FIXED POINT THEOREMS IN ORDERED METRIC SPACES AND APPLICATIONS TO NONLINEAR BOUNDARY VALUE PROBLEMS

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Abstract. In this paper, we extend the concept of mixed monotone mappings and then we consider certain fixed point theorems for a pair of mappings in metric spaces with a partial ordering. As an application, we study existence of solutions for the following fourth-order two-point boundary value problems for elastic beam equations:

\[
\begin{align*}
&u^{(4)}(t) = f(t, u(t), u''(t)), \\
u(0) = A, u'(0) = B, u''(1) = C, u'''(1) = D,
\end{align*}
\]

where \( f \) is a continuous mapping of \([0, 1] \times \mathbb{R} \times \mathbb{R} \) into \( \mathbb{R} \). Moreover, using these fixed point theorems, we prove several existence results for the solutions of various boundary value problems.

Key Words and Phrases: Fixed point theorem, partially ordered set, boundary value problem, differential equation.

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1. Introduction

A coupled fixed point theorem is a combination between fixed point results for contractive type mappings and the monotone iterative method proposed by Bhaskar and Lakshmikantham [5]. Several authors [1, 3, 4, 7, 9, 15, 18, 19, 22, 23] investigated it. It is a strong tool to study a existence and uniqueness solution of boundary value problems for several ordinary differential equations, see [5, 4, 23, 11]. Recently in [11], Jleli et.al extend and generalize several existing results in the literature [4, 5, 11, 23]. They also show the existence and uniqueness of solutions of the following fourth-order two-point boundary value problem for elastic beam equations:

\[
\begin{align*}
&u^{(4)}(t) = f(t, u(t), u'(t)), \\
u(0) = u'(0) = u''(1) = u'''(1) = 0,
\end{align*}
\]

where \( f \) is a continuous mapping of \([0, 1] \times \mathbb{R} \times \mathbb{R} \) into \( \mathbb{R} \).
We are also concerned about higher order boundary value problems. In particular, for the existence of a solution the use of a fixed point theorem is a very popular method. So, for instance, we consider the following problem,

\[
\begin{align*}
  u^{(4)}(t) &= f(t, u(t), u''(t)), \\
  u(0) &= u(1) = u''(0) = u''(1) = 0,
\end{align*}
\]  

(1.1)

or, for example, the next one (see [11]):

\[
\begin{align*}
  u^{(4)}(t) &= f(t, u(t), u''(t)), \\
  u(0) &= u'(0) = u''(0) = u''(1) = 0,
\end{align*}
\]  

(1.2)

where \( f \) is a continuous mapping of \([0, 1] \times \mathbb{R} \times \mathbb{R} \) into \( \mathbb{R} \). We will show that some coupled fixed point theorems are very useful in order to get a solution of these boundary value problems.

For the existence and uniqueness of solutions for the fourth-order two-point boundary value problem for (1.1), many researchers have studied, see [2, 10, 12, 13, 16, 17, 25, 26, 27, 8]. The proof is carried out using the Leray-Schauder fixed point theorem, etc. [2, 8, 10, 12, 13, 16, 17, 25, 26, 27]. Moreover, several authors consider the following boundary value problem, which includes (1.1).

\[
\begin{align*}
  u^{(4)}(t) &= f(t, u(t), u''(t)), \\
  u(0) &= A, u(1) = B, u''(0) = C, u''(1) = D.
\end{align*}
\]  

(1.3)

Naturally the following boundary value problem, which includes (1.2), can be considerable.

\[
\begin{align*}
  u^{(4)}(t) &= f(t, u(t), u''(t)), \\
  u(0) &= A, u'(0) = B, u''(1) = C, u'''(1) = D.
\end{align*}
\]  

(1.4)

Recently Petruşel and Petruşel improve mixed monotone property and have a fixed point theorem. Using their method they solve second-order two-point boundary value problems for system of ordinary differential equations, for detail see [20].

In this paper, using the method of coupled fixed point theorem in [5, 4, 7, 15, 11], we show the existence of solutions for (1.4). Our paper is organized as follows. In Section 2, we describe the fixed point theorem in metric spaces endowed with a order. In Section 3, let \( X \) be a metric space. And we introduce reverse mixed-monotone property for the mapping of \( X \times X \) into \( X \). We consider two mappings of \( X \times X \) into \( X \) which have mixed-monotone property and reverse mixed-monotone property and we have fixed point theorems (Theorems 3.2, 3.4). In Section 4, we show that our method can be applicable to fourth-order two-point boundary value problems (1.3), (1.4), and typical third-order two-point boundary value problems.

## 2. Fixed point theorem

First of all, we cited the following definitions and preliminary results will be useful later.
Let \((X,d)\) be a metric space endowed with a partial order \(\preceq\). We say that a mapping \(F : X \to X\) is nondecreasing if for any \(x, y \in X\),
\[
x \preceq y \Rightarrow Fx \preceq Fy.
\]
Let \(\Phi\) denote the set of all functions \(\varphi : [0, \infty) \to [0, \infty)\) satisfying
(a) \(\varphi\) is continuous and nondecreasing;
(b) \(\varphi^{-1}(\{0\}) = \{0\}\).
Let \(\Psi\) denote the set of all functions \(\psi : [0, \infty) \to [0, \infty)\) satisfying
(c) \(\lim_{t \to r^+} \psi(t) > 0\) (and finite) for all \(r > 0\);
(d) \(\lim_{t \to 0^+} \psi(t) = 0\).
Let \(\Theta\) denote the set of all functions \(\theta : [0, \infty) \times [0, \infty) \times [0, \infty) \times [0, \infty) \to [0, \infty)\) satisfying
(e) \(\theta\) is continuous;
(f) \(\theta(s_1, s_2, s_3, s_4) = 0\) if and only if \(s_1 s_2 s_3 s_4 = 0\).

Examples of functions \(\psi\) of \(\Psi\) are given in [15]; see also [4, 21]. Examples of functions \(\theta\) in \(\Theta\) are given in [11].

In [11, Theorem 3.1], the following fixed point theorem is obtained.

**Theorem 2.1.** Let \((X,d)\) be a complete metric space endowed with a partial order \(\preceq\) and \(F : X \to X\) a continuous nondecreasing mapping such that there exist \(\varphi \in \Phi\), \(\psi \in \Psi\) and \(\theta \in \Theta\) such that for any \(x, y \in X\) with \(x \preceq y\),
\[
\varphi(d(Fx, Fy)) \leq \varphi(d(x, y)) - \psi(d(x, y)) + \theta(d(x, Fx), d(y, Fy), d(x, Fy), d(y, Fx)).
\]
(2.1)
Suppose also that there exists \(x_0 \in X\) such that \(x_0 \preceq Fx_0\) (or \(x_0 \preceq Fx_0\)). Then \(F\) admits a fixed point, that is, there exists \(x \in X\) such that \(x = Fx\).

The previous result is still valid for \(F\) which is not necessarily continuous. Instead, we require an additional assumption to the metric space \(X\) with a partial order \(\preceq\):
We say that \((X,d, \preceq)\) is regular if \(\{a_n\}\) is a nondecreasing sequence in \(X\) with respect to \(\preceq\) such that \(a_n \to a \in X\) as \(n \to \infty\), then \(a_n \preceq a\) for all \(n\).

The following theorem is also obtained; see [11, Theorem 3.2].

**Theorem 2.2.** Let \((X,d)\) be a complete metric space endowed with a partial order \(\preceq\) and \(F : X \to X\) a nondecreasing mapping such that there exist \(\varphi \in \Phi\), \(\psi \in \Psi\) and \(\theta \in \Theta\) such that for any \(x, y \in X\) with \(x \preceq y\), inequality (2.1) is satisfied. Suppose also that \((X,d, \preceq)\) is regular and there exists \(x_0 \in X\) such that \(x_0 \preceq Fx_0\) (or \(x_0 \preceq Fx_0\)). Then there exists \(x \in X\) such that \(x = Fx\).

### 3. Fixed point theorem for monotone mapping

In this section, for mappings \(F\) of \(X \times X\) into \(X\), we introduce a monotone property. Moreover we consider fixed point theorems for monotone mappings which have this monotone property. We say that a mapping \(F\) of \(X \times X\) into \(X\) is mixed monotone
if $F$ is nondecreasing in its first variable and nonincreasing in its second, that is, for $x, y, u, v \in X$,

$$x \preceq u, y \succeq v \Rightarrow F(u, v) \preceq F(x, y),$$

and a mapping $\widetilde{F}$ of $X \times X$ into $X$ is reverse mixed monotone if $\widetilde{F}$ is nonincreasing in its first variable and nondecreasing in its second, that is, for $x, y, u, v \in X$,

$$x \preceq u, y \succeq v \Rightarrow \widetilde{F}(u, v) \succeq \widetilde{F}(x, y).$$

Let $(X, d)$ be a metric space, Let $F$ and $\widetilde{F}$ be mappings of $X \times X$ into $X$. We also consider the mapping $A$ of $X \times X$ into $[0, \infty)$ defined by

$$A(x, y) = \frac{d(x, F(x, y)) + d(y, \widetilde{F}(x, y))}{2}, (x, y) \in X \times X,$$

and the mapping $B$ of $X \times X \times X \times X$ into $[0, \infty)$ defined by

$$B(x, y, u, v) = \frac{d(x, F(u, v)) + d(y, \widetilde{F}(u, v))}{2}, (x, y, u, v) \in X \times X \times X \times X.$$

Definition 3.1. Mappings $F$ and $\widetilde{F}$ admit a pre-coupled fixed point, if there exists $(a, b) \in X \times X$ such that $a = F(a, b)$ and $b = \widetilde{F}(a, b)$.

Motivated by [11, Theorem 3.4], we have the following fixed point theorem.

Theorem 3.2. Let $(X, d)$ be a complete metric space endowed with a partial order $\preceq$, $F : X \times X \to X$ a continuous mixed monotone mapping and $\widetilde{F} : X \times X \to X$ a continuous reverse mixed monotone mapping. We assume that there exist $\varphi \in \Phi$, $\psi \in \Psi$ and $\theta \in \Theta$ such that for any $x, y, u, v \in X$ with $x \preceq u, y \succeq v$, the following inequality holds:

$$\varphi \left( \frac{d(F(x, y), F(u, v)) + d(\widetilde{F}(x, y), \widetilde{F}(u, v))}{2} \right)$$

$$\leq \varphi \left( \frac{d(x, u) + d(y, v)}{2} \right) - \psi \left( \frac{d(x, u) + d(y, v)}{2} \right)$$

$$\quad + \theta (A(x, y), A(u, v), B(x, y, u, v), B(u, v, x, y)).$$

(3.1)

If there exist $x_0, y_0 \in X$ such that

$$x_0 \preceq F(x_0, y_0), y_0 \succeq \widetilde{F}(x_0, y_0),$$

(3.2)

or

$$x_0 \succeq F(x_0, y_0), y_0 \preceq \widetilde{F}(x_0, y_0),$$

(3.3)

then $F$ and $\widetilde{F}$ admit a pre-coupled fixed point, that is, there exists $(a, b) \in X \times X$ such that $a = F(a, b)$ and $b = \widetilde{F}(a, b)$.

Proof. We consider the product set $Y = X \times X$ endowed with the metric $\eta$ defined by

$$\eta((x, y), (u, v)) = \frac{d(x, u) + d(y, v)}{2}, (x, y), (u, v) \in Y.$$
Since \((X,d)\) is complete, clearly \((Y,\eta)\) is also complete. We also consider the partial order \(\preccurlyeq\) in \(Y\) defined by
\[
(u,v) \preccurlyeq (x,y) \iff x \geq u, y \leq v
\]
for any \((x,y),(u,v)\) \(\in Y\). We also consider the mapping \(G\) of \(Y\) into \(Y\) defined by
\[
G(x,y) = (F(x,y), \tilde{F}(x,y)), (x,y) \in Y.
\]
Since \(F\) and \(\tilde{F}\) are continuous, \(G\) is also continuous in \((Y,\eta)\).

Now, we prove that \(G\) is nondecreasing with respect to \(\preccurlyeq\). Let \((x,y),(u,v)\) \(\in Y\) with \((u,v) \preccurlyeq (x,y)\), that is, \(x \geq u, y \leq v\). Since \(F\) is mixed monotone and \(\tilde{F}\) is reverse mixed monotone, these imply that \(F(x,y) \succeq F(u,v), \tilde{F}(x,y) \preceq \tilde{F}(u,v)\), which give us that
\[
G(u,v) = (F(u,v), \tilde{F}(u,v)) \preccurlyeq G(x,y) = (F(x,y), \tilde{F}(x,y)).
\]

Thus we can prove that \(G\) is nondecreasing with respect to \(\preccurlyeq\).

On the other hand, for any \(x,y,u,v \in X\), we can write
\[
A(x,y) = \eta((x,y),G(x,y)), B(x,y,u,v) = \eta((x,y),G(u,v)).
\]
Then, from (3.1), for any \(p = (x,y), q = (u,v) \in Y\) with \(p \succ q\), we have
\[
\varphi(\eta(Gp,Gq)) \leq \varphi(\eta(p,q)) - \psi(\eta(p,q)) + \theta(\eta(p,Gp), \eta(q,Gq), \eta(p,Gp), \eta(q,Gp)).
\]
Moreover, for \(p_0 = (x_0,y_0) \in Y\), from (3.2) and (3.3), we have \(p_0 \preccurlyeq Gp_0\) or \(p_0 \succ Gp_0\).

Now \(G\) satisfies all the hypotheses of Theorem 2.1, we deduce that \(G\) has a fixed point \(\bar{x} = (a,b) \in Y\), that is,
\[
\bar{x} = (a,b) = G\bar{x} = G(a,b) = (F(a,b), \tilde{F}(a,b)).
\]
It implies that \(a = F(a,b), b = \tilde{F}(a,b)\), that is, \(F\) and \(\tilde{F}\) admit a pre-coupled fixed point \((a,b)\).

The previous result is still valid for \(F\) and \(\tilde{F}\) which are not necessarily continuous. Instead, we require additional assumptions to the metric space \((X,d)\) with a partial order \(\preceq\):

**Definition 3.3.** Let \((X,d)\) be a complete metric space endowed with a partial order \(\preceq\). We say that

(i) \((X,d,\preceq)\) is nondecreasing-regular (\(\uparrow\)-regular) if a nondecreasing sequence \(\{x_n\} \subset X\) converges to \(x\), then \(x_n \preceq x\) for all \(n\);

(ii) \((X,d,\preceq)\) is nonincreasing-regular (\(\downarrow\)-regular) if a nonincreasing sequence \(\{x_n\} \subset X\) converges to \(x\), then \(x_n \succeq x\) for all \(n\).

Motivated by [11, Theorem 3.5], we have the following result.

**Theorem 3.4.** Let \((X,d)\) be a complete metric space endowed with a partial order \(\preceq\), \(F : X \times X \to X\) a mixed monotone mapping, and \(\tilde{F} : X \times X \to X\) a reverse mixed monotone mapping. We assume that there exist \(\varphi \in \Phi, \psi \in \Psi\) and \(\theta \in \Theta\) such that for any \(x,y,u,v \in X\) with \(x \succeq u, y \preceq v\), inequality (3.1) holds. We also assume that \((X,d,\preceq)\) is nondecreasing-regular and nonincreasing-regular (\(\uparrow\downarrow\)-regular), and there
exist $x_0, y_0 \in X$ such that (3.2) or (3.3) hold. Then $F$ and $\tilde{F}$ admit a pre-coupled fixed point, that is, there exists $(a, b) \in X \times X$ such that $a = F(a, b)$ and $b = \tilde{F}(a, b)$.

**Proof.** It is sufficient to show that if $(X, d, \preceq)$ is nondecreasing-regular and nonincreasing-regular $(\uparrow \downarrow$-regular), then $(Y, \eta, \ll)$ is regular. The proof of this claim follows immediately from Theorem 2.2. In detail, also see the proof of [5, Theorem 2.2]. \qed

4. Applications

In this section, we study the existence of solutions of two types fourth-order two-point boundary value problems for elastic beam equations and two types third-order two-point boundary value problems. In particular, a theorem in the subsection 4.1 (Type I) is an extension of the result in [11].

4.1. Type I. First of all, we study the existence of solutions of the following fourth-order two-point boundary value problem for elastic beam equations:

\[
\begin{cases}
  u''''(t) = f(t, u(t), u''(t)), \\
  u(0) = A, \ u'(0) = B, \ u''(1) = C, \ u'''(1) = D,
\end{cases}
\]  

(4.1)

with $I = [0, 1]$ and $f \in C(I \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$, where $C(I \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$ is a set of continuous mappings of $I \times \mathbb{R} \times \mathbb{R}$ into $\mathbb{R}$. Let $\Omega$ be a set of functions $\omega$ of $[0, \infty)$ into $[0, \infty)$ satisfying

(i) $\omega$ is nondecreasing;

(ii) there exists $\psi \in \Psi$ such that $\omega(r) = \frac{r^2}{2} - \psi(\frac{r}{2})$ for all $r \in [0, \infty)$.

For examples of such functions, see [15].

Next we consider the following assumptions (A1) and (A2).

(A1) There exists $\omega \in \Omega$ such that for all $t \in I$ and for all $a, b, c, e \in \mathbb{R}$, with $a \geq c$ and $b \leq e$,

\[
0 \leq f(t, a, b) - f(t, c, e) \leq \omega(a - c) + \omega(e - b).
\]  

(4.2)

(A2) There exist $\alpha, \beta \in C(I, \mathbb{R})$ which are solutions of

\[
\alpha(t) \leq Bt + A - \int_0^1 H_2(t, s)(C - D + Ds)ds + \int_0^1 G(t, s)f(s, \alpha(s), \beta(s))ds, t \in I,
\]  

(4.3)

and

\[
\beta(t) \geq - \left( C - D + Dt + \int_0^1 H_1(t, s)f(s, \alpha(s), \beta(s))ds \right), t \in I,
\]  

(4.4)

where the Green functions $G$ and $H_1$ are defined by

\[
G(t, s) = \begin{cases}
  \frac{1}{6} s^2(3t - s), & (0 \leq s \leq t \leq 1), \\
  \frac{1}{6} t^2(3s - t), & (0 \leq t \leq s \leq 1),
\end{cases}
\]

and

\[
H_1(t, s) = \begin{cases}
  \frac{1}{6} s^2(3t - s), & (0 \leq s \leq t \leq 1), \\
  \frac{1}{6} t^2(3s - t), & (0 \leq t \leq s \leq 1),
\end{cases}
\]
and
\[ H_1(t, s) = \begin{cases} 0, & (0 \leq s \leq t \leq 1), \\ s - t, & (0 \leq t \leq s \leq 1). \end{cases} \]

Note that
\[ \int_0^1 G(t, s) f(s, u(s), v(s)) ds = \int_0^1 H_2(t, s) \int_0^1 H_1(s, r) f(r, u(r), v(r)) dr ds, \]
where the green function \( H_2 \) is defined by
\[ H_2(t, s) = \begin{cases} t - s, & (0 \leq s \leq t \leq 1), \\ 0, & (0 \leq t \leq s \leq 1). \end{cases} \]

It is easy to show that
\[ 0 \leq G(t, s) \leq \frac{1}{2} t^2 s \text{ for all } t, s \in I, \quad (4.5) \]
and
\[ 0 \leq H_1(t, s) \leq \min\{s, t\} \text{ for all } t, s \in I. \quad (4.6) \]

Now we have the following theorem.

**Theorem 4.1.** Under the assumptions (A1) and (A2), the fourth-order two-point boundary value problem (4.1) has a solution.

**Proof.** Consider the natural partial order relation \( \preceq \) on \( X = C(I, R) \), that is,
\[ u, v \in X, u \preceq v \Leftrightarrow u(t) \leq v(t) \text{ for all } t \in I. \]

It is well known that \( X \) is a complete metric space with respect to the metric
\[ d(u, v) = \max_{t \in I} |u(t) - v(t)| := \| u - v \|_\infty, u, v \in C(I, R). \]

It is easy to show that \((X, d, \preceq)\) is nondecreasing-regular and nonincreasing-regular (\( \uparrow \downarrow \)-regular), and that every pair of elements in \( X \times X \) has either a lower bound or an upper bound. Solving problem (4.1) is equivalent to finding \( u \in C(I, R) \) which is a solution of
\[ u(t) = Bt + A - \int_0^1 H_2(t, s)(C - D + Ds) ds \]
\[ + \int_0^1 G(t, s) f(s, u(s), v(s)) ds, t \in I, \]
where \( v = u'' \). Moreover the boundary value problem (4.1) can be written as
\[ \begin{align*}
    u''(t) &= v(t), \\
    v''(t) &= f(t, u(t), v(t)), \\
    u(0) &= A, u'(0) = B, v(1) = C, v'(1) = D,
\end{align*} \]
and it is equivalent to the following integral equations,

\[
\begin{cases}
    u(t) = Bt + A - \int_0^1 H_2(t, s)v(s)ds \\
    = Bt + A - \int_0^1 H_2(t, s)(C - D + Ds)ds \\
    + \int_0^1 G(t, s)f(s, u(s), v(s))ds, t \in I, \\
    v(t) = v(t) = C - D + Dt + \int_0^1 H_1(t, s)f(s, u(s), v(s))ds, t \in I.
\end{cases}
\]

Let \( F \) and \( \bar{F} \) be mappings of \( C(I, \mathbb{R}) \times C(I, \mathbb{R}) \) into \( C(I, \mathbb{R}) \) defined by

\[
F(x, y)(t) = Bt + A - \int_0^1 H_2(t, s)(C - D + Ds)ds + \int_0^1 G(t, s)f(s, x(s), y(s))ds, t \in I, x, y \in C(I, \mathbb{R}),
\]

and

\[
\bar{F}(x, y)(t) = - \left( C - D + Dt + \int_0^1 H_1(t, s)f(s, x(s), y(s))ds \right), t \in I, x, y \in C(I, \mathbb{R}).
\]

By the assumption (A1), we can show that the mapping \( F \) is mixed monotone and the mapping \( \bar{F} \) is reverse mixed monotone. In fact, for all \( t \in I \) and for all \( x, y, u, v \in C(I, \mathbb{R}) \) with \( x \geq u \) and \( y \leq v \), we have

\[
0 \leq f(t, x(t), y(t)) - f(t, u(t), v(t)).
\]

Thus we have

\[
F(x, y)(t) - F(u, v)(t) = \int_0^1 G(t, s)(f(s, x(s), y(s)) - f(s, u(s), v(s)))ds \geq 0,
\]

and

\[
\bar{F}(x, y)(t) - \bar{F}(u, v)(t) = - \int_0^1 H_1(t, s)(f(s, x(s), y(s)) - f(s, u(s), v(s)))ds \leq 0.
\]

Again, since \( \omega \) is nondecreasing and from (4.2) and (4.5), we have

\[
F(x, y)(t) - F(u, v)(t)
= \int_0^1 G(t, s)(f(s, x(s), y(s)) - f(s, u(s), v(s)))ds \\
\leq \int_0^1 G(t, s)(\omega(x(s) - u(s)))ds + \int_0^1 G(t, s)(\omega(v(s) - y(s)))ds \\
\leq \int_0^1 G(t, s)ds(\omega(\| x - u \|_\infty) + \omega(\| v - y \|_\infty)) \\
\leq \omega(\| x - u \|_\infty) + \omega(\| v - y \|_\infty)
\]
for all \( t \in I \) and for all \( x, y, u, v \in C(I, \mathbb{R}) \) with \( x \geq u \) and \( y \leq v \). Also from (4.2) and (4.6), we have

\[
|\tilde{F}(x, y)(t) - \tilde{F}(u, v)(t)| \\
\leq \int_0^1 H_1(t, s)|f(s, x(s), y(s)) - f(s, u(s), v(s))| \, ds \\
\leq \frac{\omega(\|x - u\|_\infty) + \omega(\|y - v\|_\infty)}{2}
\]

for all \( t \in I \) and for all \( x, y, u, v \in C(I, \mathbb{R}) \) with \( x \geq u \) and \( y \leq v \). By (4.7) and (4.8), we get

\[
\frac{d(F(x, y), F(u, v)) + d(\tilde{F}(x, y), \tilde{F}(u, v))}{2} \leq \frac{3(\omega(\|x - u\|_\infty) + \omega(\|y - v\|_\infty))}{8}.
\]

On the other hand, since \( \omega \) is nondecreasing and condition (ii) for \( \Omega \), we have

\[
\frac{3(\omega(\|x - u\|_\infty) + \omega(\|y - v\|_\infty))}{8} \leq \omega(\|x - u\|_\infty + \|y - v\|_\infty) \\
= \frac{d(x, u) + d(y, v)}{2} - \psi \left( \frac{d(x, u) + d(y, v)}{2} \right).
\]

Thus we finally get

\[
\eta((F(x, y), \tilde{F}(x, y)), (F(u, v), \tilde{F}(u, v))) = \frac{d(F(x, y), F(u, v)) + d(\tilde{F}(x, y), \tilde{F}(u, v))}{2} \\
\leq \frac{d(x, u) + d(y, v)}{2} - \psi \left( \frac{d(x, u) + d(y, v)}{2} \right).
\]

Now, let \( \alpha, \beta \in C(I, \mathbb{R}) \) be solutions of (4.3) and (4.4). By the assumption (A2), we have \( \alpha \preceq F(\alpha, \beta) \) and \( \beta \succeq \tilde{F}(\alpha, \beta) \). We also take \( \varphi(t) = t \) for any \( t \in [0, \infty) \) and \( \theta \equiv 0 \). Thus all the hypotheses of Theorem 3.4 are satisfied. Therefore \( u, v \in C(I, \mathbb{R}) \) are solution of the problem \( F(u, v) = u \) and \( \tilde{F}(u, v) = v \). These prove that \( u \in C(I, \mathbb{R}) \) is a solution of (4.1).

4.2. Type II. Next as an application of our results, we study the existence of solutions of the following fourth-order two-point boundary value problem, see [2, 25, 27]:

\[
\begin{cases}
\alpha'''(t) = f(t, u(t), u''(t)), \\
u(0) = A, u(1) = B, u''(0) = C, u''(1) = D,
\end{cases}
\]

with \( I = [0, 1] \) and \( f \in C(I \times \mathbb{R} \times \mathbb{R}, \mathbb{R}) \). We take the set of functions \( \Omega \) same way as in Type I.

The assumptions (A1) and (A2) are same as those of Type I with respect to the following Green functions \( G \) and \( H \).

\[
G(t, s) = \begin{cases}
\frac{1}{6} s(1-t)(2t-s^2-t^2), & (0 \leq s \leq t \leq 1), \\
\frac{1}{6} t(1-s)(2s-t^2-s^2), & (0 \leq t \leq s \leq 1),
\end{cases}
\]

\[
H(t, s) = \begin{cases}
\frac{1}{6} s(1-t)(2t-s^2-t^2), & (0 \leq s \leq t \leq 1), \\
\frac{1}{6} t(1-s)(2s-t^2-s^2), & (0 \leq t \leq s \leq 1),
\end{cases}
\]
and

\[ H(t, s) = \begin{cases} 
  s(t-1) & (0 \leq s \leq t \leq 1), \\
  t(s-1) & (0 \leq t \leq s \leq 1).
\end{cases} \]

Note that

\[ \int_0^1 G(t,s) f(s,u(s),v(s))ds = \int_0^1 H(t,s) \int_0^1 H(s,r) f(r,u(r),v(r))dr ds, t \in I. \]

It is easy to show that

\[ 0 \leq G(t,s) \leq \frac{1}{3} st \text{ for all } t, s \in I, \quad (4.10) \]

and

\[ 0 \leq -H(t,s) \leq \min\{s,t\} \text{ for all } t, s \in I. \quad (4.11) \]

**Theorem 4.2.** Under the assumptions (\(A_1\)) and (\(A_2\)), the fourth-order two-point boundary value problem (4.9) has a solution.

**Proof.** From the same argument in Theorem 4.1, we consider the natural partial order relation \(\preceq\) on \(X = C(I, \mathbb{R})\), that is,

\[ u, v \in X, u \preceq v \Leftrightarrow u(t) \leq v(t) \text{ for all } t \in I. \]

It is well known that \(X\) is a complete metric space with respect to the metric

\[ d(u,v) = \max_{t \in I} |u(t) - v(t)| := \|u - v\|_\infty, u, v \in C(I, \mathbb{R}). \]

It is easy to show that \((X, d, \preceq)\) is nondecreasing-regular and nonincreasing-regular (\(\uparrow\downarrow\)-regular), and that every pair of elements in \(X \times X\) has either a lower bound or an upper bound. Solving problem (4.9) is equivalent to finding \(u \in C(I, \mathbb{R})\) which is a solution of

\[ u(t) = (B - A)t + A + \int_0^1 H(t,s)((D - C)s + C)ds \]
\[ + \int_0^1 G(t,s)f(s,u(s),v(s))ds, t \in I, \]

where \(v = u''\). Moreover equation (4.9) can be written as

\[
\begin{aligned}
\begin{cases}
    u''(t) = v(t), \\
    v''(t) = f(t, u(t), v(t)), \\
    u(0) = A, u(1) = B, v(0) = C, v(1) = D,
\end{cases}
\end{aligned}
\]
and it is also equivalent to the following integral equations,

\[
\begin{cases}
    u(t) = (B - A)t + A - \int_0^1 H(t, s)v(s)ds \\
    \quad = (B - A)t + A - \int_0^1 H(t, s)((D - C)s + C)ds \\
    \quad \quad + \int_0^1 G(t, s)f(s, u(s), v(s))ds, t \in I, \\
    v(t) = (D - C)t + C - \int_0^1 H(t, s)f(s, u(s), v(s))ds, t \in I.
\end{cases}
\]

Let \( F \) and \( \tilde{F} \) be mappings of \( C(I, \mathbb{R}) \times C(I, \mathbb{R}) \) into \( C(I, \mathbb{R}) \) defined by

\[
F(x, y)(t) = (B - A)t + A - \int_0^1 H(t, s)((D - C)s + C)ds \\
\quad \quad + \int_0^1 G(t, s)f(s, x(s), y(s))ds, t \in I, x, y \in C(I, \mathbb{R}),
\]

and

\[
\tilde{F}(x, y)(t) = (D - C)t + C - \int_0^1 H(t, s)f(s, x(s), y(s))ds, t \in I, x, y \in C(I, \mathbb{R}).
\]

By the assumption (A1), we can show that the mapping \( F \) is mixed monotone and the mapping \( \tilde{F} \) is reverse mixed monotone. In fact, for all \( t \in I \) and for all \( x, y, u, v \in C(I, \mathbb{R}) \), with \( x \succeq u \) and \( y \preceq v \),

\[
0 \leq f(t, x(t), y(t)) - f(t, u(t), v(t)).
\]

Thus we have

\[
F(x, y)(t) - F(u, v)(t) = \int_0^1 G(t, s)(f(s, x(s), y(s)) - f(s, u(s), v(s)))ds \geq 0
\]

and

\[
\tilde{F}(x, y)(t) - \tilde{F}(u, v)(t) = - \int_0^1 H(t, s)(f(s, x(s), y(s)) - f(s, u(s), v(s)))ds \leq 0.
\]

The remaining parts of the proof are same as that of Type I using properties (4.10), (4.11), and the assumption (A2).

4.3. **Type III.** In this subsection, we consider the solutions of the following third-order two-point boundary value problem, see [6, 24, 28]:

\[
\begin{cases}
    u''''(t) = f(t, u(t), u''(t)), \\
    u(0) = A, u(1) = B, u''(0) = C,
\end{cases}
\]

(4.12)
with \( I = [0,1] \) and \( f \in C(I \times \mathbb{R} \times \mathbb{R}, \mathbb{R}) \). The assumptions \((A1)\) and \((A2)\) are same as that of Type I using the following Green functions \( G \) and \( H \).

\[
G(t,s) = \begin{cases} 
-(t-s)^2 + t(1-s)^2, & (0 \leq s \leq t \leq 1), \\
\frac{t(1-s)^2}{2}, & (0 \leq t \leq s \leq 1),
\end{cases}
\]

and

\[
H(t,s) = \begin{cases} 
(1-t)s, & (0 \leq s \leq t \leq 1), \\
(1-s)t, & (0 \leq t \leq s \leq 1).
\end{cases}
\]

Note that

\[
\int_0^1 G(t,s)f(s,u(s),v(s))ds = \int_0^1 H(t,s) \int_0^s f(r,u(r),v(r))drds.
\]

It is easy to show that

\[
0 \leq G(t,s) \leq \frac{1}{2} t(1-s)^2 \text{ for all } t,s \in I,
\]

and

\[
0 \leq H(t,s) \leq \min\{s(1-t),t(1-s)\} \text{ for all } t,s \in I.
\]

**Theorem 4.3.** Under the assumptions \((A1)\) and \((A2)\), the third-order two-point boundary value problem \((4.12)\) has a solution.

**Proof.** From the same argument in Theorem 4.1, we consider the natural partial order relation \(\preceq\) in \(X = C(I,\mathbb{R})\). Then \((X,d,\preceq)\) is complete, nondecreasing-regular, nonincreasing-regular (\(\uparrow\downarrow\)-regular), and every pair of elements in \(X \times X\) has either a lower bound or an upper bound. Solving problem \((4.12)\) is equivalent to finding \(u \in C(I,\mathbb{R})\) which is a solution of

\[
u(t) = (B-A)t + A - \int_0^1 H(t,s)ds + \int_0^1 G(t,s)f(s,u(s),v(s))ds, \quad t \in I,\]

where \(v = u''\). Moreover the boundary value problem \((4.12)\) can be written as follows

\[
\begin{align*}
u''(t) &= v(t), \\
v'(t) &= -f(t,u(t),v(t)), \\
u(0) = A, u(0) = B, v(0) = C,
\end{align*}
\]
and it is also equivalent to the following,
\[
\begin{align*}
    u(t) &= (B - A)t + A - \int_0^1 H(t, s)v(s)ds \\
    &= (B - A)t + A - \int_0^1 H(t, s)C ds \\
    &\quad + \int_0^1 G(t, s)f(s, u(s), v(s))ds, t \in I, \\
    v(t) &= C - \int_0^t f(s, u(s), v(s))ds, t \in I.
\end{align*}
\]

Let \( F \) and \( \tilde{F} \) be mappings of \( C(I, \mathbb{R}) \times C(I, \mathbb{R}) \) into \( C(I, \mathbb{R}) \) defined by
\[
F(x, y)(t) = (B - A)t + A - \int_0^1 H(t, s)C ds \\
\quad + \int_0^1 G(t, s)f(s, x(s), y(s))ds, t \in I, x, y \in C(I, \mathbb{R}),
\]
and
\[
\tilde{F}(x, y)(t) = C - \int_0^t f(s, x(s), y(s))ds, t \in I, x, y \in C(I, \mathbb{R}).
\]

By the assumption (A1), we can show that the mapping \( F \) is mixed monotone and the mapping \( \tilde{F} \) is reverse mixed monotone. The remaining parts of the proof are same as that of Type I using properties (4.13), (4.14), and the assumption (A2).

4.4. Type IV. Finally we consider the solutions of the following third-order two-point boundary value problem, see [14]:
\[
\begin{align*}
    u'''(t) &= f(t, u(t), u''(t)), \\
    u(0) &= A, u(1) = B, u''(1) = C,
\end{align*}
\]
with \( I = [0, 1] \) and \( f \in C(I \times \mathbb{R} \times \mathbb{R}, \mathbb{R}) \). The assumptions (A1) and (A2) are same as those of Type I using the following Green functions \( G \) and \( H \).
\[
G(t, s) = \begin{cases}
    \frac{1}{2} s^2 (1 - t), & (0 \leq s \leq t \leq 1), \\
    \frac{1}{2} t [(1 - t) - (1 - s)^2], & (0 \leq t \leq s \leq 1),
\end{cases}
\]
and
\[
H(t, s) = \begin{cases}
    (1 - t)s & (0 \leq s \leq t \leq 1), \\
    (1 - s)t & (0 \leq t \leq s \leq 1).
\end{cases}
\]

Note that
\[
\int_0^1 G(t, s)f(s, u(s), v(s))ds = \int_0^1 H(t, s) \int_s^1 f(r, u(r), v(r))dr ds.
\]
It is easy to show that
\[ 0 \leq G(t, s) \leq \frac{1}{2} s^2 (1 - t) \text{ for all } t, s \in I, \]
and
\[ 0 \leq H(t, s) \leq \min\{s(1 - t), t(1 - s)\} \text{ for all } t, s \in I. \]

**Theorem 4.4.** Under the assumptions (A1) and (A2), the third-order two-point boundary value problem (4.15) has a solution.

**Proof.** From the same argument in Theorem 4.1, we consider the natural partial order relation \( \preceq \) on \( X = C(I, \mathbb{R}) \). Then \( (X, d, \preceq) \) is complete, nondecreasing-regular and nonincreasing-regular (\( \uparrow \downarrow \)-regular), and every pair of elements in \( X \times X \) has either a lower bound or an upper bound. Solving problem (4.15) is equivalent to finding \( u \in C(I, \mathbb{R}) \) which is a solution of
\[ u(t) = (B - A)t + A - \int_0^1 H(t, s) C ds + \int_0^1 G(t, s) f(s, u(s), v(s)) ds, \quad t \in I, \]
where \( v = u'' \). Moreover the boundary value problem (4.15) can be written as
\[ \begin{cases} u''(t) = v(t), \\ v'(t) = f(t, u(t), v(t)), \\ u(0) = u(1) = v(1) = 0, \end{cases} \]
and it is also equivalent to the following integral equations,
\[ \begin{cases} u(t) = (B - A)t + A - \int_0^1 H(t, s) v(s) ds, \\ = (B - A)t + A - \int_0^1 H(t, s) C ds \\ + \int_0^1 G(t, s) f(s, u(s), v(s)) ds, \quad t \in I, \\ v(t) = C - \int_t^1 f(s, u(s), v(s)) ds, \quad t \in I. \end{cases} \]

Let \( F \) and \( \tilde{F} \) be mappings of \( C(I, \mathbb{R}) \times C(I, \mathbb{R}) \) into \( C(I, \mathbb{R}) \) defined by
\[ F(x, y)(t) = (B - A)t + A - \int_0^1 H(t, s) C ds \]
\[ + \int_0^1 G(t, s) f(s, x(s), y(s)) ds, \quad t \in I, x, y \in C(I, \mathbb{R}), \]
and
\[ \tilde{F}(x, y)(t) = C - \int_t^1 f(s, u(s), v(s)) ds, \quad t \in I, x, y \in C(I, \mathbb{R}). \]
By the assumption (A1), we can show that the mapping $F$ is mixed monotone and the mapping $\tilde{F}$ is reverse mixed monotone. The remaining parts of the proof are same as that of Type I using properties (4.16), (4.17), and the assumption (A2).

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