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FIXED POINT THEOREMS IN ORDERED METRIC SPACES AND APPLICATIONS TO NONLINEAR BOUNDARY VALUE PROBLEMS

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Abstract. In this paper, we extend the concept of mixed monotone mappings and then we consider certain fixed point theorems for a pair of mappings in metric spaces with a partial ordering. As an application, we study existence of solutions for the following fourth-order two-point boundary value problems for elastic beam equations:

$$\begin{cases} u''''(t) = f(t, u(t), u''(t)), \\ u(0) = A, u'(0) = B, u''(1) = C, u'''(1) = D, \end{cases}$$

where f is a continuous mapping of $[0,1] \times \mathbb{R} \times \mathbb{R}$ into $\mathbb{R}$. Moreover, using these fixed point theorems, we prove several existence results for the solutions of various boundary value problems.
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1. Introduction

A coupled fixed point theorem is a combination between fixed point results for contractive type mappings and the monotone iterative method proposed by Bhaskar and Lakshmikantham [5]. Several authors [1, 3, 4, 7, 9, 15, 18, 19, 22, 23] investigated it. It is a strong tool to study a existence and uniqueness solution of boundary value problems for several ordinary differential equations, see [5, 4, 23, 11]. Recently in [11], Jleli et.al extend and generalize several existing results in the literature [4, 5, 11, 23]. They also show the existence and uniqueness of solutions of the following fourth-order two-point boundary value problem for elastic beam equations:

$$\begin{cases} u''''(t) = f(t, u(t), u(t)), \\ u(0) = u'(0) = u''(1) = u'''(1) = 0, \end{cases}$$

where f is a continuous mapping of $[0,1] \times \mathbb{R} \times \mathbb{R}$ into $\mathbb{R}$.

We are also concerned about higher order boundary value problems. In particular, for the existence of a solution the use of a fixed point theorem is a very popular method. So, for instance, we consider the following problem,

$$\begin{cases} u''''(t) = f(t, u(t), u''(t)), \\ u(0) = u(1) = u''(0) = u''(1) = 0, \end{cases} \tag{1.1}$$

or, for example, the next one (see [11]):

$$\begin{cases} u''''(t) = f(t, u(t), u''(t)), \\ u(0) = u'(0) = u''(1) = u'''(1) = 0, \end{cases} \tag{1.2}$$

where f is a continuous mapping of $[0,1] \times \mathbb{R} \times \mathbb{R}$ into $\mathbb{R}$. We will show that some coupled fixed point theorems are very useful in order to get a solution of these boundary value problems.

For the existence and uniqueness of solutions for the fourth-order two-point boundary value problem for (1.1), many researchers have studied, see [2, 10, 12, 13, 16, 17, 25, 26, 27, 8]. The proof is carried out using the Leray-Schauder fixed point theorem, etc.[2, 8, 10, 12, 13, 16, 17, 25, 26, 27]. Moreover, several authors consider the following boundary value problem, which includes (1.1).

$$\begin{cases} u''''(t) = f(t, u(t), u''(t)), \\ u(0) = A, u(1) = B, u''(0) = C, u''(1) = D. \end{cases} \tag{1.3}$$

Naturally the following boundary value problem, which includes (1.2), can be considerable.

$$\begin{cases} u''''(t) = f(t, u(t), u''(t)), \\ u(0) = A, u'(0) = B, u''(1) = C, u'''(1) = D. \end{cases} \tag{1.4}$$

Recently Petruşel and Petruşel improve mixed monotone property and have a fixed point theorem. Using their method they solve second-order two-point boundary value problems for system of ordinary differential equations, for detail see [20].

In this paper, using the method of coupled fixed point theorem in [5, 4, 7, 15, 11], we show the existence of solutions for (1.4). Our paper is organized as follows. In Section 2, we describe the fixed point theorem in metric spaces endowed with a order. In Section 3, let X be a metric space. And we introduce reverse mixed-monotone property for the mapping of $X \times X$ into X. We consider two mappings of $X \times X$ into X which have mixed-monotone property and reverse mixed-monotone property and we have fixed point theorems (Theorems 3.2, 3.4). In Section 4, we show that our method can be applicable to fourth-order two-point boundary value problems (1.3), (1.4), and typical third-order two-point boundary value problems.

2. Fixed point theorem

First of all, we cited the following definitions and preliminary results will be useful later.

Let (X, d) be a metric space endowed with a partial order $\preceq$. We say that a mapping $F : X \to X$ is nondecreasing if for any $x, y \in X$,

$$x \preceq y \Rightarrow Fx \preceq Fy.$$

Let Φ denote the set of all functions $\varphi : [0, \infty) \to [0, \infty)$ satisfying

(a) φ is continuous and nondecreasing;
(b) $\varphi^{-1}(\{0\}) = \{0\}$.

Let Ψ denote the set of all functions $\psi : [0, \infty) \to [0, \infty)$ satisfying

(c) $\lim_{t \to r+} \psi(t) > 0$ (and finite) for all $r > 0$;
(d) $\lim_{t \to 0+} \psi(t) = 0$.

Let Θ denote the set of all functions $\theta : [0, \infty) \times [0, \infty) \times [0, \infty) \times [0, \infty) \to [0, \infty)$ satisfying

(e) θ is continuous;
(f) $\theta(s1, s2, s3, s4) = 0$ if and only if $s1s2s3s4 = 0$.

Examples of functions ψ of Ψ are given in [15]; see also [4, 21]. Examples of functions θ in Θ are given in [11].

In [11, Theorem 3.1], the following fixed point theorem is obtained.

Theorem 2.1. *Let (X, d) be a complete metric space endowed with a partial order $\preceq$ and $F : X \to X$ a continuous nondecreasing mapping such that there exist $\varphi \in \Phi$, $\psi \in \Psi$ and $\theta \in \Theta$ such that for any $x, y \in X$ with $x \succeq y$,*

$$\begin{aligned} \varphi(d(Fx, Fy)) \le \varphi(d(x, y)) - \psi(d(x, y)) \\ + \theta(d(x, Fx), d(y, Fy), d(x, Fy), d(y, Fx)). \end{aligned} \tag{2.1}$$

Suppose also that there exists $x_0 \in X$ such that $x_0 \preceq Fx_0$ (or $x_0 \succeq Fx_0$). Then F admits a fixed point, that is, there exists $\overline{x} \in X$ such that $\overline{x} = F\overline{x}$.

The previous result is still valid for F which is not necessarily continuous. Instead, we require an additional assumption to the metric space X with a partial order $\preceq$: We say that $(X, d, \preceq)$ is regular if $\{a_n\}$ is a nondecreasing sequence in X with respect to $\preceq$ such that $a_n \to a \in X$ as $n \to \infty$, then $a_n \preceq a$ for all n.

The following theorem is also obtained; see [11, Theorem 3.2].

Theorem 2.2. *Let (X, d) be a complete metric space endowed with a partial order $\preceq$ and $F : X \to X$ a nondecreasing mapping such that there exist $\varphi \in \Phi$, $\psi \in \Psi$ and $\theta \in \Theta$ such that for any $x, y \in X$ with $x \succeq y$, inequality (2.1) is satisfied. Suppose also that $(X, d, \preceq)$ is regular and there exists $x_0 \in X$ such that $x_0 \preceq Fx_0$ (or $x_0 \succeq Fx_0$). Then there exists $\overline{x} \in X$ such that $\overline{x} = F\overline{x}$.*

3. Fixed point theorem for monotone mapping

In this section, for mappings F of $X \times X$ into X, we introduce a monotone property. Moreover we consider fixed point theorems for monotone mappings which have this monotone property. We say that a mapping F of $X \times X$ into X is mixed monotone if F is nondecreasing in its first variable and nonincreasing in its second, that is, for $x, y, u, v \in X$,

$$x \succeq u, y \preceq v \Rightarrow F(u,v) \preceq F(x,y),$$

and a mapping $\widetilde{F}$ of $X \times X$ into X is reverse mixed monotone if $\widetilde{F}$ is nonincreasing in its first variable and nondecreasing in its second, that is, for $x, y, u, v \in X$,

$$x \succeq u, y \preceq v \Rightarrow \widetilde{F}(u,v) \succeq \widetilde{F}(x,y).$$

Let (X, d) be a metric space, Let F and $\widetilde{F}$ be mappings of $X \times X$ into X. We also consider the mapping A of $X \times X$ into $[0, \infty)$ defined by

$$A(x,y) = \frac{d(x, F(x,y)) + d(y, \widetilde{F}(x,y))}{2}, (x,y) \in X \times X,$$

and the mapping B of $X \times X \times X \times X$ into $[0, \infty)$ defined by

$$B(x,y,u,v) = \frac{d(x, F(u,v)) + d(y, \widetilde{F}(u,v))}{2}, (x,y,u,v) \in X \times X \times X \times X.$$

Definition 3.1. Mappings F and $\widetilde{F}$ admit a pre-coupled fixed point, if there exists $(a,b) \in X \times X$ such that $a = F(a,b)$ and $b = \widetilde{F}(a,b)$.

Motivated by [11, Theorem 3.4], we have the following fixed point theorem.

Theorem 3.2. *Let (X, d) be a complete metric space endowed with a partial order $\preceq$, $F : X \times X \to X$ a continuous mixed monotone mapping and $\widetilde{F} : X \times X \to X$ a continuous reverse mixed monotone mapping. We assume that there exist $\varphi \in \Phi$, $\psi \in \Psi$ and $\theta \in \Theta$ such that for any $x, y, u, v \in X$ with $x \succeq u$, $y \preceq v$, the following inequality holds:*

$$\begin{aligned} &\varphi\left(\frac{d(F(x,y), F(u,v)) + d(\widetilde{F}(x,y), \widetilde{F}(u,v))}{2}\right) \\ &\leq \varphi\left(\frac{d(x,u) + d(y,v)}{2}\right) - \psi\left(\frac{d(x,u) + d(y,v)}{2}\right) \\ &+ \theta\left(A(x,y), A(u,v), B(x,y,u,v), B(u,v,x,y)\right). \end{aligned} \tag{3.1}$$

If there exist $x_0, y_0 \in X$ such that

$$x_0 \preceq F(x_0, y_0), y_0 \succeq \widetilde{F}(x_0, y_0), \tag{3.2}$$

or

$$x_0 \succeq F(x_0, y_0), y_0 \preceq \widetilde{F}(x_0, y_0), \tag{3.3}$$

then F and $\widetilde{F}$ admit a pre-coupled fixed point, that is, there exists $(a,b) \in X \times X$ such that $a = F(a,b)$ and $b = \widetilde{F}(a,b)$.

Proof. We consider the product set $Y = X \times X$ endowed with the metric η defined by

$$\eta((x,y),(u,v)) = \frac{d(x,u) + d(y,v)}{2}, (x,y),(u,v) \in Y.$$

Since (X,d) is complete, clearly (Y,η) is also complete. We also consider the partial order $\ll$ in Y defined by

$$(u,v) \ll (x,y) \Leftrightarrow x \succeq u, y \preceq v$$

for any $(x,y),(u,v) \in Y$. We also consider the mapping G of Y into Y defined by

$$G(x,y) = (F(x,y), \widetilde{F}(x,y)), (x,y) \in Y.$$

Since F and $\widetilde{F}$ are continuous, G is also continuous in (Y,η).

Now, we prove that G is nondecreasing with respect to $\ll$. Let $(x,y),(u,v) \in Y$ with $(u,v) \ll (x,y)$, that is, $x \succeq u, y \preceq v$. Since F is mixed monotone and $\widetilde{F}$ is reverse mixed monotone, these imply that $F(x,y) \succeq F(u,v), \widetilde{F}(x,y) \preceq \widetilde{F}(u,v)$, which give us that

$$G(u,v) = (F(u,v), \widetilde{F}(u,v)) \ll G(x,y) = (F(x,y), \widetilde{F}(x,y)).$$

Thus we can prove that G is nondecreasing with respect to $\ll$.

On the other hand, for any $x,y,u,v \in X$, we can write

$$A(x,y) = \eta((x,y), G(x,y)), B(x,y,u,v) = \eta((x,y), G(u,v)).$$

Then, from (3.1), for any $p = (x,y), q = (u,v) \in Y$ with $p \gg q$, we have

$$\varphi\left(\eta(Gp,Gq)\right) \leq \varphi(\eta(p,q)) - \psi(\eta(p,q)) + \theta\left(\eta(p,Gp), \eta(q,Gq), \eta(p,Gq), \eta(q,Gp)\right).$$

Moreover, for $p_0 = (x_0,y_0) \in Y$, from (3.2) and (3.3), we have $p_0 \ll Gp_0$ or $p_0 \gg Gp_0$.

Now G satisfies all the hypotheses of Theorem 2.1, we deduce that G has a fixed point $\overline{x} = (a,b) \in Y$, that is,

$$\overline{x} = (a,b) = G\overline{x} = G(a,b) = (F(a,b), \widetilde{F}(a,b)).$$

It implies that $a = F(a,b), b = \widetilde{F}(a,b)$, that is, F and $\widetilde{F}$ admit a pre-coupled fixed point (a,b). □

The previous result is still valid for F and $\widetilde{F}$ which are not necessarily continuous. Instead, we require additional assumptions to the metric space X with a partial order $\preceq$:

Definition 3.3. Let (X,d) be a complete metric space endowed with a partial order $\preceq$. We say that

(i) $(X,d,\preceq)$ is nondecreasing-regular ($\uparrow$-regular) if a nondecreasing sequence $\{x_n\} \subset X$ converges to x, then $x_n \preceq x$ for all n;

(ii) $(X,d,\preceq)$ is nonincreasing-regular ($\downarrow$-regular) if a nonincreasing sequence $\{x_n\} \subset X$ converges to x, then $x_n \succeq x$ for all n.

Motivated by [11, Theorem 3.5], we have the following result.

Theorem 3.4. *Let (X,d) be a complete metric space endowed with a partial order $\preceq$, $F : X \times X \to X$ a mixed monotone mapping, and $\widetilde{F} : X \times X \to X$ a reverse mixed monotone mapping. We assume that there exist $\varphi \in \Phi$, $\psi \in \Psi$ and $\theta \in \Theta$ such that for any $x,y,u,v \in X$ with $x \succeq u$, $y \preceq v$, inequality (3.1) holds. We also assume that $(X,d,\preceq)$ is nondecreasing-regular and nonincreasing-regular ($\uparrow\downarrow$-regular), and there* *exist* $x_0, y_0 \in X$ *such that (3.2) or (3.3) hold. Then* F *and* $\widetilde{F}$ *admit a pre-coupled fixed point, that is, there exists* $(a,b) \in X \times X$ *such that* $a = F(a,b)$ *and* $b = \widetilde{F}(a,b)$.

Proof. It is sufficient to show that if $(X, d, \preceq)$ is nondecreasing-regular and nonincreasing-regular ($\uparrow\downarrow$-regular), then $(Y, \eta, \ll)$ is regular. The proof of this claim follows immediately from Theorem 2.2. In detail, also see the proof of [5, Theorem 2.2]. □

4. Applications

In this section, we study the existence of solutions of two types fourth-order two-point boundary value problems for elastic beam equations and two types third-order two-point boundary value problems. In particular, a theorem in the subsection 4.1 (Type I) is an extension of the result in [11].

4.1. **Type I.** First of all, we study the existence of solutions of the following fourth-order two-point boundary value problem for elastic beam equations:

$$\begin{cases} u''''(t) = f(t, u(t), u''(t)), \\ u(0) = A,\ u'(0) = B,\ u''(1) = C,\ u'''(1) = D, \end{cases} \tag{4.1}$$

with $I = [0,1]$ and $f \in C(I \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$, where $C(I \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$ is a set of continuous mappings of $I \times \mathbb{R} \times \mathbb{R}$ into $\mathbb{R}$. Let Ω be a set of functions ω of $[0,\infty)$ into $[0,\infty)$ satisfying

(i) ω is nondecreasing;
(ii) there exists $\psi \in \Psi$ such that $\omega(r) = \frac{r}{2} - \psi(\frac{r}{2})$ for all $r \in [0,\infty)$.

For examples of such functions, see [15].

Next we consider the following assumptions $(A1)$ and $(A2)$.

(A1) There exists $\omega \in \Omega$ such that for all $t \in I$ and for all $a, b, c, e \in \mathbb{R}$, with $a \geq c$ and $b \leq e$,

$$0 \leq f(t,a,b) - f(t,c,e) \leq \omega(a-c) + \omega(e-b). \tag{4.2}$$

(A2)There exist $\alpha, \beta \in C(I, \mathbb{R})$ which are solutions of

$$\alpha(t) \leq Bt + A - \int_0^1 H_2(t,s)(C - D + Ds)ds + \int_0^1 G(t,s) f(s, \alpha(s), \beta(s))ds, t \in I, \tag{4.3}$$

and

$$\beta(t) \geq -\left(C - D + Dt + \int_0^1 H_1(t,s) f(s, \alpha(s), \beta(s))ds\right), t \in I, \tag{4.4}$$

where the Green functions G and H_1 are defined by

$$G(t,s) = \begin{cases} \dfrac{1}{6}s^2(3t - s), & (0 \leq s \leq t \leq 1), \\ \dfrac{1}{6}t^2(3s - t), & (0 \leq t \leq s \leq 1), \end{cases}$$

and

$$H_1(t,s) = \begin{cases} 0, & (0 \le s \le t \le 1), \\ s-t, & (0 \le t \le s \le 1). \end{cases}$$

Note that

$$\int_0^1 G(t,s) f(s,u(s),v(s))ds = \int_0^1 H_2(t,s) \int_0^1 H_1(s,r) f(r,u(r),v(r))drds,$$

where the green function H_2 is defined by

$$H_2(t,s) = \begin{cases} t-s, & (0 \le s \le t \le 1), \\ 0, & (0 \le t \le s \le 1). \end{cases}$$

It is easy to show that

$$0 \le G(t,s) \le \frac{1}{2} t^2 s \text{ for all } t,s \in I, \tag{4.5}$$

and

$$0 \le H_1(t,s) \le \min\{s,t\} \text{ for all } t,s \in I. \tag{4.6}$$

Now we have the following theorem.

Theorem 4.1. *Under the assumptions* $(A1)$ *and* $(A2)$, *the fourth-order two-point boundary value problem (4.1) has a solution.*

Proof. Consider the natural partial order relation $\preceq$ on $X = C(I,R)$, that is,

$$u,v \in X, u \preceq v \Leftrightarrow u(t) \le v(t) \text{ for all } t \in I.$$

It is well known that X is a complete metric space with respect to the metric

$$d(u,v) = \max_{t \in I} |u(t) - v(t)| :=\| u-v \|_\infty, u,v \in C(I,\mathbb{R}).$$

It is easy to show that $(X,d,\preceq)$ is nondecreasing-regular and nonincreasing-regular ($\uparrow\downarrow$-regular), and that every pair of elements in $X \times X$ has either a lower bound or an upper bound. Solving problem (4.1) is equivalent to finding $u \in C(I,\mathbb{R})$ which is a solution of

$$\begin{aligned} u(t) = Bt + A - &\int_0^1 H_2(t,s)(C-D+Ds)ds \\ &+ \int_0^1 G(t,s) f(s,u(s),v(s))ds, t \in I, \end{aligned}$$

where $v = u''$. Moreover the boundary value problem (4.1) can be written as

$$\begin{cases} u''(t) = v(t), \\ v''(t) = f(t,u(t),v(t)), \\ u(0) = A, u'(0) = B, v(1) = C, v'(1) = D, \end{cases}$$

and it is equivalent to the following integral equations,

$$\begin{cases} u(t) = Bt + A - \int_0^1 H_2(t,s)v(s)ds \\ \qquad = Bt + A - \int_0^1 H_2(t,s)(C - D + Ds)ds \\ \qquad\quad + \int_0^1 G(t,s)f(s,u(s),v(s))ds, t \in I, \\ v(t) = v(t) = C - D + Dt + \int_0^1 H_1(t,s)f(s,u(s),v(s))ds, t \in I. \end{cases}$$

Let F and $\widetilde{F}$ be mappings of $C(I,\mathbb{R}) \times C(I,\mathbb{R})$ into $C(I,\mathbb{R})$ defined by

$$F(x,y)(t) = Bt + A - \int_0^1 H_2(t,s)(C - D + Ds)ds$$
$$+ \int_0^1 G(t,s)f(s,x(s),y(s))ds, t \in I, x,y \in C(I,\mathbb{R}),$$

and

$$\tilde{F}(x,y)(t) = -\left(C - D + Dt + \int_0^1 H_1(t,s)f(s,x(s),y(s))ds\right), t \in I, x,y \in C(I,\mathbb{R}).$$

By the assumption $(A1)$, we can show that the mapping F is mixed monotone and the mapping $\widetilde{F}$ is reverse mixed monotone. In fact, for all $t \in I$ and for all $x,y,u,v \in C(I,\mathbb{R})$ with $x \succeq u$ and $y \preceq v$, we have

$$0 \leq f(t,x(t),y(t)) - f(t,u(t),v(t)).$$

Thus we have

$$F(x,y)(t) - F(u,v)(t) = \int_0^1 G(t,s)(f(s,x(s),y(s)) - f(s,u(s),v(s)))ds \geq 0,$$

and

$$\widetilde{F}(x,y)(t) - \widetilde{F}(u,v)(t) = -\int_0^1 H_1(t,s)(f(s,x(s),y(s)) - f(s,u(s),v(s)))ds \leq 0.$$

Again, since ω is nondecreasing and from (4.2) and (4.5), we have

$$\begin{aligned} & F(x,y)(t) - F(u,v)(t) \\ &= \int_0^1 G(t,s)(f(s,x(s),y(s)) - f(s,u(s),v(s)))ds \\ &\leq \int_0^1 G(t,s)(\omega(x(s) - u(s))ds + \int_0^1 G(t,s)\omega(v(s) - y(s))ds \\ &\leq \int_0^1 G(t,s)ds(\omega(\| x - u \|_\infty) + \omega(\| v - y \|_\infty)) \\ &\leq \frac{\omega(\| x - u \|_\infty) + \omega(\| v - y \|_\infty)}{4} \end{aligned} \tag{4.7}$$

for all $t \in I$ and for all $x, y, u, v \in C(I, \mathbb{R})$ with $x \succeq u$ and $y \preceq v$. Also from (4.2) and (4.6), we have

$$\begin{aligned} & |\widetilde{F}(x,y)(t) - \widetilde{F}(u,v)(t)| \\ \leq & \int_0^1 H_1(t,s)\,|f(s,x(s),y(s)) - f(s,u(s),v(s))|\,ds \\ \leq & \frac{\omega\left(\| x-u \|_\infty\right) + \omega\left(\| v-y \|_\infty\right)}{2} \end{aligned} \tag{4.8}$$

for all $t \in I$ and for all $x, y, u, v \in C(I, \mathbb{R})$ with $x \succeq u$ and $y \preceq v$. By (4.7) and (4.8), we get

$$\frac{d\left(F(x,y),F(u,v)\right) + d\left(\widetilde{F}(x,y),\widetilde{F}(u,v)\right)}{2} \leq \frac{3\left(\omega\left(\|x-u\|_\infty\right) + \omega\left(\|v-y\|_\infty\right)\right)}{8}.$$

On the other hand, since ω is nondecreasing and condition (ii) for Ω, we have

$$\begin{aligned} \frac{3\left(\omega(\|x-u\|_\infty) + \omega(\|v-y\|_\infty)\right)}{8} &\leq \omega(\|x-u\|_\infty + \|v-y\|_\infty) \\ &= \frac{d(x,u)+d(y,v)}{2} - \psi\left(\frac{d(x,u)+d(y,v)}{2}\right). \end{aligned}$$

Thus we finally get

$$\begin{aligned} \eta((F(x,y),\widetilde{F}(x,y)),(F(u,v),\widetilde{F}(u,v))) &= \frac{d\left(F(x,y),F(u,v)\right) + d\left(\widetilde{F}(x,y),\widetilde{F}(u,v)\right)}{2} \\ &\leq \frac{d(x,u)+d(y,v)}{2} - \psi\left(\frac{d(x,u)+d(y,v)}{2}\right). \end{aligned}$$

Now, let $\alpha, \beta \in C(I, \mathbb{R})$ be solutions of (4.3) and (4.4). By the assumption $(A2)$, we have $\alpha \preceq F(\alpha, \beta)$ and $\beta \succeq \widetilde{F}(\alpha, \beta)$. We also take $\varphi(t) = t$ for any $t \in [0, \infty)$ and $\theta \equiv 0$. Thus all the hypotheses of Theorem 3.4 are satisfied. Therefore $u, v \in C(I, \mathbb{R})$ are solution of the problem $F(u, v) = u$ and $\widetilde{F}(u, v) = v$. These prove that $u \in C(I, \mathbb{R})$ is a solution of (4.1). □

4.2. **Type II.** Next as an application of our results, we study the existence of solutions of the following fourth-order two-point boundary value problem, see [2, 25, 27]:

$$\begin{cases} u''''(t) = f(t, u(t), u''(t)), \\ u(0) = A, u(1) = B, u''(0) = C, u''(1) = D, \end{cases} \tag{4.9}$$

with $I = [0, 1]$ and $f \in C(I \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$. We take the set of functions Ω same way as in Type I.

The assumptions $(A1)$ and $(A2)$ are same as those of Type I with respect to the following Green functions G and H.

$$G(t,s) = \begin{cases} \dfrac{1}{6}s(1-t)(2t - s^2 - t^2), & (0 \leq s \leq t \leq 1), \\ \dfrac{1}{6}t(1-s)(2s - t^2 - s^2), & (0 \leq t \leq s \leq 1), \end{cases}$$

and

$$H(t,s) = \begin{cases} s(t-1) & (0 \le s \le t \le 1), \\ t(s-1) & (0 \le t \le s \le 1). \end{cases}$$

Note that

$$\int_0^1 G(t,s)f(s,u(s),v(s))ds = \int_0^1 H(t,s)\int_0^1 H(s,r)f(r,u(r),v(r))drds, t \in I.$$

It is easy to show that

$$0 \le G(t,s) \le \frac{1}{3}st \text{ for all } t,s \in I, \tag{4.10}$$

and

$$0 \le -H(t,s) \le \min\{s,t\} \text{ for all } t,s \in I. \tag{4.11}$$

Theorem 4.2. *Under the assumptions* $(A1)$ *and* $(A2)$, *the fourth-order two-point boundary value problem (4.9) has a solution.*

Proof. From the same argument in Theorem 4.1, we consider the natural partial order relation $\preceq$ on $X = C(I,R)$, that is,

$$u,v \in X, u \preceq v \Leftrightarrow u(t) \le v(t) \text{ for all } t \in I.$$

It is well known that X is a complete metric space with respect to the metric

$$d(u,v) = \max_{t \in I} |u(t) - v(t)| := \| u - v \|_\infty, u,v \in C(I,R).$$

It is easy to show that $(X,d,\preceq)$ is nondecreasing-regular and nonincreasing-regular ($\uparrow\downarrow$-regular), and that every pair of elements in $X \times X$ has either a lower bound or an upper bound. Solving problem (4.9) is equivalent to finding $u \in C(I,\mathbb{R})$ which is a solution of

$$\begin{aligned} u(t) = (B-A)t + A &+ \int_0^1 H(t,s)((D-C)s + C)ds \\ &+ \int_0^1 G(t,s)f(s,u(s),v(s))ds, t \in I, \end{aligned}$$

where $v = u''$. Moreover equation (4.9) can be written as

$$\begin{cases} u''(t) = v(t), \\ v''(t) = f(t,u(t),v(t)), \\ u(0) = A, u(1) = B, v(0) = C, v(1) = D, \end{cases}$$

and it is also equivalent to the following integral equations,

$$\begin{cases} u(t) = (B-A)t + A - \int_0^1 H(t,s)v(s)ds \\ \quad = (B-A)t + A - \int_0^1 H(t,s)((D-C)s+C)ds \\ \quad\quad + \int_0^1 G(t,s)f(s,u(s),v(s))ds, t \in I, \\ v(t) = (D-C)t + C - \int_0^1 H(t,s)f(s,u(s),v(s))ds, t \in I. \end{cases}$$

Let F and $\widetilde{F}$ be mappings of $C(I,\mathbb{R}) \times C(I,\mathbb{R})$ into $C(I,\mathbb{R})$ defined by

$$F(x,y)(t) = (B-A)t + A - \int_0^1 H(t,s)((D-C)s+C)ds \\ + \int_0^1 G(t,s)f(s,x(s),y(s))ds, t \in I, x, y \in C(I,\mathbb{R}),$$

and

$$\widetilde{F}(x,y)(t) = (D-C)t + C - \int_0^1 H(t,s)f(s,x(s),y(s))ds, t \in I, x, y \in C(I,\mathbb{R}).$$

By the assumption (A1), we can show that the mapping F is mixed monotone and the mapping $\widetilde{F}$ is reverse mixed monotone. In fact, for all $t \in I$ and for all $x, y, u, v \in C(I,\mathbb{R})$, with $x \succeq u$ and $y \preceq v$,

$$0 \leq f(t,x(t),y(t)) - f(t,u(t),v(t)).$$

Thus we have

$$F(x,y)(t) - F(u,v)(t) = \int_0^1 G(t,s)(f(s,x(s),y(s)) - f(s,u(s),v(s)))ds \geq 0$$

and

$$\widetilde{F}(x,y)(t) - \widetilde{F}(u,v)(t) = -\int_0^1 H(t,s)(f(s,x(s),y(s)) - f(s,u(s),v(s)))ds \leq 0.$$

The remaining parts of the proof are same as that of Type I using properties (4.10), (4.11), and the assumption (A2). □

4.3. **Type III.** In this subsection, we consider the solutions of the following third-order two-point boundary value problem, see [6, 24, 28]:

$$\begin{cases} u'''(t) = f(t,u(t),u''(t)), \\ u(0) = A, u(1) = B, u''(0) = C, \end{cases} \tag{4.12}$$

with $I = [0,1]$ and $f \in C(I \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$. The assumptions $(A1)$ and $(A2)$ are same as that of Type I using the following Green functions G and H.

$$G(t,s) = \begin{cases} \dfrac{-(t-s)^2 + t(1-s)^2}{2}, & (0 \leq s \leq t \leq 1), \\ \dfrac{t(1-s)^2}{2}, & (0 \leq t \leq s \leq 1), \end{cases}$$

and

$$H(t,s) = \begin{cases} (1-t)s, & (0 \leq s \leq t \leq 1), \\ (1-s)t, & (0 \leq t \leq s \leq 1). \end{cases}$$

Note that

$$\int_0^1 G(t,s) f(s, u(s), v(s)) ds = \int_0^1 H(t,s) \int_0^s f(r, u(r), v(r)) dr ds.$$

It is easy to show that

$$0 \leq G(t,s) \leq \frac{1}{2} t(1-s)^2 \text{ for all } t, s \in I, \tag{4.13}$$

and

$$0 \leq H(t,s) \leq \min\{s(1-t), t(1-s)\} \text{ for all } t, s \in I. \tag{4.14}$$

Theorem 4.3. *Under the assumptions* $(A1)$ *and* $(A2)$*, the third-order two-point boundary value problem (4.12) has a solution.*

Proof. From the same argument in Theorem 4.1, we consider the natural partial order relation $\preceq$ in $X = C(I, R)$. Then $(X, d, \preceq)$ is complete, nondecreasing-regular, nonincreasing-regular ($\uparrow\downarrow$-regular), and every pair of elements in $X \times X$ has either a lower bound or an upper bound. Solving problem (4.12) is equivalent to finding $u \in C(I, \mathbb{R})$ which is a solution of

$$u(t) = (B - A)t + A - \int_0^1 H(t,s) C ds + \int_0^1 G(t,s) f(s, u(s), v(s)) ds, t \in I,$$

where $v = u''$. Moreover the boundary value problem (4.12) can be written as follows

$$\begin{cases} u''(t) = v(t), \\ v'(t) = -f(t, u(t), v(t)), \\ u(0) = A, u(0) = B, v(0) = C, \end{cases}$$

and it is also equivalent to the following,

$$\begin{cases} u(t) = (B-A)t + A - \int_0^1 H(t,s)v(s)ds \\ \quad\;\; = (B-A)t + A - \int_0^1 H(t,s)Cds \\ \qquad + \int_0^1 G(t,s)f(s,u(s),v(s))ds, t \in I, \\ v(t) = C - \int_0^t f(s,u(s),v(s))ds, t \in I. \end{cases}$$

Let F and $\widetilde{F}$ be mappings of $C(I,\mathbb{R}) \times C(I,\mathbb{R})$ into $C(I,\mathbb{R})$ defined by

$$F(x,y)(t) = (B-A)t + A - \int_0^1 H(t,s)Cds \\ + \int_0^1 G(t,s)f(s,x(s),y(s))ds, t \in I, x, y \in C(I,\mathbb{R}),$$

and

$$\widetilde{F}(x,y)(t) = C - \int_0^t f(s,x(s),y(s))ds, t \in I, x, y \in C(I,\mathbb{R}).$$

By the assumption (A1), we can show that the mapping F is mixed monotone and the mapping $\widetilde{F}$ is reverse mixed monotone. The remaining parts of the proof are same as that of Type I using properties (4.13), (4.14), and the assumption (A2). □

4.4. **Type IV.** Finally we consider the solutions of the following third-order two-point boundary value problem, see [14]:

$$\begin{cases} u'''(t) = f(t,u(t),u''(t)), \\ u(0) = A, u(1) = B, u''(1) = C, \end{cases} \tag{4.15}$$

with $I = [0,1]$ and $f \in C(I \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$. The assumptions $(A1)$ and $(A2)$ are same as those of Type I using the following Green functions G and H.

$$G(t,s) = \begin{cases} \frac{1}{2}s^2(1-t), & (0 \le s \le t \le 1), \\ \frac{1}{2}t((1-t)-(1-s)^2), & (0 \le t \le s \le 1), \end{cases}$$

and

$$H(t,s) = \begin{cases} (1-t)s & (0 \le s \le t \le 1), \\ (1-s)t & (0 \le t \le s \le 1). \end{cases}$$

Note that

$$\int_0^1 G(t,s)f(s,u(s),v(s))ds = \int_0^1 H(t,s)\int_s^1 f(r,u(r),v(r))drds.$$

It is easy to show that

$$0 \le G(t,s) \le \frac{1}{2}s^2(1-t) \text{ for all } t, s \in I, \tag{4.16}$$

and

$$0 \le H(t,s) \le \min\{s(1-t), t(1-s)\} \text{ for all } t, s \in I. \tag{4.17}$$

Theorem 4.4. *Under the assumptions* $(A1)$ *and* $(A2)$*, the third-order two-point boundary value problem (4.15) has a solution.*

Proof. From the same argument in Theorem 4.1, we consider the natural partial order relation $\preceq$ on $X = C(I, R)$. Then $(X, d, \preceq)$ is complete, nondecreasing-regular and nonincreasing-regular ($\uparrow\downarrow$-regular), and every pair of elements in $X \times X$ has either a lower bound or an upper bound. Solving problem (4.15) is equivalent to finding $u \in C(I, \mathbb{R})$ which is a solution of

$$u(t) = (B-A)t + A - \int_0^1 H(t,s)Cds + \int_0^1 G(t,s)f(s,u(s),v(s))ds, t \in I,$$

where $v = u''$. Moreover the boundary value problem (4.15) can be written as

$$\begin{cases} u''(t) = v(t), \\ v'(t) = f(t, u(t), v(t)), \\ u(0) = u(1) = v(1) = 0, \end{cases}$$

and it is also equivalent to the following integral equations,

$$\begin{cases} u(t) = (B-A)t + A - \displaystyle\int_0^1 H(t,s)v(s)ds, \\ \qquad = (B-A)t + A - \displaystyle\int_0^1 H(t,s)Cds \\ \qquad\quad + \displaystyle\int_0^1 G(t,s)f(s,u(s),v(s))ds, t \in I, \\ v(t) = C - \displaystyle\int_t^1 f(s,u(s),v(s))ds, t \in I. \end{cases}$$

Let F and $\widetilde{F}$ be mappings of $C(I, \mathbb{R}) \times C(I, \mathbb{R})$ into $C(I, \mathbb{R})$ defined by

$$F(x,y)(t) = (B-A)t + A + \int_0^1 H(t,s)Cds + \int_0^1 G(t,s)f(s,x(s),y(s))ds, t \in I, x, y \in C(I, \mathbb{R}),$$

and

$$\widetilde{F}(x,y)(t) = C - \int_t^1 f(s,u(s),v(s))ds, t \in I, x, y \in C(I, \mathbb{R}).$$

By the assumption (A1), we can show that the mapping F is mixed monotone and the mapping $\widetilde{F}$ is reverse mixed monotone. The remaining parts of the proof are same as that of Type I using properties (4.16), (4.17), and the assumption (A2). $\square$

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