Fixed Point Theory, 20(2019), No. 2, 417-430
DOI: 10.24193/fpt-ro.2019.2.27
http://www.math.ubbcluj.ro/~nodeacj/sfptcj.html

ON THE HYPERSTABILITY OF JENSEN FUNCTIONAL EQUATION IN 2-BANACH SPACES

MUAADH ALMAHALEBI*, SAMIR KABBAJ** AND GWANG HUI KIM***,[1]

*Department of Mathematics, Faculty of Sciences
Ibn Tofail University, BP:14000, Kenitra, Morocco
E-mail: muaadh1979@hotmail.fr

**Department of Mathematics, Faculty of Sciences
Ibn Tofail University, BP:14000, Kenitra, Morocco
E-mail: samkabbaj@yahoo.fr

***Department of Mathematics, Kangnam University
Yongin, Gyeonggi, 16979, Republic of Korea
E-mail: ghkim@kangnam.ac.kr

Abstract. In this paper, we make 2-Banach version of hyperstability results for the Jensen equation. Indeed, by using Brzdęk's fixed point theorem [15], we present some hyperstability results for the Jensen equation in 2-Banach spaces.

Key Words and Phrases: Stability, hyperstability, 2-Banach space, Jensen functional equation.
2010 Mathematics Subject Classification: 39B82, 39B52, 47H10.

1. Introduction

The concept of linear 2-normed spaces was introduced by Gähler ([18], [19]) in the middle of 1960s.

We need to recall some basic facts concerning 2-normed spaces and some preliminary results.

Definition 1.1. let X be a real linear space with $dimX > 1$ and $\|\cdot,\cdot\| : X \times X \longrightarrow \mathbb{R}$ be a function satisfying the following properties:

(1) $\|x, y\| = 0$ if and only if x and y are linearly dependent,
(2) $\|x, y\| = \|y, x\|$,
(3) $\|\lambda x, y\| = |\lambda|\|x, y\|$,
(4) $\|x, y + z\| \leq \|x, y\| + \|x, z\|$,

for all $x, y, z \in X$ and $\lambda \in \mathbb{R}$. Then the function $\|\cdot,\cdot\|$ is called *a 2-norm* on X and the pair $(X, \|\cdot,\cdot\|)$ is called a *linear 2-normed space*. Sometimes the condition (4) called the *triangle inequality*.

[1] Corresponding author.

Example 1.2. For $x = (x_1, x_2), y = (y_1, y_2) \in E = \mathbb{R}^2$, the Euclidean 2-norm $\|x, y\|_E$ is defined by

$$\|x, y\|_E = |x_1 y_2 - x_2 y_1| .$$

Definition 1.3. A sequence $\{x_k\}$ in a 2-normed space X is called a *convergent sequence* if there is an $x \in X$ such that

$$\lim_{k\to\infty} \|x_k - x, y\| = 0,$$

for all $y \in X$. If $\{x_k\}$ converges to x, write $x_k \longrightarrow x$ with $k \longrightarrow \infty$ and call x the limit of $\{x_k\}$. In this case, we also write $\lim_{k\to\infty} x_k = x$.

Definition 1.4. A sequence $\{x_k\}$ in a 2-normed space X is said to be a *Cauchy sequence* with respect to the 2-norm if

$$\lim_{k,l\to\infty} \|x_k - x_l, y\| = 0,$$

for all $y \in X$. If every Cauchy sequence in X converges to some $x \in X$, then X is said to be *complete* with respect to the 2-norm. Any complete 2-normed space is said to be a *2-Banach space.*

Now, we state the following results as lemma (See [21] for the details).

Lemma 1.5. *Let X be a 2-normed space. Then,*
(1) $\big|\|x, z\| - \|y, z\|\big| \leq \|x - y, z\|$ for all $x, y, z \in X$,
(2) if $\|x, z\| = 0$ for all $z \in X$, then $x = 0$,
(3) for a convergent sequence x_n in X,

$$\lim_{n\longrightarrow\infty} \|x_n, z\| = \left\|\lim_{n\longrightarrow\infty} x_n, z\right\|$$

for all $z \in X$.

Throughout this paper, we will denote the set of natural numbers by $\mathbb{N}$ and the set of real numbers by $\mathbb{R}$. By $\mathbb{N}_m$, $m \in \mathbb{N}$, we will denote the set of all natural numbers greater than or equal to m.

Let $\mathbb{R}_+ = [0, \infty)$ the set of nonnegative real numbers. We write B^A to mean the family of all functions mapping from a nonempty set A into a nonempty set B and we use the notation X_0 for the set $X \setminus \{0\}$.

The problem of the stability of functional equations was first raised by Ulam [26]. This included the following question concerning the stability of group homomorphisms.

Let $(G_1, *_1)$ be a group and let $(G_2, *_2)$ be a metric group with a metric $d(.,.)$. Given $\varepsilon > 0$, does there exists a $\delta > 0$ such that if a mapping $h : G_1 \to G_2$ satisfies the inequality

$$d\big(h(x *_1 y), h(x) *_2 h(y)\big) < \delta$$

for all $x, y \in G_1$, then there exists a homomorphism $H : G_1 \to G_2$ with

$$d\big(h(x), H(x)\big) < \varepsilon$$

for all $x \in G_1$?

If the answer is affirmative, we say that the equation of homomorphism

$$h(x *_1 y) = h(x) *_2 H(y)$$

is stable.

The first partial answer to Ulam's question was given by Hyers [20] and he established the stability result as follows:

Theorem 1.6. [20] *Let E_1 and E_2 be two Banach spaces and $f : E_1 \to E_2$ be a function such that*

$$\|f(x+y) - f(x) - f(y)\| \leq \delta$$

for some $\delta > 0$ and for all $x, y \in E_1$. Then the limit

$$A(x) := \lim_{n\to\infty} 2^{-n} f(2^n x)$$

exists for each $x \in E_1$, and $A : E_1 \to E_2$ is the unique additive function such that

$$\|f(x) - A(x)\| \leq \delta$$

for all $x \in E_1$. Moreover, if $f(tx)$ is continuous in t for each fixed $x \in E_1$, then the function A is linear.

Later, Aoki [8] and Bourgin [9] considered the problem of stability with unbounded Cauchy differences. Rassias [23] attempted to weaken the condition for the bound of the norm of Cauchy difference

$$\|f(x+y) - f(x) - f(y)\|$$

and proved a generalization of Theorem 1.6 using a direct method (cf. Theorem 1.7):

Theorem 1.7. [23] *Let E_1 and E_2 be two Banach spaces. If $f : E_1 \to E_2$ satisfies the inequality*

$$\|f(x+y) - f(x) - f(y)\| \leq \theta\big(\|x\|^p + \|y\|^p\big)$$

for some $\theta \geq 0$, for some $p \in \mathbb{R}$ with $0 \leq p < 1$, and for all $x, y \in E_1$, then there exists a unique additive function $A : E_1 \to E_2$ such that

$$\|f(x) - A(x)\| \leq \frac{2\theta}{2 - 2^p}\|x\|^p$$

for each $x \in E_1$. If, in addition, $f(tx)$ is continuous in t for each fixed $x \in E_1$, then the function A is linear.

Later, Rassias [24], [25] motivated Theorem 1.7 as follows:

Theorem 1.8. [24], [25] *Let E_1 be a normed space, E_2 be a Banach space, and $f : E_1 \to E_2$ be a function. If f satisfies the inequality*

$$\|f(x+y) - f(x) - f(y)\| \leq \theta\big(\|x\|^p + \|y\|^p\big) \tag{1.1}$$

for some $\theta \geq 0$, for some $p \in \mathbb{R}$ with $p \neq 1$, and for all $x, y \in E_1 - \{0_{E_1}\}$, then there exists a unique additive function $A : E_1 \to E_2$ such that

$$\|f(x) - A(x)\| \leq \frac{2\theta}{|2 - 2^p|}\|x\|^p \tag{1.2}$$

for each $x \in E_1 - \{0_{E_1}\}$.

Note that Theorem 1.8 reduces to Theorem 1.6 when $p = 0$. For $p = 1$, the analogous result is not valid. Also, Brzdęk [10] showed that estimation (1.2) is optimal for $p \geq 0$ in the general case.

Recently, Brzdęk [12] showed that Theorem 1.8 can be significantly improved; namely, in the case $p < 0$, each $f : E_1 \to E_2$ satisfying (1.1) must actually be additive, and the assumption of completeness of E_2 is not necessary. Unfortunately, this result does not remain valid if we restrict the domain of f (see the further detail in [16]). On the other hand, several mathematicians showed that the fixed point method is an another very efficient and convenient tool for proving the Hyers-Ulam stability for a quite wide class of functional equations (see [13]). Brzdęk et al. [14] proved the fixed point theorem for a nonlinear operator in metric spaces and used this result to study the Hyers-Ulam stability of some functional equations in non-Archimedean metric spaces. In this work, they also obtained the fixed point result in arbitrary metric spaces as follows:

By using this theorem, Brzdęk [11] improved, extended and complemented several earlier classical stability results concerning the additive Cauchy equation (in particular Theorem 1.8). During the past few years many mathematicians have investigated various generalizations, extensions and applications of the Hyers-Ulam stability of a number of functional equations (see, for instance, [4, 6, 5, 7, 3, 1, 2, 17, 13, 16] and references therein).

Now, we will introduce the fixed point theorem, which is main tool theorem by Brzdęk and Ciepliński [Theorem 1, [15]]. That is following :

Let us introduce the following three hypotheses:

(H1) E is a nonempty set, $(Y, \|\cdot,\cdot\|)$ is a 2-Banach space, Y_0 is a subset of Y containing two linearly independent vectors, $j \in \mathbb{N}, f_i : E \to E, g_i : Y_0 \to Y_0$, and $L_i : E \times Y_0 \to \mathbb{R}_+$ for $i = 1, \cdots, j$;

(H2) $\mathcal{T} : Y^E \to Y^E$ is an operator satisfying the inequality

$$\big\|\mathcal{T}\xi(x) - \mathcal{T}\mu(x), y\big\| \leq \sum_{i=1}^{j} L_i(x,y)\Big\|\xi\big(f_i(x)\big) - \mu\big(f_i(x)\big), g_i(y)\Big\|, \ \xi, \mu \in Y^E, \tag{1.3}$$

for all $x \in E, y \in Y_0$.

(H3) $\Lambda : \mathbb{R}_+^{E\times Y_0} \to \mathbb{R}_+^{E\times Y_0}$ is an operator defined by

$$\Lambda\delta(x,y) := \sum_{i=1}^{j} L_i(x,y)\delta\big(f_i(x), g_i(y)\big), \quad \delta \in \mathbb{R}_+^{E\times Y_0}, \quad x \in E, y \in Y_0. \tag{1.4}$$

Theorem 1.9. [15] *Let hypotheses* ***(H1)*** *-* ***(H3)*** *hold and functions* $\varepsilon : E \times Y_0 \to \mathbb{R}_+$ *and* $\varphi : E \to Y$ *fulfill the following two conditions:*

$$\big\|\mathcal{T}\varphi(x) - \mathcal{T}\varphi(x), y\big\| \leq \varepsilon(x,y) \quad x \in E, y \in Y_0, \tag{1.5}$$

$$\varepsilon^*(x,y) := \sum_{n=0}^{\infty} \big(\Lambda^n\varepsilon\big)(x,y) < \infty \quad x \in E, y \in Y_0. \tag{1.6}$$

Then, there exists a unique fixed point ψ *of* $\mathcal{T}$ *for which*

$$\big\|\varphi(x) - \psi(x), y\big\| \leq \varepsilon^*(x,y) \quad x \in E, y \in Y_0. \tag{1.7}$$

Moreover, the function $\psi \in Y^E$ *defined by*

$$\psi(x) := \lim_{n\to\infty} \big((\mathcal{T}^n\varphi)\big)(x) \quad x \in E. \tag{1.8}$$

Let X, Y be normed spaces. A function $f : X \to Y$ is Jensen provided it satisfies the functional equation

$$2f\left(\frac{x+y}{2}\right) = f(x) + f(y) \quad \text{for all } \ x, y \in X, \tag{1.9}$$

and we can say that $f : X \to Y$ is Jensen on X_0 if it satisfies (1.9) for all $x, y \in X_0 := X \setminus \{0\}$ such that $x + y \neq 0$.

2. Main results

In this section, we prove some hyperstability results for the Jensen equation (1.9) in 2-Banach spaces by using Theorem 1.9. In what follows $(X, \|\cdot,\cdot\|)$ is a real 2-Banach space.

Theorem 2.1. *Let* $c \geq 0$, $p, q \in \mathbb{R}$, $p + q < 0$ *and* $f : X \to Y$ *satisfy*

$$\left\|2f\left(\frac{x+y}{2}\right) - f(x) - f(y), z\right\| \leq c\ \|x, z\|^p\ \|y, z\|^q, \tag{2.1}$$

for all $x, y \in X_0$ *such that* $x + y \neq 0$ *and* $z \in Y_0$. *Then* f *is Jensen on* X_0.

Proof. Observe that there exists $m_0 \in \mathbb{N}$ such that

$$\alpha_m := 2\left(\frac{m+1}{2}\right)^{p+q} + m^{p+q} < 1 \ \text{ and } \ m \geq m_0.$$

Since $p + q < 0$, one of p, q must be negative. Assume that $q < 0$, fix $m \in \mathbb{N}_{m_0}$ and replace y by mx in (2.1) we get

$$\left\|2f\left(\left(\frac{m+1}{2}\right)x\right) - f(mx) - f(x), z\right\| \leq c\ m^q \|x, z\|^{p+q}, \quad x \in X_0, z \in Y_0 \tag{2.2}$$

For each $m \in \mathbb{N}$, we define the operators

$$\mathcal{T}_m : Y^{X_0} \to Y^{X_0} \text{ and } \Lambda_m : \mathbb{R}_+^{X_0 \times Y_0} \to \mathbb{R}_+^{X_0 \times Y_0}$$

by

$$\mathcal{T}_m\xi(x) := 2\xi\left(\left(\frac{m+1}{2}\right)x\right) - \xi(mx), \quad \xi \in X^{X_0},\ x \in X_0, \tag{2.3}$$

$$\Lambda_m\delta(x, z) := 2\delta\left(\left(\frac{m+1}{2}\right)x, z\right) + \delta(mx, z), \quad \delta \in \mathbb{R}_+^{X_0},\ x \in X_0, z \in Y_0 \tag{2.4}$$

and write

$$\varepsilon_m(x, z) :=\ c\ m^q \|x, z\|^{p+q}, \quad x \in X_0, z \in Y_0. \tag{2.5}$$

It is easily seen that Λ_m has the form described in (1.4) with $j = 2$,

$$f_1(x) = \left(\frac{m+1}{2}\right)x,\ \ f_2(x) = mx$$

and $L_1(x,z)=2$, $L_2(x,z)=1$. Further, (2.2) can be written in the following way

$$\left\|\mathcal{T}_m f(x)-f(x),z\right\|\leq \varepsilon_m(x,z),\quad x\in X_0, z\in Y_0.$$

Moreover, for every $\xi,\mu\in X^{X_0}$, $x\in X_0$,

$$\left\|\mathcal{T}_m\xi(x)-\mathcal{T}_m\mu(x),z\right\| = \left\|2\xi\left(\left(\frac{m+1}{2}\right)x\right)-\xi(mx)-2\mu\left(\left(\frac{m+1}{2}\right)x\right)+\mu(mx),z\right\|$$

$$\leq 2\left\|\xi\left(\left(\frac{m+1}{2}\right)x\right)-\mu\left(\left(\frac{m+1}{2}\right)x\right),z\right\|+\left\|\xi(mx)-\mu(mx),z\right\|$$

$$=\sum_{i=1}^{2}L_i(x,z)\left\|\xi\big(f_i(x)\big)-\mu\big(f_i(x)\big),z\right\|.$$

Consequently, for each $m\in\mathbb{N}$, (1.3) is valid with $\mathcal{T}:=\mathcal{T}_m$. Next, it easy to show that

$$\Lambda_m^n\varepsilon_m(x,z)=\alpha_m^n\ c\ m^q\|x,z\|^{p+q}, \tag{2.6}$$

for all $x\in X_0$, $z\in Y_0$, $n\in\mathbb{N}_0$ and $m\in\mathbb{N}_{m_0}$. Therefore, we obtain

$$\begin{aligned}\varepsilon_m^*(x,z): &= \sum_{n=0}^{\infty}\big(\Lambda_m^n\varepsilon_m\big)(x,z)\\ &= \varepsilon_m(x,z)\sum_{n=0}^{\infty}\alpha_m^n\\ &= \frac{c\ m^q\|x,z\|^{p+q}}{1-\alpha_m}\end{aligned}$$

for all $x\in X_0$, $z\in Y_0$ and $m\in\mathbb{N}_{m_0}$.
By using Theorem 1.9 with $\varphi=f$, we get that the limit

$$J_m(x):=\lim_{n\to\infty}\big(\mathcal{T}_m^n f\big)(x)$$

exists for each $x\in X_0$ and $m\in\mathbb{N}_{m_0}$, and

$$\left\|f(x)-J_m(x),z\right\|\leq\frac{c\ m^q\|x,z\|^{p+q}}{1-\alpha_m} \tag{2.7}$$

for all $x\in X_0$, $z\in Y_0$ and $m\in\mathbb{N}_{m_0}$. Next, we show that

$$\left\|2\mathcal{T}_m^n f\left(\frac{x+y}{2}\right)-\mathcal{T}_m^n f(x)-\mathcal{T}_m^n f(y),z\right\|\leq c\ \alpha_m^n\|x,z\|^p\ \|y,z\|^q, \tag{2.8}$$

for every $x,y\in X_0$ such that $x+y\neq 0$ and all $z\in Y_0$. Since the case $n=0$ is just (2.1), take $k\in\mathbb{N}$ and assume that (2.8) holds for $n=k$ and every $x,y\in X_0$ such that $x+y\neq 0$.

Then

$$\begin{aligned}
&\left\|2\mathcal{T}_m^{k+1}f\left(\frac{x+y}{2}\right)-\mathcal{T}_m^{k+1}f(x)-\mathcal{T}_m^{k+1}f(y),z\right\| \\
&=\left\|4\mathcal{T}_m^{k}f\left(\left(\frac{m+1}{2}\right)\left(\frac{x+y}{2}\right)\right)-2\mathcal{T}_m^{k}f\left(m\left(\frac{x+y}{2}\right)\right)\right. \\
&\qquad\left.-2\mathcal{T}_m^{k}f\left(\left(\frac{m+1}{2}\right)x\right)+\mathcal{T}_m^{k}f(mx)-2\mathcal{T}_m^{k}f\left(\left(\frac{m+1}{2}\right)y\right)+\mathcal{T}_m^{k}f(my),z\right\| \\
&\le 2\left\|2\mathcal{T}_m^{k}f\left(\left(\frac{m+1}{2}\right)\left(\frac{x+y}{2}\right)\right)-\mathcal{T}_m^{k}f\left(\left(\frac{m+1}{2}\right)x\right)-\mathcal{T}_m^{k}f\left(\left(\frac{m+1}{2}\right)y\right),z\right\| \\
&\qquad+\left\|2\mathcal{T}_m^{k}f\left(m\left(\frac{x+y}{2}\right)\right)-\mathcal{T}_m^{k}f(mx)-\mathcal{T}_m^{k}f(my),z\right\| \\
&\le\ c\left(2\left(\frac{m+1}{2}\right)^{p+q}+m^{p+q}\right)\|x,z\|^p\ \|y,z\|^q \\
&=c\ \alpha_m^n\|x,z\|^p\ \|y,z\|^q,
\end{aligned}$$

for all $x,y\in X_0$ such that $x+y\neq 0$ and all $z\in Y_0$. Thus, by induction we have shown that (2.8) holds for every $n\in\mathbb{N}$. Letting $n\to\infty$ in (2.8), we obtain that

$$2J_m\left(\frac{x+y}{2}\right)=J_m(x)+J_m(y),$$

for all $x,y\in X_0$ such that $x+y\neq 0$. In this way we obtain a sequence $\{J_m\}_{m\ge m_0}$ of Jensen functions on X_0 such that

$$\|f(x)-J_m(x),z\|\le\frac{c\ m^q\|x,z\|^{p+q}}{1-\alpha_m},$$

for all $x\in X_0$ and all $z\in Y_0$. It follows, with $m\to\infty$, that f is Jensen on X_0. □

In similar way we can prove the following theorem in which we consider the case when $p+q>0$. Then obviously at least one of p and q must be positive and without loss of generality we can assume that $q>0$.

Theorem 2.2. *Let $c\ge 0$, $p,q\in\mathbb{R}$, $p+q>0$ and $q>0$. If there exists two sequences $\{e_m\}_{m\in\mathbb{N}}$, $\{g_m\}_{m\in\mathbb{N}}$ of real numbers such that $\{e_m\}_{m\in\mathbb{N}}$ is bounded, $\lim_{m\to\infty}g_m=0$ and there exists a positive integer n_0 such that one of the conditions is satisfied:*

(C_1) $e_m\equiv 1$ *and* $\lim_{m\to\infty}\lambda_m^1<1$ *where*

$$\lambda_m^1:=2\left|\frac{e_m+g_m}{2}\right|^{p+q}+|g_m|^{p+q},$$

(C_2) $\frac{e_m+g_m}{2}\equiv 1$ *and* $\lim_{m\to\infty}\lambda_m^2<1$ *where*

$$\lambda_m^1:=\frac{1}{2}\,|e_m|^{p+q}+|g_m|^{p+q},$$

and $f:X\to Y$ satisfies (2.1) then f is Jensen on X_0.

Proof. Replacing in (2.1) x by $e_m x$ and y by $g_m x$, where

$$m \in \mathbb{N}_{n_0} := \{m \in \mathbb{N} : m \geq n_0\},$$

we get

$$\left\| 2f\left(\left(\frac{e_m + g_m}{2}\right)x\right) - f(e_m x) - f(g_m x), z\right\| \leq c\,|e_m|^p\,|g_m|^q\,\|x, z\|^{p+q}, \tag{2.9}$$

for all $x \in X_0, z \in Y_0$.

Let the case (C_i) holds, where $i \in \{1, 2\}$. For $x \in X_0$ and $z \in Y_0$, we define

$$\mathcal{T}_m\xi(x) := k_1^i \xi\left(\left(\frac{e_m + g_m}{2}\right)x\right) - k_2^i\xi(e_m x) - k_3^i\xi(g_m x), \tag{2.10}$$

$$\Lambda_m\delta(x, z) := |k_1^i|\delta\left(\left(\frac{e_m + g_m}{2}\right)x, z\right) + |k_2^i|\delta(e_m x, z) + |k_3^i|\delta(g_m x, z), \tag{2.11}$$

$$\varepsilon_m(x, z) := \ c\,k_0^i\,|e_m|^p\,|g_m|^q\,\|x, z\|^{p+q}, \tag{2.12}$$

where $k_1^1 = 2$, $k_2^1 = 0$, $k_3^1 = 1$, $k_1^2 = 0$, $k_2^2 = -\frac{1}{2}$, $k_3^2 = -\frac{1}{2}$, $k_0^1 = 1$, $k_0^2 = \frac{1}{2}$.
As in proof of Theorem 2.1 we observe that (2.9) takes form

$$\left\|\mathcal{T}_m f(x) - f(x), z\right\| \leq \varepsilon_m(x, z), \quad x \in X_0, z \in Y_0.$$

and Λ_m has the form described in (1.4) and (1.3) is valid for every $\xi, \mu \in X^{X_0}$, $x \in X_0$ and $z \in Y_0$.
Next we can find $m_0 \in \mathbb{N}$, such that $m_0 \geq n_0$ and $\lambda_m^i < 1$ for $m \in \mathbb{N}_{m_0}$. Therefore

$$\varepsilon_m^*(x, z) : \ = \ \sum_{n=0}^{\infty}\left(\Lambda_m^n \varepsilon_m\right)(x, z) = \frac{\varepsilon_m(x, z)}{1 - \lambda_m^i}$$

for $m = m_0$, $x \in X_0$ and $z \in Y_0$. Hence, according to Theorem 1.9, for each $m \in \mathbb{N}_{m_0}$ there exists a unique solution $J_m : X \to Y$ of the equation

$$J_m(x) := k_1^i J_m\left(\left(\frac{e_m + g_m}{2}\right)x\right) - k_2^i J_m(e_m x) - k_3^i J_m(g_m x),$$

such that

$$\|f(x) - J_m(x), z\| \leq \varepsilon_m^*(x, z), \tag{2.13}$$

for all $x \in X_0$ and all $z \in Y_0$. Moreover,

$$2J_m\left(\frac{x + y}{2}\right) = J_m(x) + J_m(y),$$

for all $x, y \in X_0$ such that $x + y \neq 0$ and $z \in Y_0$. In this way we obtain a sequence $\{J_m\}_{m \geq m_0}$ of Jensen functions on X_0 such that (2.13) holds. It follows, with $m \to \infty$ that f is Jensen because

$$\lim_{m\to\infty} \varepsilon_m^*(x, z) = \|x, z\|^{p+q} \lim_{m\to\infty} \frac{c\,k_0^i\,|e_m|^p\,|g_m|^q}{1 - \lambda_m^i} = 0. \qquad \square$$

From the Theorem 2.2, we deduce in particular the following corollaries.

Corollary 2.3. *Let $c \geq 0$, $p, q \in \mathbb{R}$, $p+q>0$ and $q>0$. If there exits a positive integer n_0 such that*

$$2\left|\frac{m-1}{2m}\right|^{p+q}+\left|\frac{1}{m}\right|^{p+q}<1 \quad m \in \mathbb{N}_{n_0},$$

and $f : X \to Y$ fulfills (2.1) then f is Jensen on X_0.

Proof. Putting $g_m = \frac{-1}{m}$ and using Theorem 2.2 (C_1), we have

$$\lambda_m^1 := 2\left|\frac{m-1}{2m}\right|^{p+q}+\left|\frac{1}{m}\right|^{p+q},$$

hence

$$\lim_{m\to\infty} \lambda_m^1 < 1,$$

so the function f is Jensen on X_0. □

Corollary 2.4. *Let $c \geq 0$, $p, q \in \mathbb{R}$, $p+q>0$ and $q>0$. If there exits a positive integer n_0 such that*

$$\frac{1}{2}\left|\frac{m-1}{m}\right|^{p+q}+\left|\frac{1}{m}\right|^{p+q}<1 \quad m \in \mathbb{N}_{n_0},$$

and $f : X \to Y$ fulfills (2.1) then f is Jensen on X_0.

Proof. Sitting $e_m = 1-\frac{1}{m}$, $g_m = \frac{1}{m}$ and using Theorem 2.2 (C_2), we have

$$\lambda_m^1 := \frac{1}{2}\left|\frac{m-1}{m}\right|^{p+q}+\left|\frac{1}{m}\right|^{p+q},$$

hence

$$\lim_{m\to\infty} \lambda_m^2 < 1,$$

so the function f is Jensen on X_0. □

In the following theorem, we investigate the generalized hyperstability results of Jensen equation (1.9) in 2-Banach spaces. In the rest of the paper, $\{\alpha\}_n$ is a sequence of real numbers such that $\lim_{n\to\infty} \alpha_n = 0$.

Theorem 2.5. *Let $\varphi : X \times X \times Y_0 \to [0,+\infty)$ be a function fulfils the following two conditions:*

$$\lim_{n\to\infty} \sum_{i=0}^{n} \binom{n}{i} 2^{n-i} \varphi\Big(\beta_m^{n-i}\alpha_m^i x, \beta_m^{n-i}\alpha_m^i y, z\Big) = 0, \tag{2.14}$$

$$\lim_{m\to\infty} \sum_{n=0}^{\infty}\sum_{i=0}^{n} \binom{n}{i} 2^{n-i} \varphi\Big(\beta_m^{n-i}\alpha_m^i x, \beta_m^{n-i}\alpha_m^{i+1} x, z\Big) = 0, \tag{2.15}$$

for all $x, y \in X_0$, $z \in Y_0$ and for sufficiently large integers m, where

$$\beta_m = \frac{1+\alpha_m}{2}.$$

Assume that $f : X \to Y$ satisfies

$$\left\|2f\left(\frac{x+y}{2}\right)-f(x)-f(y), z\right\| \leq \varphi(x,y,z), \tag{2.16}$$

for all $x, y \in X_0$ and all $z \in Y_0$ such that $x + y \neq 0$. Then f is Jensen on X_0.

Proof. Replacing y by $\alpha_m x$ in (2.16), where $\alpha_m \in \mathbb{R}$, we get

$$\left\|2f\big(\beta_m x\big) - f(\alpha_m x) - f(x), z\right\| \leq \varphi(x, \alpha_m x, z) \tag{2.17}$$

for all $x \in X_0$ and all $z \in Y_0$, where

$$\beta_m = \frac{1 + \alpha_m}{2}.$$

Define operators $\mathcal{T}_m : Y^{X_0} \to Y^{X_0}$ and $\Lambda_m : \mathbb{R}_+^{X_0 \times Y_0} \to \mathbb{R}_+^{X_0 \times Y_0}$ by

$$\mathcal{T}_m \xi(x) := 2\xi\big(\beta_m x\big) - \xi(\alpha_m x), \quad \xi \in X^{X_0},\ x \in X_0, \tag{2.18}$$

$$\Lambda_m \delta(x, z) := 2\delta\big(\beta_m x, z\big) + \delta(\alpha_m x, z), \quad \delta \in \mathbb{R}_+^{X_0},\ x \in X_0, z \in Y_0, \tag{2.19}$$

and write

$$\varepsilon_m(x, z) := \varphi(x, \alpha_m x, z), \quad x \in X_0, z \in Y_0. \tag{2.20}$$

It is easily seen that Λ_m has the form described in (1.4) with $j = 2$, $f_1(x) = \beta_m x$, $f_2(x) = \alpha_m x$, $L_1(x, z) = 2$ and $L_2(x, z) = 1$. Further, (2.17) can be written in the following way

$$\left\|\mathcal{T}_m f(x) - f(x), z\right\| \leq \varepsilon_m(x, z), \quad x \in X_0, z \in Y_0.$$

Moreover, for every $\xi, \mu \in X^{X_0}$, $x \in X_0$ and $z \in Y_0$

$$\left\|\mathcal{T}_m \xi(x) - \mathcal{T}_m \mu(x), z\right\| \leq L_1(x, z)\left\|(\xi - \mu)\big(f_1(x)\big), z\right\| + L_2(x, z)\left\|(\xi - \mu)\big(f_2(x)\big), z\right\|$$

So, for each $m \in \mathbb{N}$, (1.3) is valid with $\mathcal{T} := \mathcal{T}_m$. It is not hard to show that

$$\Lambda_m^n \varepsilon_m(x, z) = \sum_{i=0}^{n} \tbinom{n}{i} 2^{n-i} \varphi\Big(\beta_m^{n-i} \alpha_m^i x, \beta_m^{n-i} \alpha_m^{i+1} x, z\Big), \tag{2.21}$$

for all $x \in X_0$, $z \in Y_0$, $n \in \mathbb{N}_0$ and $m \in \mathbb{N}_{n_0}$. Therefore,

$$\varepsilon_m^*(x, z) := \sum_{n=0}^{\infty} \sum_{i=0}^{n} \tbinom{n}{i} 2^{n-i} \varphi\Big(\beta_m^{n-i} \alpha_m^i x, \beta_m^{n-i} \alpha_m^{i+1} x, z\Big), \tag{2.22}$$

for all $x \in X_0$, $z \in Y_0$ and $m \in \mathbb{N}_{m_0}$. By (2.14), we get $\varepsilon_m^*(x, z) < \infty$ for all $x \in X_0$ and all $z \in Y_0$. Hence, according to Theorem 1.9, for each $m > n_0$ the limit

$$J_m(x) := \lim_{n \to \infty} \big(\mathcal{T}_m^n f\big)(x)$$

exists for each $x \in X_0$ and $m \in \mathbb{N}_{m_0}$, and

$$\left\|f(x) - J_m(x), z\right\| \leq \varepsilon_m^*(x, z) \tag{2.23}$$

for all $x \in X_0$, $z \in Y_0$ and $m \in \mathbb{N}_{n_0}$.

By similar method in proof of Theorem 2.1, we can prove that

$$\left\|2\mathcal{T}_m^n f\left(\frac{x+y}{2}\right) - \mathcal{T}_m^n f(x) - \mathcal{T}_m^n f(y), z\right\| \leq \sum_{i=0}^{n} \tbinom{n}{i} 2^{n-i} \varphi\Big(\beta_m^{n-i} \alpha_m^i x, \beta_m^{n-i} \alpha_m^i y, z\Big), \tag{2.24}$$

for every $x, y \in X_0$ such that $x + y \neq 0$ and all $z \in Y_0$. Indeed, if $n = 0$, then (2.24) is simply (2.16). So, take $k \in \mathbb{N}_0$ and suppose that (2.24) holds for $n = k$ and every $x, y \in X_0$ such that $x + y \neq 0$. Then

$$\begin{aligned}
&\left\|2\mathcal{T}_m^{k+1} f\left(\frac{x+y}{2}\right) - \mathcal{T}_m^{k+1} f(x) - \mathcal{T}_m^{k+1} f(y), z\right\| \\
&= \left\|4\mathcal{T}_m^k f\left(\beta_m\left(\frac{x+y}{2}\right)\right) - 2\mathcal{T}_m^k f\left(\alpha_m\left(\frac{x+y}{2}\right)\right)\right. \\
&\qquad \left. -2\mathcal{T}_m^k f(\beta_m x) + \mathcal{T}_m^k f(\alpha_m x) - 2\mathcal{T}_m^k f(\beta_m y) + \mathcal{T}_m^k f(\alpha_m y), z\right\| \\
&\le 2\left\|2\mathcal{T}_m^k f\left(\beta_m\left(\frac{x+y}{2}\right)\right) - \mathcal{T}_m^k f(\beta_m x) - \mathcal{T}_m^k f(\beta_m y), z\right\| \\
&\qquad + \left\|2\mathcal{T}_m^k f\left(\alpha_m\left(\frac{x+y}{2}\right)\right) - \mathcal{T}_m^k f(\alpha_m x) - \mathcal{T}_m^k f(\alpha_m y), z\right\| \\
&\le 2\sum_{i=0}^{k} \binom{k}{i} 2^{k-i} \varphi\Big(\beta_m^{k+1-i}\alpha_m^i x, \beta_m^{k+1-i}\alpha_m^i y, z\Big) \\
&\qquad + \sum_{i=0}^{k} \binom{k}{i} 2^{k-i} \varphi\Big(\beta_m^{n-i}\alpha_m^{i+1} x, \beta_m^{n-i}\alpha_m^{i+1} y, z\Big) \\
&= \sum_{i=0}^{k+1} \binom{k+1}{i} 2^{k+1-i} \varphi\Big(\beta_m^{k+1-i}\alpha_m^i x, \beta_m^{k+1-i}\alpha_m^i y, z\Big)
\end{aligned}$$

for all $x, y \in X_0$ such that $x + y \neq 0$ and all $z \in Y_0$. Thus, by induction we have shown that (2.24) holds for every $n \in \mathbb{N}$.

Letting $n \to \infty$ in (2.24) and using (2.14) and (2.15), we obtain

$$2J_m\left(\frac{x+y}{2}\right) = J_m(x) + J_m(y) \quad x, y \in X_0,\ x + y \neq 0,\ m > n_0. \tag{2.25}$$

Since $\lim_{m\to\infty} \varepsilon_m^*(x, z) = 0$, it follows from the inequality in (2.23) that

$$\lim_{m\to\infty} J_m(x) = f(x)$$

for all $x \in X_0$. Therefore we get, with $m \to \infty$, from (2.25) that f is Jensen on X_0. □

Corollary 2.6. *Let $c \geq 0$, $p, q \in \mathbb{R}$ and $f : X \to Y$ satisfy (2.1). Moreover, assume that there exits a positive integer n_0 such that one of the following conditions is satisfied:*

(D_1) $p + q < 0$, $q < 0$ *and for each* $m \geq n_0$,

$$2\left(\frac{m+1}{2}\right)^{p+q} + m^{p+q} < 1,$$

(D_2) $p+q>0$, $q>0$ *and for each* $m \geq n_0$,

$$2\left|\frac{m-1}{2m}\right|^{p+q}+\left(\frac{1}{m}\right)^{p+q}<1,$$

(D_3) $p+q>0$, $q>0$ *and for each* $m \geq n_0$,

$$\frac{1}{2}\left|\frac{m-1}{m}\right|^{p+q}+\left(\frac{1}{m}\right)^{p+q}<1$$

then f *is Jensen on* X_0.

Theorem 2.5 implies the following corollary, which shows its simple application.

Corollary 2.7. *Let* $\varphi : X \times X \times Y_0 \to [0,+\infty)$ *be a function fulfils (2.14) and (2.15). Assume that* $G : X \times X \to Y$ *and* $f : X \to Y$ *satisfy the inequality*

$$\left\|2f\left(\frac{x+y}{2}\right)-f(x)-f(y)-G(x,y),z\right\| \leq \varphi(x,y,z) \tag{2.26}$$

for all $x, y \in X_0$ *and all* $z \in Y_0$. *If the functional equation*

$$2g\left(\frac{x+y}{2}\right)=g(x)+g(y)+G(x,y), \quad x,y \in X \tag{2.27}$$

has a solution $f_0 : X \to Y$, *then* f *is a solution to (2.27).*

Proof. From (2.26) we get that $h := f - f_0$ satisfies (2.16). Consequently, Theorem 2.5 implies that h is Jensen on X_0 which means f is a solution to (2.27). □

References

[1] L. Aiemsomboon, W. Sintunavarat, *On new stability results for generalized Cauchy functional equations on groups by using Brzdęk's fixed point theorem*, J. Fixed Point Theory Appl., **18**(2016), 45-59.

[2] L. Aiemsomboon, W. Sintunavarat, *On generalized hyperstability of a general linear equation*, Acta Math. Hungar., **149**(2016), 413-422.

[3] M. Almahalebi, A. Chahbi, *Hyperstability of the Jensen functional equation in ultrametric spaces*, Aequat. Math., **91**(2017), no. 4, 647-661.

[4] M. Almahalebi, A. Charifi, S. Kabbaj, *Hyperstability of a monomial functional equation*, J. Scientific Research Reports, **3**(2014), no. 20, 2685-2693.

[5] M. Almahalebi, A. Charifi, S. Kabbaj, *Hyperstability of a Cauchy functional equation*, J. Comput. Anal. Appl., **6**(2015), no. 2, 127-137.

[6] M. Almahalebi, S. Kabbaj, *Hyperstability of a Cauchy-Jensen type functional equation*, Advances in Research, **2**(2014), no. 12, 1017-1025.

[7] M. Almahalebi, C. Park, *On the hyperstability of a functional equation in commutative groups*, J. Comput. Anal. Appl., **20**(2016), no. 1, 826-833.

[8] T. Aoki, *On the stability of the linear transformation in Banach spaces*, J. Math. Soc. Japan, **2**(1950), 64-66.

[9] D.G. Bourgin, *Classes of transformations and bordering transformations*, Bull. Amer. Math. Soc., **57**(1951), 223-237.

[10] J. Brzdęk, *A note on stability of additive mappings*, in: Stability of Mappings of Hyers-Ulam Type, Rassias, T.M., Tabor, J. (eds.), Hadronic Press, Palm Harbor, 1994, 19-22.

[11] J. Brzdęk, *Stability of additivity and fixed point methods*, Fixed Point Theory Appl., **2013**, 2013:265, pp.9.

[12] J. Brzdęk, *Hyperstability of the Cauchy equation on restricted domains*, Acta Math. Hungar., **141**(2013), 58-67.
[13] J. Brzdęk, L. Cadăriu, K. Ciepliński, *Fixed point theory and the Ulam stability*, J. Funct. Spaces, **2014**(2014), Article ID 829419, pp. 16.
[14] J. Brzdęk, K. Ciepliński, *A fixed point approach to the stability of functional equations in non-Archimedean metric spaces*, Nonlinear Anal., **74**(2011), 6861-6867.
[15] J. Brzdęk, K. Ciepliński, *On a fixed point theorm in 2-Banach spaces and some of its applications*, Acta Math. Scientia, **38**(2018), 377–390.
[16] J. Brzdęk, W. Fechner, M.S. Moslehian, J. Sikorska, *Recent developments of the conditional stability of the homomorphism equation*, Banach J. Math. Anal., **9**(2015), 278-327.
[17] J. Brzdęk, D. Popa, I. Raşa, B. Xu, *Ulam Stability of Operators*, Mathematical Analysis and its Applications, vol. 1, Academic Press, Elsevier, Oxford 2018.
[18] S. Gähler, *2-metrische Räume und ihre topologische Struktur,* Math. Nachr., **26**(1963), 115-148.
[19] S. Gähler, *Linear 2-normiete Räumen,* Math. Nachr., **28**(1964), 1-43.
[20] D.H. Hyers, *On the stability of the linear functional equation*, Proc. Natl. Acad. Sci. U.S.A., **27**(1941), 222-224.
[21] W.-G. Park, *Approximate additive mappings in 2-Banach spaces and related topics,* J. Math. Anal. Appl., **376**(2011), 193-202.
[22] W. Park, J. Bae, B. Chung, *On an additive-quadratic functional equation and its stability*, J. Appl. Math. Computing, **18**(2005), 563-572.
[23] Th. M. Rassias, *On the stability of the linear mapping in Banach spaces*, Proc. Amer. Math. Soc., **72**(1978), 297-300.
[24] Th. M. Rassias, *Problem 16; 2. Report of the 27th International Symposium on Functional Equations*, Aequationes Math., **39**(1990), 292-293.
[25] Th. M. Rassias, *On a modified Hyers-Ulam sequence*, J. Math. Anal. Appl., **158**(1991), 106-113.
[26] S.M. Ulam, *Problems in Modern Mathematics*, Science Editions, John-Wiley & Sons Inc., New York, 1964.

Received: April 16, 2018; Accepted: October 18, 2018.