal covering space Nil_4 is

$$\tilde{F} = \left\{ \begin{pmatrix} 1 & 0 & x_5 & \frac{1}{4} \\ 0 & 1 & 0 & x_4 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} : \ x_4, x_5 \in \mathbb{R} \right\} \cong \mathbb{R}^2.$$

All the other components of fixed points are just translates of $\tilde{F}$ by Π. Clearly, the fixed point set of the map on $\Pi\backslash\mathrm{Nil}_4$ is $F = \Pi\backslash(\Pi \cdot \tilde{F}) = T^2$, a 2-torus. This T^2 can be surgered out. More precisely, one can find a tubular neighborhood V of T^2 which is invariant by f. Now a result due to R. Brown (see Theorem 3 in [1]) enables us to homotope f to a map f' which is fixed point free on V and $f' = f$ on the boundary of V. Then this new map is homotopic to the original f and has no fixed points. Consequently, $N(f) = L(f) = 0$.

7. Nielsen theory of infra-nilmanifolds modeled on Nil_m $(m \geq 4)$

Write

$$E_{i,j} = \exp \mathbf{e}_{i,j} \ (i < j).$$

It is easy to observe that

$$[E_{i,j}, E_{p,q}] = \begin{cases} E_{i,q} & \text{if } j = p \\ E_{p,j}^{-1} & \text{if } i = q \\ e\,(= \text{identity matrix in } \mathrm{Nil}_m) & \text{otherwise.} \end{cases} \tag{7.1}$$

Recall that the lattice Γ_m of Nil_m with integer entries is generated by the $E_{i,j}$'s.

By the unique extension property, we have

$$\mathrm{Endo}(\Gamma_m) \subset \mathrm{Endo}(\mathrm{Nil}_m) = \mathrm{Endo}(\mathfrak{nil}_m).$$

A Lie algebra endomorphism of $\mathfrak{nil}_m$ is a linear transformation of the linear space $\mathfrak{nil}_m$ preserving all Lie brackets

$$[\mathbf{e}_{i,j}, \mathbf{e}_{p,q}] = \begin{cases} \mathbf{e}_{i,q} & \text{if } j = p \\ -\mathbf{e}_{p,j} & \text{if } i = q \\ \mathbf{0} & \text{otherwise.} \end{cases} \tag{7.2}$$

A Lie algebra endomorphism of $\mathfrak{nil}_m$ is an endomorphism of the lattice Γ_m if and only if if preserves the $E_{i,j}$'s.

Let ϕ be an endomorphism of Γ_m. Since $\phi = \mathrm{d}\phi$ preserves the lower central series $\mathfrak{nil}_m = \mathfrak{nil}^{(1)} \supset \mathfrak{nil}^{(2)} \supset \mathfrak{nil}^{(3)} \supset \cdots \supset \mathfrak{nil}^{(m-1)} \supset \mathfrak{nil}^{(m)} = \{0\}$ of $\mathfrak{nil}_m$, ϕ must be a lower block triangular matrix of the form

$$\phi = P = \left(\begin{array}{c:c:c:c} P_{1,1} & & & \\ \hdashline * & P_{2,2} & & \\ \hdashline \vdots & \vdots & \ddots & \\ \hdashline * & * & \cdots & P_{m-1,m-1} \end{array}\right).$$

Among the Lie brackets (7.2),

- the nontrivial Lie brackets will determine the matrices
$$P_{2,2}, P_{3,2}, \cdots, P_{m-1,2}; P_{3,3}, \cdots, P_{m-1,3}; \cdots; P_{m-1,m-1},$$
and
- the trivial Lie brackets will give rise to the whole conditions for the matrices $P_{1,1}, P_{2,1}, \cdots, P_{m-1,1}$ to be satisfied.

Let us look at an example to make this clear.

Example 7.1. Let $m = 4$. The nontrivial Lie brackets are
$$[\mathbf{e}_{1,2}, \mathbf{e}_{2,3}] = \mathbf{e}_{1,3},\ [\mathbf{e}_{2,3}, \mathbf{e}_{3,4}] = \mathbf{e}_{2,4},\ [\mathbf{e}_{1,2}, \mathbf{e}_{2,4}] = \mathbf{e}_{1,4} = [\mathbf{e}_{1,3}, \mathbf{e}_{3,4}].$$
These identities are preserved by ϕ and as a result they determine the matrices $P_{2,2}, P_{3,2}$ and $P_{3,3}$ as follows:
$$\begin{aligned}
&p_{44} = p_{11}p_{22} - p_{21}p_{12}, \qquad p_{45} = p_{12}p_{23} - p_{22}p_{13},\\
&p_{54} = p_{21}p_{32} - p_{31}p_{22}, \qquad p_{55} = p_{22}p_{33} - p_{32}p_{23},\\
&p_{64} = (p_{11}p_{52} - p_{51}p_{12}) - (p_{31}p_{42} - p_{41}p_{32}),\\
&p_{65} = (p_{12}p_{53} - p_{52}p_{13}) - (p_{32}p_{43} - p_{42}p_{33}),\\
&p_{66} = p_{11}p_{55} - p_{31}p_{45} = p_{33}p_{44} - p_{13}p_{54}.
\end{aligned}$$
Next, the trivial Lie brackets
$$\begin{aligned}
&[\mathbf{e}_{1,2}, \mathbf{e}_{3,4}] = \mathbf{0}, \quad [\mathbf{e}_{1,2}, \mathbf{e}_{1,3}] = \mathbf{0}, \quad [\mathbf{e}_{2,3}, \mathbf{e}_{1,3}] = \mathbf{0},\\
&[\mathbf{e}_{2,3}, \mathbf{e}_{2,4}] = \mathbf{0}, \quad [\mathbf{e}_{3,4}, \mathbf{e}_{2,4}] = \mathbf{0}, \quad [\mathbf{e}_{1,3}, \mathbf{e}_{2,4}] = \mathbf{0}
\end{aligned}$$
will determine the conditions on the matrices $P_{1,1}, P_{2,1}, P_{3,1}$ to be satisfied. Indeed, the Lie bracket $[\mathbf{e}_{1,2}, \mathbf{e}_{3,4}] = \mathbf{0}$ yields that
$$\begin{aligned}
&[\phi(\mathbf{e}_{1,2}), \phi(\mathbf{e}_{3,4})]\\
&= [p_{11}\mathbf{e}_{1,2} + p_{21}\mathbf{e}_{2,3} + p_{31}\mathbf{e}_{3,4} + p_{41}\mathbf{e}_{1,3} + p_{51}\mathbf{e}_{2,4} + p_{61}\mathbf{e}_{1,4},\\
&\qquad p_{13}\mathbf{e}_{1,2} + p_{23}\mathbf{e}_{2,3} + p_{33}\mathbf{e}_{3,4} + p_{43}\mathbf{e}_{1,3} + p_{53}\mathbf{e}_{2,4} + p_{63}\mathbf{e}_{1,4}] = \mathbf{0},
\end{aligned}$$
hence
$$0 = p_{11}p_{23} - p_{21}p_{13}, \tag{7.3}$$
$$0 = p_{21}p_{33} - p_{31}p_{23}, \tag{7.4}$$
$$0 = (p_{11}p_{53} - p_{51}p_{13}) - (p_{31}p_{43} - p_{41}p_{33}).$$
Similarly, from the remaining trivial Lie brackets, we obtain
$$0 = p_{11}p_{54} - p_{31}p_{44}, \tag{7.5}$$
$$0 = p_{12}p_{54} - p_{32}p_{44}, \tag{7.6}$$
$$0 = p_{12}p_{55} - p_{32}p_{45}, \tag{7.7}$$
$$0 = p_{13}p_{55} - p_{33}p_{45}. \tag{7.8}$$
If $p_{21} \neq 0$ or $p_{23} \neq 0$ then (7.3) and (7.4) imply that
$$(p_{11}, p_{13}) = k_1(p_{21}, p_{23}),\ (p_{31}, p_{33}) = k_2(p_{21}, p_{23})$$

for some $k_1, k_2 \in \mathbb{R}$. Thus $P_{1,1}$ must be one of the following forms

$$\begin{pmatrix} k_1p_{21}^* & p_{12} & 0 \\ p_{21}^* & p_{22} & 0 \\ k_2p_{21}^* & p_{32} & 0 \end{pmatrix}, \qquad \begin{pmatrix} 0 & p_{12} & k_1p_{23}^* \\ 0 & p_{22} & p_{23}^* \\ 0 & p_{32} & k_2p_{23}^* \end{pmatrix}, \qquad \begin{pmatrix} k_1p_{21}^* & p_{12} & k_1p_{23}^* \\ p_{21}^* & p_{22} & p_{23}^* \\ k_2p_{21}^* & p_{32} & k_2p_{23}^* \end{pmatrix}.$$

Here, p_{ij}^*'s appearing in the above matrices denote nonzero numbers. By the identities (7.5) $\sim$ (7.8), all of the above matrices for $P_{1,1}$ must satisfy

$$k_1(p_{32} - k_2p_{22}) + k_2(p_{12} - k_1p_{22}) = 0, \tag{7.9}$$

$$p_{12}(p_{32} - k_2p_{22}) + p_{32}(p_{12} - k_1p_{22}) = 0. \tag{7.10}$$

Remark that one of these matrices $P_{1,1}$ is singular.

Now we consider the case where $p_{21} = p_{23} = 0$. The identities (7.3) and (7.4) are automatically true and so our matrix $P_{1,1}$ is

$$P_{1,1} = \begin{pmatrix} p_{11} & p_{12} & p_{13} \\ 0 & p_{22} & 0 \\ p_{31} & p_{32} & p_{33} \end{pmatrix}.$$

From the identities (7.5) $\sim$ (7.8), we have

$$p_{11}p_{22}p_{31} = 0,\ p_{13}p_{22}p_{33} = 0, \tag{7.11}$$

$$p_{22}(p_{11}p_{32} + p_{12}p_{31}) = 0,\ p_{22}(p_{12}p_{33} + p_{13}p_{32}) = 0. \tag{7.12}$$

If $P_{1,1}$ is nonsingular, then $p_{22} \neq 0$ and $\begin{pmatrix} p_{11} & p_{13} \\ p_{31} & p_{33} \end{pmatrix}$ is nonsingular. By (7.11),

$$p_{11}p_{31} = 0 = p_{13}p_{33},$$

and by (7.12),

$$p_{12} = p_{32} = 0.$$

Consequently, if $P_{1,1}$ is nonsingular then $P_{1,1}$ is either diagonal or anti-diagonal. This fact was proved in [3, Proposition 3.2] and reminded in the previous section.

For another remark, we can see that there exists a self-map f of $M = \Gamma_4\backslash\mathrm{Nil}_4$ whose linearization is

$$P = \left(\begin{array}{ccc:cc:c} 0 & 0 & 0 & & & \\ 0 & p_{22} & 0 & & & \\ 0 & 0 & 0 & & & \\ \hdashline * & * & * & 0 & 0 & \\ * & * & * & 0 & 0 & \\ \hdashline * & * & * & * & * & 0 \end{array}\right) \in \mathrm{Endo}(\Gamma_4)$$

because our P satisfies the identities (7.11) and (7.12). Thus

$$L(f) = \det(I - P) = 1 - p_{22}.$$

This proves Theorem 7.2 below for $M = \Gamma_4\backslash\mathrm{Nil}_4$.

Let M be an infra-nilmanifold which is essentially covered by $\Gamma_m\backslash\mathrm{Nil}_m$. In the following *we will find a family of self-maps* $\{f\}$ *of* M *such that*

$$\{L(f)\} = \mathbb{Z},\ \{N(f)\} = \mathbb{N} \cup \{0\},\ \{R(f)\} = \mathbb{N} \cup \{\infty\}.$$

Consequently this will prove one of our main results:

Theorem 7.2. *Let M be an infra-nilmanifold which is essentially covered by $\Gamma_m\backslash\mathrm{Nil}_m$. Then*

$$\begin{aligned}\mathfrak{L}(M) &= \mathbb{Z},\\ \mathfrak{N}(M) &= \mathbb{N}\cup\{0\},\\ \mathfrak{R}(M) &= \mathbb{N}\cup\{\infty\}.\end{aligned}$$

The proof of the theorem goes as follows: Let $m = 2n \geq 4$ be even. Just like the case when $m = 4$, we can see that there is an endomorphism $P \in \mathrm{Endo}(\Gamma_m)$ such that $P_{1,1} = (p_{ij})$ is a matrix of the form

$$P_{1,1} = \begin{pmatrix} 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \iddots & \vdots \\ 0 & \cdots & p_{n,n} & \cdots & 0 \\ \vdots & \iddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & 0 \end{pmatrix}; \tag{7.13}$$

$p_{ij} = 0$ for all (i,j) except for $(i,j) = (n,n)$. Thus $P_{2,2}, \cdots, P_{m-1,m-1}$ are trivial matrices. Hence $L(f) = \det(I - P) = 1 - p_{n,n}$. Therefore Theorem 7.2 is proved for $M = \Gamma_m\backslash\mathrm{Nil}_m$ with $m \geq 4$ even.

By Theorem 2.2, there are 2^{n-1} infra-nilmanifolds $\Pi\backslash\mathrm{Nil}_m$ which are essentially covered by the nilmanifold $\Gamma_m\backslash\mathrm{Nil}_m$. All of these have the covering group $\mathbb{Z}_2$ generated by

$$A = \begin{pmatrix} 0 & 0 & \cdots & 0 & \epsilon_1 \\ 0 & 0 & \cdots & \epsilon_2 & 0 \\ \vdots & \vdots & \iddots & \vdots & \vdots \\ 0 & \epsilon_{m-2} & \cdots & 0 & 0 \\ \epsilon_{m-1} & 0 & \cdots & 0 & 0 \end{pmatrix} \in \mathrm{GL}(m-1, \mathbb{Z}),$$

$\epsilon_j = \pm 1$, where the signs are taken in such a way that the number of -1's is even, and the matrix is symmetric. Remark that this matrix representation for A is obtained by considering $A \in \mathrm{Aut}(\Gamma_m)$.

Lemma 7.3. *Let $m = 2n \geq 4$. Then there exists a fully invariant subgroup $\Lambda_m \subset \Gamma_m$ of Π so that $\Lambda_m/[\Lambda_m, \Lambda_m]$ is generated by*

$$E_{1,2}, \cdots, E_{n-1,n}, E^2_{n,n+1}, E_{n+1,n+2}, \cdots,, E_{m-1,m}$$

modulo $[\Lambda_m, \Lambda_m]$. In particular, these elements of Λ_m generate the Lie group Nil_m.

Proof. Every element x of Γ_m can be written uniquely as

$$x = E^{k_1}_{1,2} E^{k_2}_{2,3} \cdots E^{k_{m-1}}_{m-1,m} \cdot \tilde{x}$$

where $\tilde{x} \in [\Gamma_m, \Gamma_m]$. Hence by (7.1), x^2 has a unique expression

$$x^2 = E^{2k_1}_{1,2} E^{2k_2}_{2,3} \cdots E^{2k_{m-1}}_{m-1,m} \cdot \tilde{x}' \tag{7.14}$$

where $\tilde{x}' \in [\Gamma_m, \Gamma_m]$.

Consider $x\alpha \in \mathit{\Pi}$ with $x \in \Gamma_m$ and $\alpha = (a, A)$. Remark that the action of A on xa is as follows:

$$
\begin{aligned}
A(xa) &= A(E_{1,2}^{k_1} E_{2,3}^{k_2} \cdots E_{m-1,m}^{k_{m-1}} \cdot \tilde{x}a) \\
&= E_{1,2}^{\epsilon_1 k_{m-1}} E_{2,3}^{\epsilon_2 k_{m-2}} \cdots E_{m-1,m}^{\epsilon_{m-1} k_1} \cdot \tilde{x}'' \quad \text{with } \tilde{x}'' \in [\mathrm{Nil}_m, \mathrm{Nil}_m].
\end{aligned}
$$

Hence by (7.1) again,

$$
\begin{aligned}
(x\alpha)^2 &= xa \cdot A(xa) \\
&= \left(E_{1,2}^{k_1} E_{2,3}^{k_2} \cdots E_{m-1,m}^{k_{m-1}} \cdot \tilde{x}a\right) \left(E_{1,2}^{\epsilon_1 k_{m-1}} E_{2,3}^{\epsilon_2 k_{m-2}} \cdots E_{m-1,m}^{\epsilon_{m-1} k_1} \cdot \tilde{x}''\right) \\
&= E_{1,2}^{k_1 + \epsilon_1 k_{m-1}} \cdots E_{n,n+1}^{k_n + \epsilon_n k_{m-n}} \cdots E_{m-1,m}^{k_{m-1} + \epsilon_{m-1} k_1} \cdot \tilde{x}'''
\end{aligned}
$$

where $\tilde{x}''' \in [\mathrm{Nil}_m, \mathrm{Nil}_m]$. Recall that $\epsilon_j = \pm 1$, and the signs are taken in such a way that the number of -1's is even, and the sequence $\{\epsilon_j\}$ is symmetric

$$
(\epsilon_1, \cdots, \epsilon_{n-1}, \epsilon_n, \epsilon_{n+1}, \cdots, \epsilon_{m-1}) = (\epsilon_1, \cdots, \epsilon_{n-1}, 1, \epsilon_{n-1}, \cdots, \epsilon_1).
$$

Thus

$$
\begin{aligned}
(x\alpha)^2 = E_{1,2}^{k_1 + \epsilon_1 k_{m-1}} \cdots E_{n-1,n}^{k_{n-1} + \epsilon_{n-1} k_{m-(n-1)}} E_{n,n+1}^{2k_n} \\
E_{n+1,n+2}^{-\epsilon_{n-1}(k_{n-1} + \epsilon_{n-1} k_{m-(n-1)})} \cdots E_{m-1,m}^{-\epsilon_1 (k_1 + \epsilon_1 k_{m-1})} \cdot \tilde{x}'''.
\end{aligned}
\tag{7.15}
$$

Recall also that Γ_m is of index 2 in $\mathit{\Pi}$. By the proof of [10, Lemma 3.1], we can choose Λ_m as the subgroup of $\mathit{\Pi}$ generated by the set of all elements x^2 and $(x\alpha)^2$ where $x \in \Gamma_m$. Therefore by (7.14) and (7.15) we can see that

$$
\Lambda_m = \langle E_{1,2}, \cdots, E_{n-1,n}, E_{n,n+1}^2, E_{n+1,n+2}, \cdots, E_{m-1,m} \rangle
$$

modulo $[\Lambda_m, \Lambda_m]$. □

Let f be a self-map of $\mathit{\Pi} \backslash \mathrm{Nil}_m$ whose linearization P has $P_{1,1}$ as given in (7.13), but $P \in \mathrm{Endo}(\Lambda_m)$. Remark also that the anti-diagonal matrix

$$
A = \mathrm{adiag}(\epsilon_1, \cdots, \epsilon_{m-1}) \in \mathrm{Aut}(\Gamma_m)
$$

is the same as

$$
A = \mathrm{adiag}(\epsilon_1, \cdots, \epsilon_{m-1}) \in \mathrm{Aut}(\Lambda_m)
$$

because of the symmetry of the exponents $(1, \cdots, 1, 2, 1, \cdots, 1)$ in the generators

$$
E_{1,2}, \cdots, E_{n-1,n}, E_{n,n+1}^2, E_{n+1,n+2}, \cdots, , E_{m-1,m}
$$

modulo $[\Lambda_m, \Lambda_m]$ of $\Lambda_m / [\Lambda_m, \Lambda_m]$.

By the averaging formula, we have

$$
\begin{aligned}
L(f) &= \frac{1}{2}\left(\det(I - P) + \det(I - AP)\right) \\
&= \frac{1}{2}\left(\det(I - P) + \det(I - P)\right) \\
&= \det(I - P) = 1 - p_{n,n}
\end{aligned}
$$

Therefore we have proved Theorem 7.2 for $M = \mathit{\Pi} \backslash \mathrm{Nil}_m$ with $m \geq 4$ even.

Now assume $m = 2n+1 \geq 5$. Then we can see that there is an endomorphism $P \in \mathrm{Endo}(\Gamma_m)$ such that $P_{1,1} = (p_{ij})$ has a submatrix of the form

$$\begin{pmatrix} p_{n,n} & p_{n,n+1} \\ p_{n+1,n} & p_{n+1,n+1} \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & k-1 \end{pmatrix}$$

and all other entries are zero. Furthermore, we can see that $P_{2,2}$ is a $2n \times 2n$ matrix with all entries 0 except the (n,n)-entry which is k. The remaining diagonal block matrices of P are zero matrices. Hence

$$L(f) = \det(I - P) = \det(I - P_{1,1}) \det(I - P_{2,2}) = 1(1-k).$$

we have proved Theorem 7.2 for $M = \Gamma_m \backslash \mathrm{Nil}_m$ with $m \geq 4$ odd. Finally we recall from Theorem 2.2 again that since $m \geq 4$ is odd, there is no infra-nilmanifold which is essentially covered by $\Gamma_m \backslash \mathrm{Nil}_m$.

Finally we consider the Nielsen coincidence theory on infra-nilmanifolds which are essentially covered by $\Gamma_m \backslash \mathrm{Nil}_m$. However these infra-nilmanifolds with nontrivial holonomy are not orientable. Thus the Lefschetz and the Nielsen coincidence numbers are not defined. So, for coincidence theory we shall consider only the nilmanifold $\Gamma_m \backslash \mathrm{Nil}_m$ and the coincidence result on the nilmanifold $\Gamma_m \backslash \mathrm{Nil}_m$ follows immediately from the fixed point result, Theorem 7.2.

Corollary 7.4. *We have*

$$\begin{aligned} \mathfrak{LC}(\Gamma_m \backslash \mathrm{Nil}_m) &= \mathbb{Z}, \\ \mathfrak{NC}(\Gamma_m \backslash \mathrm{Nil}_m) &= \mathbb{N} \cup \{0\}, \\ \mathfrak{RC}(\Gamma_m \backslash \mathrm{Nil}_m) &= \mathbb{N} \cup \{\infty\}. \end{aligned}$$

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