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# A GENERAL VISCOSITY IMPLICIT ITERATIVE ALGORITHM FOR SPLIT VARIATIONAL INCLUSIONS WITH HIERARCHICAL VARIATIONAL INEQUALITY CONSTRAINTS

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**Abstract.** The purpose of this paper is to introduce a general viscosity implicit iterative method for finding a solution of a split variational inclusion problem (SVIP) with a hierarchical variational inequality (HVI) constraint for a countable family of nonexpansive mappings in Hilbert spaces. Strong convergence theorem is obtained under some mild assumptions.
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## 1. Introduction

Let $\mathcal{H}$ be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. Let $\mathcal{C}$ be a nonempty closed convex subset of $\mathcal{H}$, and let $P_{\mathcal{C}}$ be the metric projection from $\mathcal{H}$ onto $\mathcal{C}$. Let $\mathcal{T} : \mathcal{C} \to \mathcal{H}$ be a nonlinear mapping. Denote by $\mathrm{Fix}(\mathcal{T})$ the set of fixed points of $\mathcal{T}$. We use the notations $\to$ and $\rightharpoonup$ to indicate the strong convergence and the weak convergence, respectively. Let $\mathcal{A} : \mathcal{C} \to \mathcal{H}$ be a nonlinear mapping.
The classical variational inequality (VI) is to find $x^* \in \mathcal{C}$ such that

$$\langle \mathcal{A}x^*, x - x^* \rangle \geq 0, \quad \forall x \in \mathcal{C}. \tag{1.1}$$

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We denote by VI($\mathcal{C}, A$) the solution set of VI (1.1). As a very effective and powerful tool, variational inequalities have been applied to study a wide range of problems arising in differential equations, mechanics, contact problems in elasticity, optimization and control problems, management science, etc. A set-valued mapping $M : H_1 \to 2^{H_1}$ is said to be monotone if, for all $x, y \in H_1,\ f \in Mx$ and $g \in My$ imply $\langle x - y, f - g\rangle \geq 0$. A monotone mapping $M : H_1 \to 2^{H_1}$ is maximal if the graph Gph($M$) of $M$ is not properly contained in the graph of any other monotone mapping. It is known that a monotone mapping $M$ is maximal if and only if for $(x, f) \in H_1 \times H_1,\ \langle x - y, f - g\rangle \geq 0$ for every $(y, g) \in \mathrm{Gph}(M)$ implies $f \in Mx$. Let $M : H_1 \to 2^{H_1}$ be a multi-valued maximal monotone mapping. For any positive number $\lambda$ and identity operator $I$ on $H_1$, the single-valued mapping $J_\lambda^M : H_1 \to H_1$ defined by $J_\lambda^M(x) := (I + \lambda M)^{-1}(x)\ \forall x \in H_1$, is called the resolvent operator associated with $M$. It is known that the resolvent operator $J_\lambda^M$ is firmly nonexpansive and hence in particular nonexpansive.

Let $C_1, C_2, ..., C_m$ be nonempty closed convex subsets of $H_1$. The convex feasibility problem (CFP) is to find $x^* \in H_1$ such that $x^* \in C_1 \cap C_2 \cap \cdots \cap C_m$. The convex feasibility problem (CFP) has received a lot of attention due to its diverse applications in mathematics, approximation theory, communications, geophysics, control theory, biomedical engineering, etc.. When there are only two sets and constraints are imposed on the solutions in the domain of a linear operator as well as in this operator's range, the problem is said to be the split feasibility problem (SFP) which has the following formula:

$$x^* \in C \quad \text{such that} \quad Ax^* \in Q, \tag{1.2}$$

where $C$ and $Q$ are nonempty closed convex subset of real Hilbert spaces $H_1$ and $H_2$, respectively, and $A : H_1 \to H_2$ is a bounded linear operator. It is worth pointing out that in finite dimensional Hilbert spaces, the SFP was first introduced by Censor and Elfving [9] for medical image reconstruction. Since then, the SFP has received much attention due to its applications in signal processing, image reconstruction, with particular progress in intensity-modulated radiation therapy, approximation theory, control theory, biomedical engineering, communications, and geophysics; see e.g, [2, 3, 8, 5, 9, 19, 21] and the references therein.

Recently, Moudafi [17] introduced the following split monotone variational inclusion problem (SMVIP): find $x^* \in H_1$ such that

$$0 \in f_1(x^*) + B_1(x^*), \tag{1.3}$$

and

$$y^* = Ax^* \in H_2 \quad \text{solves} \quad 0 \in f_2(y^*) + B_2(y^*), \tag{1.4}$$

where $B_1 : H_1 \to 2^{H_1}$ and $B_2 : H_2 \to 2^{H_2}$ are multi-valued maximal monotone mappings.

Moudafi [17] introduced an iterative method for solving SMVIP (1.3)-(1.4), which can be seen as an important generalization of an iterative method given by Censor, Gibali and Reich [10] for the split variational inequality problem. SMVIP (1.3)-(1.4) includes the split common fixed-point problem, split variational inequality problem, split zero problem, and split feasibility problem as special cases; see [6, 7, 14, 20, 28] and the references therein.

If $f_1 \equiv 0$ and $f_2 \equiv 0$, then SMVIP (1.3)-(1.4) reduces to the following split variational inclusion problem (SVIP): find $x^* \in H_1$ such that

$$0 \in B_1(x^*), \tag{1.5}$$

and

$$y^* = Ax^* \in H_2 \quad \text{solves} \quad 0 \in B_2(y^*). \tag{1.6}$$

When considered separately, (1.5) is the variational inclusion problem and we denote its solution set by SOLVIP$(B_1)$. The SVIP (1.5)-(1.6) constitutes a pair of variational inclusion problems which have to be solved so that the image $y^* = Ax^*$ under a given bounded linear operator $A$ of the solution $x^*$ of VIP (1.5) in $H_1$ is the solution of the other VIP (1.6) in another space $H_2$, we denote the solution set of VIP (1.6) by SOLVIP$(B_2)$. The solution set of SVIP (1.5)-(1.6) is denoted by $\Gamma$.

Byrne et al. [4] studied the weak and strong convergence of the following iterative method for SVIP (1.5)-(1.6): for given $x_1 \in H_1$, compute the iterative sequence $\{x_n\}$ generated by the following scheme:

$$x_{n+1} = J_\lambda^{B_1}(x_n + \gamma A^*(J_\lambda^{B_2} - I)Ax_n), \forall n \geq 1, \ \exists \lambda > 0.$$

For other recent results on this topic see Sitthithakerngkiet *et al.* [23].

In this paper, we introduce a general viscosity implicit iterative method for finding a solution of the SVIP (1.5)-(1.6) with a hierarchical variational inequality (HVI) constraint for a countable family of nonexpansive mappings in the framework of real Hilbert spaces. Strong convergence theorem of the sequences generated by the proposed iterative algorithm is established under some suitable assumptions. Our results improve, extend and develop the corresponding ones in the recent literature.

## 2. Preliminaries

Now we recall some basic concepts and facts. Let $H_1$ be a real Hilbert space and $C$ be a nonempty closed convex subset of $H_1$. A mapping $F : C \to H_1$ is said to be $\kappa$-Lipschitzian if there exists a constant $\kappa > 0$ such that $\|F(x)-F(y)\| \leq \kappa\|x-y\| \ \forall x, y \in C$. In particular, if $\kappa = 1$, then $F$ is said to be nonexpansive. If $\kappa < 1$, then $F$ is said to be a contraction mapping. A mapping $F : C \to H_1$ is said to be $\eta$-strongly monotone if there exists a constant $\eta > 0$ such that $\langle x - y, Fx - Fy\rangle \geq \eta\|x - y\|^2 \ \forall x, y \in C$. A mapping $F : H_1 \to H_1$ is said to be a strongly positive bounded linear operator if there exists a constant $\bar{\gamma} > 0$ such that $\langle Fx, x\rangle \geq \bar{\gamma}\|x\|^2 \ \forall x \in H_1$. It is easy to see that strongly positive bounded linear operator $F$ is a $\|F\|$-Lipschitzian and $\bar{\gamma}$-strongly monotone operator. In Hilbert spaces, it is well known that

$$\|\lambda x + (1 - \lambda)y\|^2 = \lambda\|x\|^2 + (1 - \lambda)\|y\|^2 - \lambda(1 - \lambda)\|x - y\|^2, \quad \forall x, y \in H_1, \lambda \in [0, 1].$$

For any $x \in H_1$, there exists a unique nearest point in the nonempty closed convex subset $C$ denoted by $P_C x$ such that $\|x - P_C x\| \leq \|x - y\| \ \forall y \in C$. The mapping $P_C$ is called the metric projection of $H_1$ onto $C$. We know that $P_C$ is a nonexpansive mapping from $H_1$ onto $C$. The metric projection $P_C$ can be characterized by $P_C x \in C$ and

$$\langle x - y, P_C x - P_C y\rangle \geq \|P_C x - P_C y\|^2, \quad \forall x, y \in H_1. \tag{2.1}$$

Moreover, for all $x \in H_1$ and $y \in C$, $P_C x$ is characterized by

$$\langle x - P_C x, y - P_C x\rangle \le 0. \tag{2.2}$$

It is easy to see that (2.2) is equivalent to the following inequality:

$$\|x-y\|^2 \ge \|x - P_C x\|^2 + \|y - P_C x\|^2, \quad \forall x \in H_1, y \in C. \tag{2.3}$$

It is not hard to find that every nonexpansive mapping $S : H_1 \to H_1$ satisfies the following inequality

$$\langle (I-S)x-(I-S)y, Sy-Sx\rangle \le \frac{1}{2}\|(I-S)x-(I-S)y\|^2, \quad \forall (x,y) \in H_1 \times H_1, \tag{2.4}$$

and hence

$$\langle (I-S)x, y-Sx\rangle \le \frac{1}{2}\|(I-S)x\|^2, \quad \forall (x,y) \in H_1 \times \mathrm{Fix}(S). \tag{2.5}$$

A mapping $T : H_1 \to H_1$ is said to be averaged if it can be written as the average of mappings $I, S : H_1 \to H_1$, that is, $T \equiv (1-\alpha)I + \alpha S$, where $\alpha \in (0,1)$ and $S$ is nonexpansive. We note that averaged mappings are nonexpansive. Further, firmly nonexpansive mappings (in particular, projections on nonempty closed and convex subsets and resolvent operators of maximal monotone operators) are averaged; see [1, 12, 13, 26] and the references therein.

We need the following propositions and lemmas for proving our main results.

**Proposition 2.1.** (see [16]) *(i) If $T = (1-\alpha)S + \alpha V$, where $S : H_1 \to H_1$ is averaged, $V : H_1 \to H_1$ is nonexpansive and $\alpha \in (0,1)$, then $T$ is averaged.*

*(ii) The composite of finitely many averaged mappings is averaged.*

*(iii) If the mappings $\{T_i\}_{i=1}^N$ are averaged and have a common fixed point, then*

$$\bigcap_{i=1}^{N} \mathrm{Fix}(T_i) = \mathrm{Fix}(T_1, T_2, ..., T_N).$$

*(iv) If $T$ is $\tau$-ism, then for $\gamma > 0$, $\gamma T$ is $\frac{\tau}{\gamma}$-ism.*

*(v) $T$ is averaged if and only if, its complement $I - T$ is $\tau$-ism for some $\tau > \frac{1}{2}$.*

**Proposition 2.2.** (see [27]) *Let $\lambda$ be a number in $(0,1]$ and $T : H_1 \to H_1$ be a nonexpansive mapping, we define the mapping $T^\lambda : H_1 \to H_1$ by*

$$T^\lambda x := Tx - \lambda\mu F(Tx) \ \forall x \in H_1,$$

*where $F : H_1 \to H_1$ is $\kappa$-Lipschitzian and $\eta$-strongly monotone. Then $T^\lambda$ is a contraction provided $0 < \mu < \frac{2\eta}{\kappa^2}$; that is,*

$$\|T^\lambda x - T^\lambda y\| \le (1-\lambda\tau)\|x-y\|, \ \forall x, y \in H_1,$$

*where $\tau = 1 - \sqrt{1-\mu(2\eta - \mu\kappa^2)} \in (0,1]$.*

The following lemmas are well-known.

**Lemma 2.1.** *Let $\mathcal{C}$ be a nonempty closed convex subset of a real Hilbert space $\mathcal{H}$, and let $\mathcal{A} : \mathcal{C} \to \mathcal{H}$ be a monotone and hemicontinuous mapping. Then the following hold:*

*(i) $\mathrm{VI}(\mathcal{C}, \mathcal{A}) = \mathrm{Fix}(P_{\mathcal{C}}(I - \lambda\mathcal{A}))$ for all $\lambda > 0$;*

*(ii) $\mathrm{VI}(\mathcal{C}, \mathcal{A})$ consists of one point, if $\mathcal{A}$ is strongly monotone and Lipschitz continuous.*

**Lemma 2.2.** (see [15]) *SVIP (1.5)-(1.6) is equivalent to find* $x^* \in H_1$ *with* $x^* = J_\lambda^{B_1}(x^*)$ *such that* $y^* = Ax^* \in H_2$ *and* $y^* = J_\lambda^{B_2}(y^*)$*, for some* $\lambda > 0$.

**Lemma 2.3.** (see [25]) *Let* $\{a_n\}$ *be a sequence of nonnegative numbers satisfying the conditions* $a_{n+1} \leq (1-\lambda_n)a_n + \lambda_n\gamma_n$, $\forall n \geq 0$, *where* $\{\lambda_n\}$ *and* $\{\gamma_n\}$ *are sequences of real sequences such that:*

*(i)* $\{\lambda_n\} \subset [0,1]$ *and* $\sum_{n=0}^{\infty} \lambda_n = \infty$, *or equivalently,*

$$\prod_{n=0}^{\infty}(1-\lambda_n) := \lim_{n\to\infty}\prod_{k=0}^{n}(1-\lambda_k) = 0;$$

*(ii)* $\limsup_{n\to\infty} \gamma_n \leq 0$ *or* $\sum_{n=0}^{\infty}|\lambda_n\gamma_n| < \infty$. *Then* $\lim_{n\to\infty} a_n = 0$.

Let $\{S_i\}_{i=1}^{\infty}$ be a countable family of nonexpansive self-mappings on a real Hilbert space $H_1$, and $\{\zeta_i\}_{i=1}^{\infty}$ be a sequence in $[0,1]$. For any $n \geq 1$, we define a mapping $W_n$ as follows:

$$\begin{cases} U_{n,n+1} = I, \\ U_{n,n} = \zeta_n S_n U_{n,n+1} + (1-\zeta_n)I, \\ U_{n,n-1} = \zeta_{n-1} S_{n-1} U_{n,n} + (1-\zeta_{n-1})I, \\ \cdots \\ U_{n,k} = \zeta_k S_k U_{n,k+1} + (1-\zeta_k)I, \\ \cdots \\ U_{n,2} = \zeta_2 S_2 U_{n,3} + (1-\zeta_2)I, \\ W_n = U_{n,1} = \zeta_1 S_1 U_{n,2} + (1-\zeta_1)I. \end{cases} \tag{2.6}$$

Such a mapping $W_n$ is nonexpansive and it is called a $W$-mapping generated by $S_n, S_{n-1}, ..., S_1$ and $\zeta_n, \zeta_{n-1}, ..., \zeta_1$.

**Lemma 2.4.** (see [22]) *Let* $\{S_i\}_{i=1}^{\infty}$ *be a countable family of nonexpansive self-mappings on a real Hilbert space* $H_1$ *with* $\bigcap_{i=1}^{\infty}\mathrm{Fix}(S_i) \neq \emptyset$ *and* $\{\zeta_i\}_{i=1}^{\infty}$ *be a sequence in* $(0,1]$. *Then*

*(i)* $W_n$ *is nonexpansive and* $\mathrm{Fix}(W_n) = \bigcap_{i=1}^{n}\mathrm{Fix}(S_i)$, *for each* $n \geq 1$;

*(ii) for each* $x \in H_1$ *and for each positive integer* $k$, *the* $\lim_{n\to\infty} U_{n,k}x$ *exists;*

*(iii) the mapping* $W$ *defined by* $Wx := \lim_{n\to\infty} W_n x = \lim_{n\to\infty} U_{n,1}x$, $\forall x \in H_1$, *is a nonexpansive mapping satisfying* $\mathrm{Fix}(W) = \bigcap_{i=1}^{\infty}\mathrm{Fix}(S_i)$ *and it is called the* $W$*-mapping generated by* $S_1, S_2, ...$ *and* $\zeta_1, \zeta_2, ....$

**Lemma 2.5.** (see [11]) *Let* $\{S_i\}_{i=1}^{\infty}$ *be a countable family of nonexpansive self-mappings on a real Hilbert space* $H_1$ *with* $\bigcap_{i=1}^{\infty}\mathrm{Fix}(S_i) \neq \emptyset$ *and* $\{\zeta_i\}_{i=1}^{\infty}$ *be a sequence in* $(0,l]$ *for some* $l \in (0,1]$. *If* $C$ *is any bounded subset of* $H_1$, *then* $\lim_{n\to\infty}\sup_{x\in C}\|W_n x - Wx\| = 0$.

Throughout this paper we always assume that $\{\zeta_i\}_{i=1}^{\infty} \subset (0, l]$ for some $l \in (0, 1)$.
**Lemma 2.6.** [24] *Let $\{x_n\}$ and $\{z_n\}$ be bounded sequences in a Banach space $X$ and let $\{\beta_n\}$ be a sequence in $[0, 1]$ with $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$. Suppose*

$$x_{n+1} = \beta_n x_n + (1 - \beta_n) z_n$$

*for all integers $n \ge 0$ and $\limsup_{n\to\infty}(\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|) \le 0$. Then,*

$$\lim_{n\to\infty} \|z_n - x_n\| = 0.$$

**Lemma 2.7.** *In a real Hilbert space $H_1$, there holds the following inequality*

$$\|x + y\|^2 \le \|x\|^2 + 2\langle y, x + y\rangle, \quad \forall x, y \in H_1.$$

**Lemma 2.8.** (see [18]) *Every Hilbert space satisfies the Opial condition, that is, for any sequence $\{x_n\}$ in a Hilbert space $H$ with $x_n \rightharpoonup x$, the inequality*

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|$$

*holds for every $y \in H$ with $y \ne x$.*
**Lemma 2.9.** (see [18]) *Assume that $S$ is a nonexpansive self-mapping on a nonempty closed convex subset $C$ of a Hilbert space $H_1$. If $S$ has a fixed point, then $I - S$ is demiclosed at zero, i.e., if $\{x_n\}$ is a sequence in $C$ converging weakly to some $x \in C$ and the sequence $\{(I - S)x_n\}$ converges strongly to zero, then $(I - S)x = 0$, where $I$ is the identity mapping of $H_1$.*
**Lemma 2.10.** (see [16]) *Assume that $D : H_1 \to H_1$ is a strongly positive bounded linear operator on Hilbert space $H_1$ with coefficient $\bar{\xi} > 0$ and $0 < \rho \le \|D\|^{-1}$. Then,*

$$\|I - \rho D\| \le 1 - \rho\bar{\xi}.$$

## 3. Main results

Let $\{S_i\}_{i=1}^{\infty}$ be a countable family of nonexpansive self-mappings on a real Hilbert space $H_1$. Throughout this paper, assume that $W_n$ is the $W$-mapping generated by $S_n, S_{n-1}, ..., S_1$ and $\zeta_n, \zeta_{n-1}, ..., \zeta_1$, where $\{\zeta_n\}_{n=1}^{\infty}$ is a real sequence in $(0, l]$ for some $l \in (0, 1)$. We are now in a position to state and prove the main result in this paper.
**Theorem 3.1.** *Let $H_1$ and $H_2$ be two real Hilbert spaces and $A : H_1 \to H_2$ be a bounded linear operator. Suppose that $B_1 : H_1 \to 2^{H_1}$ and $B_2 : H_2 \to 2^{H_2}$ are maximal monotone mappings. Let $f : H_1 \to H_1$ be a nonexpansive mapping and let $F : H_1 \to H_1$ be $\kappa$-Lipschitzian and $\eta$-strongly monotone with constants $\kappa, \eta > 0$ such that $0 < \delta < \tau := 1 - \sqrt{1 - \mu(2\eta - \mu\kappa^2)} \in (0, 1]$ for $0 < \mu < \frac{2\eta}{\kappa^2}$. Assume that*

$$\Omega := \left(\bigcap_{i=1}^{\infty} \mathrm{Fix}(S_i)\right) \cap \Gamma \ne \emptyset.$$

*For an arbitrary $x_1 \in H_1$, let the sequences $\{x_n\}$ and $\{y_n\}$ be generated by*

$$\begin{cases} u_n = \gamma_n x_n + (1 - \gamma_n) W_n u_n, \\ y_n = J_\lambda^{B_1}(u_n + \gamma A^*(J_\lambda^{B_2} - I) A u_n), \\ x_{n+1} = \alpha_n \delta f(x_n) + \beta_n x_n + [(1 - \beta_n) I - \alpha_n \mu F] W_n y_n, \quad \forall n \ge 1, \end{cases} \tag{3.1}$$

*where* $\lambda > 0$ *and* $\gamma \in (0, \frac{1}{L})$, $L$ *is the spectral radius of the operator* $A^*A$ *and* $A^*$ *is the adjoint of* $A$, $\{W_n\}$ *is the sequence defined by (2.6),* $\{\alpha_n\}, \{\beta_n\}$ *and* $\{\gamma_n\}$ *are sequences in* $(0, 1]$. *Suppose the control sequences satisfy the following conditions:*

*(C1)* $\{\alpha_n + \beta_n\} \subset (0, 1]$ *and* $\{\beta_n\}_{n=1}^{\infty} \subset [a, b]$ *for some* $a, b \in (0, 1)$*;*

*(C2)* $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$*;*

*(C3)* $0 < \liminf_{n\to\infty} \gamma_n \leq \limsup_{n\to\infty} \gamma_n < 1$ *and* $\lim_{n\to\infty} |\gamma_{n+1} - \gamma_n| = 0$.

*Then* $\{x_n\}$ *and* $\{y_n\}$ *converge strongly to a point* $z \in \Omega$*, which is the unique solution to the variational inequality*

$$\langle(\mu F - \delta f)z, z - p\rangle \leq 0, \quad \forall p \in \Omega, \tag{3.2}$$

*i.e.,* $P_\Omega(z - \mu Fz + \delta f(z)) = z$.

*Proof.* Taking into account that $0 < \liminf_{n\to\infty} \gamma_n \leq \limsup_{n\to\infty} \gamma_n < 1$, we may assume, without loss of generality, that $\{\gamma_n\} \subset [c, d] \subset (0, 1)$ for some $c, d \in (0, 1)$. It is easy to see that for each $n \geq 1$ there exists a unique element $u_n \in C$ such that

$$u_n = \gamma_n x_n + (1 - \gamma_n)W_n u_n. \tag{3.2}$$

As a matter of fact, consider the mapping $F_n x = \gamma_n x_n + (1-\gamma_n)W_n x$, $\forall x \in H_1$. Since each $W_n : H_1 \to H_1$ is a nonexpansive mapping, we deduce that all $x, y \in H_1$,

$$\|F_n x - F_n y\| = (1 - \gamma_n)\|W_n x - W_n y\| \leq (1 - \gamma_n)\|x - y\|.$$

Also, from $\{\gamma_n\} \subset [c, d] \subset (0, 1)$ we get $0 < 1 - \gamma_n < 1$ for all $n \geq 1$. Thus, $F_n$ is a contraction mapping of $H_1$ into itself. By the Banach contraction mapping principle, we know that for each $n \geq 1$ there exists a unique element $u_n \in C$, satisfying (3.2).

Next, we divide the rest of the proof into several steps.

**Step 1.** We claim that $\{x_n\}$, $\{y_n\}$, $\{u_n\}$, $\{W_n u_n\}$, $\{W_n y_n\}$ and $\{F(W_n y_n)\}$ are bounded. Indeed, take an element $p \in \Omega = \bigcap_{n=1}^{\infty} \mathrm{Fix}(S_n) \cap \Gamma$ arbitrarily. Then we have $p = J_\lambda^{B_1} p, Ap = J_\lambda^{B_2}(Ap)$ and $W_n p = p$ for all $n \geq 1$.
Since each $W_n : H_1 \to H_1$ is a nonexpansive mapping, it follows from (3.2) that

$$\begin{aligned} \|u_n - p\| &\leq \gamma_n\|x_n - p\| + (1 - \gamma_n)\|W_n u_n - p\| \\ &\leq \gamma_n\|x_n - p\| + (1 - \gamma_n)\|u_n - p\|, \end{aligned}$$

which hence yields

$$\|u_n - p\| \leq \|x_n - p\|, \quad \forall n \geq 1. \tag{3.3}$$

Then we get

$$\begin{aligned} \|y_n - p\|^2 &= \|J_\lambda^{B_1}(u_n + \gamma A^*(J_\lambda^{B_2} - I)Au_n) - J_\lambda^{B_1} p\|^2 \\ &\leq \|u_n - p\|^2 + \gamma^2\|A^*(J_\lambda^{B_2} - I)Au_n\|^2 \\ &\quad + 2\gamma\langle u_n - p, A^*(J_\lambda^{B_2} - I)Au_n\rangle. \end{aligned} \tag{3.4}$$

Thus, we have

$$\begin{aligned} \|y_n - p\|^2 \leq \|u_n - p\|^2 &+ \gamma^2\langle(J_\lambda^{B_2} - I)Au_n, AA^*(J_\lambda^{B_2} - I)Au_n\rangle \\ &+ 2\gamma\langle u_n - p, A^*(J_\lambda^{B_2} - I)Au_n\rangle. \end{aligned} \tag{3.5}$$

Note that

$$\begin{aligned}\gamma^2\langle (J_\lambda^{B_2}-I)Au_n, AA^*(J_\lambda^{B_2}-I)Au_n\rangle &\le L\gamma^2\langle (J_\lambda^{B_2}-I)Au_n, (J_\lambda^{B_2}-I)Au_n\rangle \\ &= L\gamma^2\|(J_\lambda^{B_2}-I)Au_n\|^2. \end{aligned} \tag{3.6}$$

Consider the term of $2\gamma\langle u_n-p, A^*(J_\lambda^{B_2}-I)Au_n\rangle$ and using (2.5), we have

$$\begin{aligned}&2\gamma\langle u_n-p, A^*(J_\lambda^{B_2}-I)Au_n\rangle \\ &=2\gamma\langle A(u_n-p), (J_\lambda^{B_2}-I)Au_n\rangle \\ &=2\gamma\{\langle Ap-J_\lambda^{B_2}Au_n, Au_n-J_\lambda^{B_2}Au_n\rangle-\|(J_\lambda^{B_2}-I)Au_n\|^2\} \\ &\le 2\gamma\{\tfrac{1}{2}\|(J_\lambda^{B_2}-I)Au_n\|^2-\|(J_\lambda^{B_2}-I)Au_n\|^2\} \\ &=-\gamma\|(J_\lambda^{B_2}-I)Au_n\|^2.\end{aligned} \tag{3.7}$$

Using (3.5), (3.6) and (3.7), we obtain

$$\begin{aligned}\|y_n-p\|^2 &\le \|u_n-p\|^2+L\gamma^2\|(J_\lambda^{B_2}-I)Au_n\|^2-\gamma\|(J_\lambda^{B_2}-I)Au_n\|^2 \\ &=\|u_n-p\|^2+\gamma(L\gamma-1)\|(J_\lambda^{B_2}-I)Au_n\|^2.\end{aligned} \tag{3.8}$$

Since $\gamma\in(0,\frac{1}{L})$, we deduce from (3.3) that

$$\|y_n-p\|^2\le\|u_n-p\|^2\le\|x_n-p\|^2. \tag{3.9}$$

Note that $f, W_n: H_1\to H_1$ are nonexpansive for all $n\ge 1$. Therefore, from (3.1), (3.9) and Proposition 2.2, we conclude that

$$\begin{aligned}&\|x_{n+1}-p\|\le\alpha_n\delta\|f(x_n)-f(p)\|+\alpha_n\|\delta f(p)-\mu Fp\|+\beta_n\|x_n-p\| \\ &\ +(1-\beta_n)\|[I-\tfrac{\alpha_n}{1-\beta_n}\mu F]W_ny_n-[I-\tfrac{\alpha_n}{1-\beta_n}\mu F]p\| \\ &\le\alpha_n\delta\|x_n-p\|+\alpha_n\|\delta f(p)-\mu Fp\|+\beta_n\|x_n-p\|+(1-\beta_n-\alpha_n\tau)\|x_n-p\| \\ &\le\max\left\{\|x_n-p\|,\tfrac{\|\delta f(p)-\mu Fp\|}{\tau-\delta}\right\}.\end{aligned}$$

By induction, we have

$$\|x_n-p\|\le\max\left\{\|x_1-p\|,\frac{\|\delta f(p)-\mu Fp\|}{\tau-\delta}\right\},\quad\forall n\ge 1.$$

It immediately follows that $\{x_n\}$ is bounded, and so are the sequences $\{y_n\}$, $\{u_n\}$, $\{W_nu_n\}$, $\{W_ny_n\}$ and $\{F(W_ny_n)\}$ (due to (3.9) and the Lipschitz continuity of $W_n$ and $F$). Hence, we can choose a bounded subset $C\subset H_1$ such that

$$u_n, x_n, y_n\in C,\quad\forall n\ge 1. \tag{3.10}$$

**Step 2.** We claim that $\|x_{n+1}-x_n\|\to 0$ and $\|y_{n+1}-y_n\|\to 0$ as $n\to\infty$. Indeed, we set

$$x_{n+1}=\beta_nx_n+(1-\beta_n)v_n,\quad\forall n\ge 1. \tag{3.11}$$

Then it can be readily seen that

$$v_n=\frac{\alpha_n}{1-\beta_n}(\delta f(x_n)-\mu FW_ny_n)+W_ny_n. \tag{3.12}$$

Hence,

$$\begin{aligned}\|v_{n+1}-v_n\| &\le\tfrac{\alpha_{n+1}}{1-\beta_{n+1}}\|\delta f(x_{n+1})-\mu FW_{n+1}y_{n+1}\| \\ &\ +\tfrac{\alpha_n}{1-\beta_n}\|\delta f(x_n)-\mu FW_ny_n\|+\|W_{n+1}y_{n+1}-W_ny_n\|.\end{aligned} \tag{3.13}$$

Since $J_\lambda^{B_1}$ and $J_\lambda^{B_2}$ both are firmly nonexpansive, they are averaged. For $\gamma \in (0, \frac{1}{L})$, the mapping $(I+\gamma A^*(J_\lambda^{B_2}-I)A)$ is averaged; see [16]. It follows from Proposition 2.1 (ii) that the mapping $J_\lambda^{B_1}(I+\gamma A^*(J_\lambda^{B_2}-I)A)$ is averaged and hence nonexpansive. So, we obtain that

$$\begin{aligned}\|y_{n+1}-y_n\| &= \|J_\lambda^{B_1}(I+\gamma A^*(J_\lambda^{B_2}-I)A)u_{n+1} - J_\lambda^{B_1}(I+\gamma A^*(J_\lambda^{B_2}-I)A)u_n\| \\ &\leq \|u_{n+1}-u_n\|. \end{aligned} \tag{3.14}$$

On the other hand, one has

$$\begin{aligned}\|W_{n+1}y_{n+1}-W_ny_n\| &\leq \|W_{n+1}y_{n+1}-W_ny_{n+1}\| + \|W_ny_{n+1}-W_ny_n\| \\ &\leq \sup_{x\in C}[\|W_{n+1}x-Wx\| \\ &\quad + \|Wx-W_nx\|] + \|y_{n+1}-y_n\|. \end{aligned} \tag{3.15}$$

where $C$ is the bounded subset of $H_1$ defined by (3.10). In a similar way, we get

$$\|W_{n+1}u_{n+1}-W_nu_n\| \leq \sup_{x\in C}[\|W_{n+1}x-Wx\| + \|Wx-W_nx\|] + \|u_{n+1}-u_n\|. \tag{3.16}$$

It follows that

$$\begin{aligned}\|u_{n+1}-u_n\| \quad &\leq \gamma_{n+1}\|x_{n+1}-x_n\| + (1-\gamma_{n+1})\{\sup_{x\in C}[\|W_{n+1}x-Wx\| \\ &+\|Wx-W_nx\|] + \|u_{n+1}-u_n\|\} \\ &+|\gamma_{n+1}-\gamma_n|\|x_n-W_nu_n\|. \end{aligned} \tag{3.17}$$

So it follows from $\{\gamma_n\} \subset [c,d]$ that

$$\begin{aligned}\|u_{n+1}-u_n\| \quad &\leq \|x_{n+1}-x_n\| + \tfrac{1}{c}\sup_{x\in C}[\|W_{n+1}x-Wx\| + \|Wx-W_nx\|] \\ &+|\gamma_{n+1}-\gamma_n|\tfrac{\|x_n-W_nu_n\|}{c}. \end{aligned} \tag{3.18}$$

Thus, from (3.13), (3.14), (3.15) and (3.18) we deduce that

$$\begin{aligned}&\|v_{n+1}-v_n\| - \|x_{n+1}-x_n\| \\ &\leq \tfrac{\alpha_{n+1}}{1-\beta_{n+1}}\|\delta f(x_{n+1}) - \mu F W_{n+1}y_{n+1}\| + \tfrac{\alpha_n}{1-\beta_n}\|\delta f(x_n) - \mu F W_n y_n\| \\ &\ +(1+\tfrac{1}{c})\sup_{x\in C}[\|W_{n+1}x-Wx\| + \|Wx-W_nx\|] + |\gamma_{n+1}-\gamma_n|\tfrac{\|x_n-W_nu_n\|}{c}, \end{aligned}$$

It follows from the conditions (C1), (C2), (C3), and Lemma 2.5 that

$$\limsup_{n\to\infty}(\|v_{n+1}-v_n\| - \|x_{n+1}-x_n\|) \leq 0.$$

Hence, from Lemma 2.6 and (3.11), we obtain that

$$\lim_{n\to\infty}\|v_n-x_n\| = 0. \tag{3.19}$$

From (3.11), we have that

$$\|x_{n+1}-x_n\| = \|\beta_nx_n + (1-\beta_n)v_n - [\beta_nx_n + (1-\beta_n)x_n]\| = (1-\beta_n)\|v_n-x_n\|.$$

By the condition (C1) and (3.19), we get

$$\lim_{n\to\infty}\|x_{n+1}-x_n\| = 0. \tag{3.20}$$

This together with (3.14) and (3.18), implies that

$$\lim_{n\to\infty}\|u_{n+1}-u_n\|=0 \quad \text{and} \quad \lim_{n\to\infty}\|y_{n+1}-y_n\|=0. \tag{3.21}$$

**Step 3.** We claim that

$$\|x_n-u_n\|\to 0,\ \|x_n-y_n\|\to 0,\ \|x_n-W_nu_n\|\to 0,\ \|x_n-W_ny_n\|\to 0$$

and $\|y_n-W_ny_n\|\to 0$ as $n\to\infty$. Indeed, we set $f_n=\delta f(x_n)-\mu FW_ny_n$ for all $n\geq 1$. For any $p\in\Omega$ and by Lemma 2.7, we observe that

$$\begin{aligned}\|x_{n+1}-p\|^2 &= \|\alpha_n\delta f(x_n)+\beta_nx_n+(1-\beta_n)W_ny_n-\alpha_n\mu FW_ny_n-p\|^2\\ &= \|\alpha_n(\delta f(x_n)-\mu FW_ny_n)+\beta_nx_n+(1-\beta_n)W_ny_n-p\|^2\\ &= \|\alpha_nf_n+\beta_n(x_n-p)+(1-\beta_n)(W_ny_n-p)\|^2\\ &\leq \|\beta_n(x_n-p)+(1-\beta_n)(W_ny_n-p)\|^2+2\langle\alpha_nf_n,x_{n+1}-p\rangle\\ &\leq \beta_n\|x_n-p\|^2+(1-\beta_n)\|y_n-p\|^2+2\alpha_nM^2,\end{aligned} \tag{3.22}$$

where $M=\max\{\sup_{n\geq1}\|f_n\|,\sup_{n\geq1}\|x_n-p\|\}$. Substituting (3.8) for (3.22), we obtain from (3.9) that

$$\begin{aligned}\|x_{n+1}-p\|^2 &\leq \beta_n\|x_n-p\|^2+(1-\beta_n)[\|u_n-p\|^2\\ &\quad+\gamma(L\gamma-1)\|(J_\lambda^{B_2}-I)Au_n\|^2]+2\alpha_nM^2\\ &\leq \|x_n-p\|^2-\gamma(1-\beta_n)(1-L\gamma)\|(J_\lambda^{B_2}-I)Au_n\|^2+2\alpha_nM^2.\end{aligned}$$

Therefore,

$$\begin{aligned}&\gamma(1-\beta_n)(1-L\gamma)\|(J_\lambda^{B_2}-I)Au_n\|^2\\ &\leq(\|x_n-p\|+\|x_{n+1}-p\|)\|x_n-x_{n+1}\|+2\alpha_nM^2,\end{aligned}$$

and from the conditions (C1), (C2), and (3.20), we get

$$\lim_{n\to\infty}\|(J_\lambda^{B_2}-I)Au_n\|=0. \tag{3.23}$$

Since $J_\lambda^{B_1}$ is firmly nonexpansive mapping, by using the inequality (3.7) and Cauchy-Schwarz inequality we have

$$\begin{aligned}\|y_n-p\|^2 &\leq \langle y_n-p,u_n+\gamma A^*(J_\lambda^{B_2}-I)Au_n-p\rangle\\ &\leq \tfrac{1}{2}\{\|y_n-p\|^2+\|u_n-p\|^2-\gamma\|(J_\lambda^{B_2}-I)Au_n\|^2\\ &\quad+\gamma^2\|A^*(J_\lambda^{B_2}-I)Au_n\|^2-\|y_n-u_n-\gamma A^*(J_\lambda^{B_2}-I)Au_n\|^2\}\\ &\leq \tfrac{1}{2}\{\|y_n-p\|^2+\|u_n-p\|^2-\|y_n-u_n\|^2\\ &\quad+2\gamma\|A(y_n-u_n)\|\|(J_\lambda^{B_2}-I)Au_n\|\}.\end{aligned}$$

Hence, we obtain

$$\|y_n-p\|^2\leq\|u_n-p\|^2-\|y_n-u_n\|^2+2\gamma\|A(y_n-u_n)\|\|(J_\lambda^{B_2}-I)Au_n\|. \tag{3.24}$$

Substituting (3.24) for (3.22), one concludes from (3.9) that

$$\begin{aligned}\|x_{n+1}-p\|^2 &\leq \beta_n\|x_n-p\|^2+(1-\beta_n)[\|u_n-p\|^2-\|y_n-u_n\|^2\\ &\quad+2\gamma\|A(y_n-u_n)\|\|(J_\lambda^{B_2}-I)Au_n\|]+2\alpha_nM^2\\ &\leq \|x_n-p\|^2-(1-\beta_n)\|y_n-u_n\|^2\\ &\quad+2\gamma(1-\beta_n)\|A(y_n-u_n)\|\|(J_\lambda^{B_2}-I)Au_n\|+2\alpha_nM^2.\end{aligned}$$

So, we get

$$\begin{aligned}(1-\beta_n)\|y_n-u_n\|^2 &\leq (\|x_n-p\|+\|x_{n+1}-p\|)\|x_n-x_{n+1}\| \\ &\quad +2\gamma(1-\beta_n)\|A(y_n-u_n)\|\|(J_\lambda^{B_2}-I)Au_n\|+2\alpha_n M^2.\end{aligned}$$

From the conditions (C1), (C2), (3.20), and (3.23), we obtain that

$$\lim_{n\to\infty}\|y_n-u_n\|=0. \tag{3.25}$$

Also, according to (3.1) we have

$$\begin{aligned}\|u_n-p\|^2 &\leq \gamma_n\langle x_n-p,u_n-p\rangle+(1-\gamma_n)\|W_nu_n-p\|\|u_n-p\| \\ &\leq \gamma_n\langle x_n-p,u_n-p\rangle+(1-\gamma_n)\|u_n-p\|^2,\end{aligned}$$

which immediately leads to

$$\|u_n-p\|^2\leq\frac{1}{2}[\|x_n-p\|^2+\|u_n-p\|^2-\|x_n-u_n\|^2].$$

It follows from (3.9) that $\|y_n-p\|^2\leq\|u_n-p\|^2\leq\|x_n-p\|^2-\|x_n-u_n\|^2$, which, together with (3.22), yields

$$\|x_{n+1}-p\|^2\leq\|x_n-p\|^2-(1-\beta_n)\|x_n-u_n\|^2-\beta_n(1-\beta_n)\|x_n-W_ny_n\|^2+2\alpha_nM^2.$$

This implies that

$$\begin{aligned}&(1-\beta_n)\|x_n-u_n\|^2+\beta_n(1-\beta_n)\|x_n-W_ny_n\|^2 \\ &\leq (\|x_n-p\|+\|x_{n+1}-p\|)\|x_n-x_{n+1}\|+2\alpha_nM^2.\end{aligned}$$

From the conditions (C1), (C2), and (3.20), we get

$$\lim_{n\to\infty}\|x_n-u_n\|=0 \quad\text{and}\quad \lim_{n\to\infty}\|x_n-W_ny_n\|=0. \tag{3.26}$$

Noticing that $\|u_n-x_n\|=(1-\gamma_n)\|W_nu_n-x_n\|\geq(1-d)\|W_nu_n-x_n\|$,

$$\|x_n-y_n\|\leq\|x_n-u_n\|+\|u_n-y_n\|,$$

and $\|y_n-W_ny_n\|\leq\|y_n-x_n\|+\|x_n-W_ny_n\|$, we deduce from (3.25) and (3.26) that

$$\lim_{n\to\infty}\|x_n-W_nu_n\|=0,\quad \lim_{n\to\infty}\|x_n-y_n\|=0 \quad\text{and}\quad \lim_{n\to\infty}\|y_n-W_ny_n\|=0. \tag{3.27}$$

**Step 4.** We claim that $\limsup_{n\to\infty}\langle(\delta f-\mu F)z,x_n-z\rangle\leq 0$, where $z=P_\Omega(z-\mu Fz+\delta f(z))$. Indeed, we first show that $P_\Omega(I-\mu F+\delta f)$ is a contraction mapping. As a matter of fact, for any $x,y\in H_1$, by Proposition 2.2 we have

$$\begin{aligned}&\|P_\Omega(I-\mu F+\delta f)(x)-P_\Omega(I-\mu F+\delta f)(y)\| \\ &\leq\delta\|f(x)-f(y)\|+\|(I-\mu F)(x)-(I-\mu F)(y)\| \\ &\leq[1-(\tau-\delta)]\|x-y\|,\end{aligned}$$

which implies that $P_\Omega(I-\mu F+\delta f)$ is a contraction mapping. Banach's Contraction Mapping Principle guarantees that $P_\Omega(I-\mu F+\delta f)$ has a unique fixed point. Say $z\in H_1$, that is, $z=P_\Omega(z-\mu Fz+\delta f(z))$. Since $\{x_n\}$ is a bounded sequence in $H_1$, without loss of generality, we may choose a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle(\delta f-\mu F)z,x_n-z\rangle=\lim_{i\to\infty}\langle(\delta f-\mu F)z,x_{n_i}-z\rangle. \tag{3.28}$$

Since $\{x_{n_i}\}$ is bounded, there exists a subsequence $\{x_{n_{i_j}}\}$ of $\{x_{n_i}\}$ which converges weakly to $w$. Without loss of generality, we may assume that $x_{n_i} \rightharpoonup w$. From (3.26) and (3.27), we also see that $u_{n_i} \rightharpoonup w$ and $y_{n_i} \rightharpoonup w$.

Next, we will show that $w \in \Omega$.

**Step 4.1.** We will show that $w \in \bigcap_{i=1}^{\infty} \mathrm{Fix}(S_i) = \mathrm{Fix}(W)$.

Indeed, suppose to the contrary that, $w \notin \mathrm{Fix}(W)$, i.e., $Ww \neq w$ and by Lemma 2.8, we see that

$$\begin{aligned} \liminf_{i\to\infty}\|y_{n_i} - w\| &< \liminf_{i\to\infty}\|y_{n_i} - Ww\| \\ &\leq \liminf_{i\to\infty}\{\|y_{n_i} - Wy_{n_i}\| + \|y_{n_i} - w\|\}. \end{aligned} \tag{3.29}$$

On the other hand, we have

$$\|Wy_n - y_n\| \leq \|Wy_n - W_n y_n\| + \|W_n y_n - y_n\| \leq \sup_{x\in C}\|Wx - W_n x\| + \|W_n y_n - y_n\|.$$

By using Lemma 2.5 and (3.27), we obtain that $\lim_{i\to\infty}\|Wy_n - y_n\| = 0$, which together with (3.29), yields $\liminf_{i\to\infty}\|y_{n_i} - w\| < \liminf_{i\to\infty}\|y_{n_i} - w\|$. This reaches a contraction, and hence we have $w \in \mathrm{Fix}(W) = \bigcap_{i=1}^{\infty}\mathrm{Fix}(S_i)$.

**Step 4.2.** We will show that $w \in \Gamma$.

Indeed, note that $y_{n_i} = J_\lambda^{B_1}(u_{n_i} + \gamma A^*(J_\lambda^{B_2} - I)Au_{n_i})$ can be rewritten as

$$\frac{(u_{n_i} - y_{n_i}) + \gamma A^*(J_\lambda^{B_2} - I)Au_{n_i}}{\lambda} \in B_1 y_{n_i}. \tag{3.30}$$

By passing to limit $i \to \infty$ in (3.30) and by taking into account (3.23), (3.25), and the fact that the graph of a maximal monotone operator is weakly-strongly closed, we obtain $0 \in B_1(w)$, i.e., $w \in \mathrm{SOLVIP}(B_1)$. Furthermore, since $\{u_n\}$ and $\{y_n\}$ have the same asymptotic behavior, $Au_{n_i}$ converges weakly to $Aw$. Since the resolvent $J_\lambda^{B_2}$ is nonexpansive, from (3.23) and Lemma 2.9, we get $Aw = J_\lambda^{B_2}(Aw)$, i.e., $Aw \in \mathrm{SOLVIP}(B_2)$. Therefore, $w \in \Gamma$, and so $w \in \Omega$.

Since $z = P_\Omega(z - \mu Fz + \delta f(z))$ and $w \in \Omega$, by (3.28) and the property of metric projection, we obtain that

$$\begin{aligned} \limsup_{n\to\infty}\langle(\delta f - \mu F)z, x_n - z\rangle &= \lim_{i\to\infty}\langle(\delta f - \mu F)z, x_{n_i} - z\rangle \\ &= \langle(z - \mu Fz + \delta f(z)) - z, w - z\rangle \leq 0. \end{aligned} \tag{3.31}$$

**Step 5.** Finally, we claim that $x_n \to z$ and $y_n \to z$ as $n \to \infty$. Indeed, by (3.9) and Proposition 2.2 we have

$$\begin{aligned} \|x_{n+1} - z\|^2 &= \|\alpha_n\delta f(x_n) + \beta_n x_n + [(1-\beta_n)I - \alpha_n\mu F]W_n y_n - z\|^2 \\ &\leq \alpha_n\langle(\delta f - \mu F)z, x_{n+1} - z\rangle + \beta_n\langle x_n - z, x_{n+1} - z\rangle \\ &\quad + \|[(1-\beta_n)I - \alpha_n\mu F]W_n y_n - [(1-\beta_n)I - \alpha_n\mu F]z\|\|x_{n+1} - z\| \\ &\leq \alpha_n\langle(\delta f - \mu F)z, x_{n+1} - z\rangle + \tfrac{1}{2}\beta_n(\|x_n - z\|^2 + \|x_{n+1} - z\|^2) \\ &\quad + \tfrac{1}{2}(1 - \beta_n - \alpha_n\tau)(\|x_n - z\|^2 + \|x_{n+1} - z\|^2) \\ &= \alpha_n\langle(\delta f - \mu F)z, x_{n+1} - z\rangle + \tfrac{(1-\alpha_n\tau)(\|x_n - z\|^2 + \|x_{n+1} - z\|^2)}{2}. \end{aligned}$$

This immediately implies that

$$2\|x_{n+1}-z\|^2 \le 2\alpha_n\langle(\delta f-\mu F)z, x_{n+1}-z\rangle + (1-\alpha_n\tau)\|x_n-z\|^2 + \|x_{n+1}-z\|^2,$$

and hence $\|x_{n+1}-z\|^2 \le (1-\alpha_n\tau)\|x_n-z\|^2 + 2\alpha_n\langle(\delta f-\mu F)z, x_{n+1}-z\rangle$. From the condition (C2), (3.31), and Lemma 2.3, we see that $\lim_{n\to\infty}\|x_n-z\| = 0$. This completes the proof.

**Remark 3.1.** Compared with Theorem 1.2 (i.e., [[23], Theorem 3.1]), our Theorem 3.1 improves, extends and develops it in the following aspects:

(i) the general iterative algorithm (1.10) in Theorem 1.2 is extended to develop the general viscosity implicit iterative algorithm (3.1) in our Theorem 3.1 by virtue of Mann implicit iteration method, viscosity approximation method and hybrid steepest-descent method;

(ii) the SVIP (1.5)-(1.6) with a HVI (1.11) constraint for a countable family of nonexpansive mappings in Theorem 1.2 is extended to develop the SVIP (1.5)-(1.6) with a HVI (3.2) constraint for a countable family of nonexpansive mappings in our Theorem 3.1, where there is an essential difference between HVI (1.11) and HVI (3.2);

(iii) since compared with algorithm (1.10) in Theorem 1.2, algorithm (3.1) enhances one implicit iterative step $u_n = \gamma_n x_n + (1-\gamma_n)W_n u_n$, there is an additional restriction imposed on $\{\gamma_n\}$, i.e., condition (C3)

$$0 < \liminf_{n\to\infty}\gamma_n \le \limsup_{n\to\infty}\gamma_n < 1 \text{ and } \lim_{n\to\infty}|\gamma_{n+1}-\gamma_n| = 0;$$

(iv) the proof of Theorem 3.1 is very different from the proof of Theorem 3.1 in [23] because in the argument process of our Theorem 3.1, we make use of Proposition 2.2 to calculate the contraction coefficient of the mapping $[(1-\beta_n)I - \alpha_n\mu F]W_n$, but in the argument process of [[23], Theorem 3.1], only Lemma 2.10 is applied to estimating the contraction coefficient of the mapping $[(1-\beta_n)I - \alpha_n D]W_n$;

(v) algorithm (3.1) is more advantageous and more subtle than algorithm (1.10) in Theorem 1.2 because algorithm (3.1) involves the predictor-corrector algorithm for finding a common fixed point of a countable family of nonexpansive mappings $\{S_i\}_{i=1}^{\infty}$, that is, the implicit iterative step $u_n = \gamma_n x_n + (1-\gamma_n)W_n u_n$ is the predictor one for finding their common fixed point and the other explicit iterative step

$$x_{n+1} = \alpha_n\delta f(x_n) + \beta_n x_n + [(1-\beta_n)I - \alpha_n\mu F]W_n y_n$$

is the corrector one for finding their common fixed point.

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