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FIXED POINT ALGORITHMS FOR SPLIT FEASIBILITY PROBLEMS

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Abstract. The purpose of the paper is to investigate a split feasibility problem based on a fixed point algorithm. A weak convergence theorem of solutions is established in the framework of infinite dimensional Hilbert spaces. As an application, a split equality problem is also investigated.
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1. Introduction

In this paper, we always assume that H_1 and H_2 are real Hilbert spaces endowed with inner products and induced norms denoted by $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$, respectively, while H refers to as any of these spaces.

Let $Proj_C^{H_1}$ be the metric projection from H_1 onto C and let $Proj_Q^{H_2}$ be the metric projection from H_2 onto Q. Recall that the split feasibility problem is to find a point $x \in H_1$ such that

$$x \in C, \quad Ax \in Q, \tag{1.1}$$

where $A : H_1 \to H_2$ is a bounded linear operator. In this paper, we always use $Sol(SFP)$ to denote the solution set of the split feasibility problem.

The split feasibility problem, which was introduced and investigated by Censor and Elfving [9] in finite-dimensional Hilbert spaces for modeling inverse problems which arise from phase retrievals and in medical image reconstruction. It has been found that the split feasibility problem can also be used in various disciplines such as image restoration, computer tomograph and radiation therapy treatment planning [8]. It is known tat the split feasibility problem includes many problems, such as, [2, 3, 12, 13, 19, 21, 24] as special cases.

It is known if the solution set of split feasibility problem (1.1) is not empty, then the split feasibility problem is equivalent to a fixed point problem

$$P_C^{H_1}(x - \gamma A^*(I - P_Q^{H_2})Ax) = x, \tag{1.2}$$

where $\gamma > 0$ is a constant and A^* is the adjoint operator of A. Recently, many authors have investigated the split feasibility problem and fixed points of nonexpansive mappings via the Byrne's CQ iterative algorithm [6] in the setting of infinite-dimensional Hilbert spaces; see, for example, [1, 7, 10, 14, 11, 20, 23] and the references therein.

The purpose of the paper is to investigated split feasibility problem (1.1) based on a fixed point method. The paper is organized as follows. In Section 2, we provide some necessary definitions, properties and lemmas. In Section 3, a weak convergence theorem is established in the framework of infinite dimensional Hilbert spaces. In Section 4, a split equality problem is also investigated as an application of our main results.

2. Preliminaries

Recall that a mapping $T : H \to H$ is said to be monotone iff

$$\langle Fx - Fy, x - y\rangle \geq 0, \quad \forall x, y \in H.$$

$F : H \to H$ is said to be strongly monotone iff there exists a constant $\nu > 0$ such that

$$\langle Fx - Fy, x - y\rangle \geq \nu\|x - y\|^2, \quad \forall x, y \in H.$$

In such a case, we also say that F is ν-strongly monotone. $F : H \to H$ is said to be inverse-strongly monotone iff there exists a constant $\nu > 0$ such that

$$\langle Fx - Fy, x - y\rangle \geq \nu\|Fx - Fy\|^2, \quad \forall x, y \in H.$$

In such a case, we also say that F is ν-inverse-strongly monotone. It is not hard to see that F is ν-inverse-strongly monotone iff F^{-1} is ν-strongly monotone. Recall that $F : H \to H$ is said to be Lipchitz continuous iff there exits $L > 0$ such that

$$\|Fx - Fy\| \leq L\|x - y\|, \quad \forall x, y \in H.$$

In such case, we also say that F is L-Lipchitz continuous. We here remark that if F is ν-inverse-strongly monotone, then it is $\frac{1}{\nu}$-Lipschitz continuous and monotone. If $L = 1$, then F is said to be nonexpansive. For the existence of fixed points of nonexpansive mappings, one is referred to [4, 5, 15] and the references therein. In this paper, the fixed point set of mapping F is denoted by $Fix(F)$. Let F be a nonexpansive mapping and define a mapping $T : H \to H$ by $Tx = (I - F)x$, $\forall x \in H$. Then T is $\frac{1}{2}$-inverse-strongly monotone.

Recall that a mapping $T : H \to H$ is said to be firmly nonexpansive if

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 - \|(x - y) - (Tx - Ty)\|, \quad \forall x, y \in H.$$

Recall that a mapping $T : H \to H$ is said to be averaged if it can be written as the average of the identity mapping and a nonexpansive mapping, i.e., $T := (1-\alpha)I + \alpha S$ where $\alpha \in (0,1)$ and $S : H \to H$ is nonexpansive and I is the identity operator on H. We note that averaged mappings are nonexpansive. Further, firmly nonexpansive mappings (in particular, projections on nonempty closed and convex subsets and resolvent operators of maximal monotone operators) are averaged.

The class of nonexpansive mappings recently has been extensively investigated for solving various convex optimization problems. Mann iterative algorithm is an efficient tool to study fixed points of nonexpansive mappings. Recall that the Mann iterative algorithm generates a sequence $\{x_n\}$ in the following manner

$$x_1 \in H, x_{n+1} = \alpha_n T x_n + (1-\alpha_n)x_n, \quad n \geq 1,$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$. In [22], Reich proved $\{x_n\}$ generated in the Mann iterative algorithm converges weakly to some fixed point of T provide that control sequence $\{\alpha_n\}$ satisfies some conditions.

In order to obtain our main results, we need the following definitions and lemmas.

Recall that a space E is said to have the Opial's condition [18] if, for each $\{x_n\}$ in E, the condition that $\{x_n\}$ converges weakly to p implies that

$$\liminf_{n\to\infty} \|x_n - p\| < \liminf_{n\to\infty} \|x_n - p'\|,$$

$\forall p' \in E$ with $p' \neq p$. It is known that the above inequality is also equivalent to

$$\limsup_{n\to\infty} \|x_n - p\| < \limsup_{n\to\infty} \|x_n - p'\|.$$

Recall that the metric (nearest point) projection $P_C^H : H \to C$ from a Hilbert space H onto a nonempty, closed and convex subset C of H is defined as follows: for each point $x \in H$, there exists a unique point $P_C^H x \in C$ with the property:

$$\|x - P_C^H x\| \leq \|x - y\|.$$

Thus for any $x \in H$, $\tilde{x} = P_C^H x$ iff $\tilde{x} \in C$ and $\|x - \tilde{x}\| = \inf\{\|x - y\| : y \in C\}$.

Lemma 2.1. *Let $P_C^H : H \to C$ be the metric projection from H on a nonempty, closed, and convex subset C. Then the following conclusions hold true*

(a) *Given $x \in H$ and $z \in C$. Then $z = P_C^H x$ iff there holds the inequality:*

$$\langle x - z, y - z\rangle \leq 0, \ y \in C.$$

(b) $\langle P_C^H x - P_C^H y, x - y\rangle \geq \|P_C^H x - P_C^H y\|^2$, $x, y \in H$.
(c) $\langle (I - P_C^H)x - (I - P_C^H)y, x - y\rangle \geq \|(I - P_C^H)x - (I - P_C^H)y\|^2$, $\forall x, y \in H$.
(d) $\|P_C^H x - P_C^H y\|^2 \leq \|x - y\|^2 - \|(I - P_C^H)x - (I - P_C^H)y\|^2$, $\forall x, y \in H$

Lemma 2.2 ([25])**.** *Let H be a Hilbert space. Then there exists a strictly increasing continuous convex function $conf : [0,\infty) \to [0,\infty)$ with $conf(0) = 0$ such that*

$$a\|x\|^2 - a(1-a)conf(\|x - y\|) + (1-a)\|y\|^2 \geq \|ax + (1-a)y\|^2, \quad \forall a \in [0,1],$$

for all $x, y \in B_r(0) := \{x \in H : \|x\| \leq r\}$, where r is some positive real number.

Lemma 2.3 ([4]). *Let H be a Hilbert space and let T be a nonexpansive mapping on H. If $x_n \rightharpoonup p$, where $\rightharpoonup$ denotes the weak convergence, and $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$, then p is a fixed point of T, that is, $p = Tp$.*

3. Main results

Theorem 3.1. *Let H_1 and H_2 be two real Hilbert spaces. Let C be a nonempty closed and convex subset of H_1 and let Q be a nonempty closed and convex subset of H_2. Let $Proj_C^{H_1}$ be the metric projection from H_1 onto C and let $Proj_Q^{H_2}$ be the metric projection from H_2 onto Q. Let $A : H_1 \to H_2$ be a bounded linear operator and let $S : C \to C$ be a nonexpansive mapping such that $Sol(SFP) \cap Fix(S) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated in the following iterative algorithm: $x_1 \in C$ and*

$$x_{n+1} = \alpha_n Sx_n + (1-\alpha_n) Proj_C^{H_1}\big(\beta_n x_n + (1-\beta_n)(x_n - \gamma_n A^*(I - Proj_Q^{H_2})Ax_n)\big), \quad (3.1)$$

where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are three real nonnegative sequences such that

(i) $0 < \alpha \leq \alpha_n \leq \alpha' < 1$,

(ii) $0 \leq \beta_n \leq \beta < 1$,

(iii) $0 < \gamma \leq \gamma_n \leq \gamma' \leq \frac{2}{\|A\|^2}$,

where $\alpha, \alpha', \beta, \gamma$ and γ' are four real numbers. Then $\{x_n\}$ converges weakly to some point in $Sol(SFP) \cap Fix(S)$.

Proof. Define a mapping $T : C \to H_1$ by

$$Tx = A^*(I - Proj_Q^{H_2})Ax, \quad \forall x \in C.$$

Then (3.1) becomes

$$x_1 \in C, \quad x_{n+1} = \alpha_n Sx_n + (1-\alpha_n) Proj_C^{H_1}\big(\beta_n x_n + (1-\beta_n)(x_n - \gamma_n Tx_n)\big), \quad (3.2)$$

Using Lemma 2.1, we have

$$\begin{aligned} \langle Tx - Ty, x - y\rangle &= \langle A^*(I - Proj_Q^{H_2})Ax - A^*(I - Proj_Q^{H_2})Ay, x - y\rangle \\ &= \langle (I - Proj_Q^{H_2})Ax - (I - Proj_Q^{H_2})Ay, Ax - Ay\rangle \\ &\geq \|(I - Proj_Q^{H_2})Ax - (I - Proj_Q^{H_2})Ay\|^2 \\ &\geq \frac{1}{\|A\|^2}\|A^*(I - Proj_Q^{H_2})Ax - A^*(I - Proj_Q^{H_2})Ay\|^2 \\ &= \frac{1}{\|A\|^2}\|Tx - Ty\|^2. \end{aligned} \quad (3.3)$$

This shows that T is $\frac{1}{\|A\|^2}$-inverse-strongly monotone.

Next, we prove $T^{-1}(0) = A^{-1}(Q)$. Letting $x \in A^{-1}(Q)$, we find from the definition of T that $x \in T^{-1}(0)$. This proves $A^{-1}(Q) \subset T^{-1}(0)$. Letting $x \in T^{-1}(0)$, we have $Tx = 0$. Since $Sol(SFP) \cap Fix(S) \neq \emptyset$, we can take a point $y \in Sol(SFP) \cap Fix(S)$. This implies $Sy = y$ and $Ay = Proj_Q^{H_2}Ay$. Hence, $Ty = 0$. Using (3.3), we have

$$0 = \langle Tx - Ty, x - y\rangle \geq \|(I - Proj_Q^{H_2})Ax\|^2,$$

which implies that $x \in A^{-1}(Q)$, that is, $T^{-1}(0) \subset A^{-1}(Q)$.
This shows that $T^{-1}(0) = A^{-1}(Q)$.
Since T is $\frac{1}{\|A\|^2}$-inverse-strongly monotone, we have

$$\begin{aligned}\|(I-\gamma_n T)x-(I-\gamma_n T)y\|^2 &= \|\gamma_n(Tx-Ty)-(x-y)\|^2\\ &= \gamma_n^2\|Tx-Ty\|^2 - 2\gamma_n\langle x-y, Tx-Ty\rangle + \|x-y\|^2\\ &\le \gamma_n^2\|Tx-Ty\|^2 - \frac{2\gamma_n}{\|A\|^2}\|Tx-Ty\|^2 + \|x-y\|^2\\ &= \|x-y\|^2 - \gamma_n(\frac{2}{\|A\|^2}-\gamma_n)\|Tx-Ty\|^2.\end{aligned}$$

From condition (iii), we find that $(I-\gamma_n T)$ is nonexpansive.
Fix $x^* \in Sol(SFP) \cap Fix(S) = C \cap A^{-1}(Q) \cap Fix(S) = C \cap T^{-1}(0) \cap Fix(S)$, we find from (3.2) that

$$\begin{aligned}&\|x_{n+1}-x^*\|\\ &\le (1-\alpha_n)\|Proj_C^{H_1}\big(\beta_n x_n + (1-\beta_n)(x_n-\gamma_n Tx_n)\big) - x^*\| + \alpha_n\|Sx_n - x^*\|\\ &\le (1-\alpha_n)\|\big(\beta_n x_n + (1-\beta_n)(x_n-\gamma_n Tx_n)\big) - x^*\| + \alpha_n\|x_n - x^*\|\\ &\le (1-\alpha_n)\big(\beta_n\|x_n-x^*\| + (1-\beta_n)\|(I-\gamma_n T)x_n - (I-\gamma_n T)x^*\|\big) + \alpha_n\|x_n-x^*\|\\ &\le (1-\alpha_n)\beta_n\|x_n-x^*\| + (1-\alpha_n)(1-\beta_n)\|x_n-x^*\| + \alpha_n\|x_n-x^*\|\\ &= \|x_n-x^*\|.\end{aligned}$$

It follows that sequence $\{\|x_n-x^*\|\}$ is nonincreasing. This implies that $\lim_{n\to\infty}\|x_n-x^*\|$ exists. In particular, we find that $\{x_n\}$ is bounded. Using Lemma 2.2, we find that

$$\begin{aligned}&\|x_{n+1}-x^*\|^2\\ &\le (1-\alpha_n)\|Proj_C^{H_1}\big(\beta_n x_n + (1-\beta_n)(x_n-\gamma_n Tx_n)\big) - x^*\|^2 + \alpha_n\|Sx_n-x^*\|^2\\ &\quad - \alpha_n(1-\alpha_n)conf(\|Sx_n - Proj_C^{H_1}\big(\beta_n x_n + (1-\beta_n)(x_n-\gamma_n Tx_n)\big)\|)\\ &\le (1-\alpha_n)\|\big(\beta_n x_n + (1-\beta_n)(x_n-\gamma_n Tx_n)\big) - x^*\|^2 + \alpha_n\|x_n-x^*\|^2\\ &\quad - \alpha_n(1-\alpha_n)conf(\|Sx_n - Proj_C^{H_1}\big(\beta_n x_n + (1-\beta_n)(x_n-\gamma_n Tx_n)\big)\|)\\ &\le (1-\alpha_n)\big(\beta_n\|x_n-x^*\| + (1-\beta_n)\|(x_n-\gamma_n Tx_n) - x^*\|\big)^2\\ &\quad + \alpha_n\|x_n-x^*\|^2 - conf(\|Sx_n - Proj_C^{H_1}\big(\beta_n x_n + (1-\beta_n)(x_n-\gamma_n Tx_n)\big)\|)\\ &\le \|x_n-x^*\|^2 - \alpha_n(1-\alpha_n)conf(\|Sx_n - Proj_C^{H_1}\big(\beta_n x_n + (1-\beta_n)(x_n-\gamma_n Tx_n)\big)\|).\end{aligned}$$

This implies that

$$\begin{aligned}&\alpha_n(1-\alpha_n)conf(\|Sx_n - Proj_C^{H_1}\big(\beta_n x_n + (1-\beta_n)(x_n-\gamma_n Tx_n)\big)\|)\\ &\le \|x_n-x^*\|^2 - \|x_{n+1}-x^*\|^2.\end{aligned}$$

Since $\lim_{n\to\infty}\|x_n-x^*\|$ exists, we see that

$$\lim_{n\to\infty} conf(\|Sx_n - Proj_C^{H_1}\big(\beta_n x_n + (1-\beta_n)(x_n-\gamma_n Tx_n)\big)\|) = 0.$$

It follows that

$$\lim_{n\to\infty} \|Sx_n - Proj_C^{H_1}\big(\beta_n x_n + (1-\beta_n)(x_n - \gamma_n T x_n)\big)\| = 0. \tag{3.4}$$

Since T is $\frac{1}{\|A\|^2}$-inverse-strongly monotone, one sees that

$$\begin{aligned}\|x_n - \gamma_n T x_n - x^*\|^2 &= \|(x_n - x^*) - \gamma_n(Tx_n - Tx^*)\|^2 \\ &= \|x_n - x^*\|^2 - 2\langle x_n - x^*, Tx_n - Tx^*\rangle + \gamma_n^2\|Tx_n - Tx^*\|^2 \\ &\le \|x_n - x^*\|^2 - \frac{2\gamma_n}{\|A\|^2}\|Tx_n - Tx^*\|^2 + \gamma_n^2\|Tx_n - Tx^*\|^2 \\ &= \|x_n - x^*\|^2 - \gamma_n(\frac{2}{\|A\|^2} - \gamma_n)\|Tx_n\|^2.\end{aligned} \tag{3.5}$$

Note that $\|\cdot\|^2$ is convex. Using (3.2) and (3.5), we find that

$$\begin{aligned}&\|x_{n+1} - x^*\|^2 \\ &\le (1-\alpha_n)\|Proj_C^{H_1}\big(\beta_n x_n + (1-\beta_n)(x_n - \gamma_n T x_n)\big) - x^*\|^2 + \alpha_n\|Sx_n - x^*\|^2 \\ &\le (1-\alpha_n)\|\big(\beta_n x_n + (1-\beta_n)(x_n - \gamma_n T x_n)\big) - x^*\|^2 + \alpha_n\|x_n - x^*\|^2 \\ &\le (1-\alpha_n)\big(\beta_n\|x_n - x^*\|^2 + (1-\beta_n)\|(x_n - \gamma_n T x_n) - x^*\|^2\big) + \alpha_n\|x_n - x^*\|^2 \\ &\le \|x_n - x^*\|^2 - (1-\alpha_n)(1-\beta_n)\gamma_n(\frac{2}{\|A\|^2} - \gamma_n)\|Tx_n\|^2.\end{aligned}$$

It follows that

$$(1-\alpha_n)(1-\beta_n)\gamma_n(\frac{2}{\|A\|^2} - \gamma_n)\|Tx_n\|^2 \le \|x_n - p\|^2 - \|x_{n+1} - p\|^2.$$

Using conditions (i), (ii) and (iii), we find that $\lim_{n\to\infty}\|Tx_n\| = 0$. Note that

$$\begin{aligned}\|Sx_n - x_n\| &\le \|Sx_n - Proj_C^{H_1}\big(\beta_n x_n + (1-\beta_n)(x_n - \gamma_n T x_n)\big)\| \\ &\quad + \|Proj_C^{H_1}\big(\beta_n x_n + (1-\beta_n)(x_n - \gamma_n T x_n)\big) - x_n\| \\ &\le \|Sx_n - Proj_C^{H_1}\big(\beta_n x_n + (1-\beta_n)(x_n - \gamma_n T x_n)\big)\| \\ &\quad + \|\big(\beta_n x_n + (1-\beta_n)(x_n - \gamma_n T x_n)\big) - x_n\| \\ &\le \|Sx_n - Proj_C^{H_1}\big(\beta_n x_n + (1-\beta_n)(x_n - \gamma_n T x_n)\big)\| + \gamma_n\|Tx_n\|.\end{aligned}$$

In view of $\lim_{n\to\infty}\|Tx_n\| = 0$, we find from (3.4) that $\lim_{n\to\infty}\|Sx_n - x_n\| = 0$.

Since $\{x_n\}$ is bounded, we see there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ converges weakly to p in H_1. Since C is weakly closed, we see that $p \in C$. Since T is $\frac{1}{\|A\|^2}$-inverse-strongly monotone, we have

$$0 \le \frac{1}{\|A\|^2}\|Tx_{n_i} - Tp\|^2 \le \langle Tx_{n_i} - Tp, x_{n_i} - p\rangle. \tag{3.6}$$

Letting $i \to \infty$ in (3.6), we find that $Tp = 0$, that is, $p \in T^{-1}(0)$. Note that $Sx_{n_i} - x_{n_i} \to 0$ as $i \to \infty$. Using the demiclosed principal of nonexpansive mappings, we find that $p \in Fix(S)$.

Finally, we prove that $\{x_n\}$ converges weakly to $p \in Sol(SFP) \cap Fix(S)$. Assume that there exists another subsequence $\{x_{n_j}\}$ of $\{x_n\}$ converges weakly to q, where $q \neq p$.

Since H_1 has the Opial's condition, we obtain that

$$\lim_{n\to\infty} \|x_n - p\| = \liminf_{i\to\infty} \|x_{n_i} - p\| < \liminf_{i\to\infty} \|x_{n_i} - p'\|$$

$$= \lim_{n\to\infty} \|x_n - q\| = \lim_{j\to\infty} \|x_{n_j} - q\| < \lim_{j\to\infty} \|x_{n_j} - q\| = \lim_{n\to\infty} \|x_n - p\|.$$

This is a contradiction. It follows that $p = q$. This proves that $\{x_n\}$ converges weakly to p. This completes the proof. □

Using Theorem 3.1, we find the following results immediately.

Corollary 3.2. *Let H_1 and H_2 be two real Hilbert spaces. Let C be a nonempty closed and convex subset of H_1 and let Q be a nonempty closed and convex subset of H_2. Let $Proj_C^{H_1}$ be the metric projection from H_1 onto C and let $Proj_Q^{H_2}$ be the metric projection from H_2 onto Q. Let $A : H_1 \to H_2$ be a bounded linear operator such that $Sol(SFP) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated in the following iterative algorithm*

$$x_1 \in C, \quad x_{n+1} = \alpha_n x_n + (1-\alpha_n) Proj_C^{H_1}\big(\beta_n x_n + (1-\beta_n)(x_n - \gamma_n A^*(I - Proj_Q^{H_2})Ax_n)\big),$$

where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are three real nonnegative sequences such that

(i) $0 < \alpha \leq \alpha_n \leq \alpha' < 1$,
(ii) $0 \leq \beta_n \leq \beta < 1$,
(iii) $0 < \gamma \leq \gamma_n \leq \gamma' \leq \frac{2}{\|A\|^2}$,

where $\alpha, \alpha', \beta, \gamma$ and γ' are four real numbers. Then $\{x_n\}$ converges weakly to some point in $Sol(SFP)$.

Corollary 3.3. *Let H_1 and H_2 be two real Hilbert spaces. Let C be a nonempty closed and convex subset of H_1 and let Q be a nonempty closed and convex subset of H_2. Let $Proj_C^{H_1}$ be the metric projection from H_1 onto C and let $Proj_Q^{H_2}$ be the metric projection from H_2 onto Q. Let $A : H_1 \to H_2$ be a bounded linear operator such that $Sol(SFP) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated in the following iterative algorithm*

$$x_1 \in C, \quad x_{n+1} = \alpha_n x_n + (1-\alpha_n) Proj_C^{H_1}\big(x_n - \gamma_n A^*(I - Proj_Q^{H_2})Ax_n\big),$$

where $\{\alpha_n\}$ and $\{\gamma_n\}$ are two real nonnegative sequences such that

(i) $0 < \alpha \leq \alpha_n \leq \alpha' < 1$,
(ii) $0 < \gamma \leq \gamma_n \leq \gamma' \leq \frac{2}{\|A\|^2}$,

where α, α', γ and γ' are four real numbers. Then $\{x_n\}$ converges weakly to some point in $Sol(SFP)$.

Corollary 3.4. *Let H_1 and H_2 be two real Hilbert spaces. Let C be a nonempty closed and convex subset of H_1 and let Q be a nonempty closed and convex subset of H_2. Let $Proj_C^{H_1}$ be the metric projection from H_1 onto C and let $Proj_Q^{H_2}$ be the metric projection from H_2 onto Q. Let $A : H_1 \to H_2$ be a bounded linear operator* *and let $S : C \to C$ be a nonexpansive mapping such that $Sol(SFP) \cap Fix(S) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated in the following iterative algorithm*

$$x_1 \in C, \quad x_{n+1} = \alpha_n S x_n + (1-\alpha_n) Proj_C^{H_1}\big(x_n - \gamma_n A^*(I - Proj_Q^{H_2})Ax_n\big),$$

where $\{\alpha_n\}$ and $\{\gamma_n\}$ are two real nonnegative sequences such that
(i) $0 < \alpha \leq \alpha_n \leq \alpha' < 1$,
(ii) $0 < \gamma \leq \gamma_n \leq \gamma' \leq \frac{2}{\|A\|^2}$,
where α, α', γ and γ' are four real numbers. Then $\{x_n\}$ converges weakly to some point in $Sol(SFP) \cap Fix(S)$.

4. Applications

Let H_1, H_2 and H_3 be three real Hilbert spaces. Let C be a nonempty closed and convex subsets of Hilbert space H_1 and let Q be a nonempty closed and convex subsets of Hilbert space H_2. Let $A : H_1 \to H_3$ and $B : H_2 \to H_3$ be two bounded linear operators. Recall that the split equality problem is to

$$\text{find } x \in C \text{ and } y \in Q \text{ such that } Ax = By. \tag{4.1}$$

Next, we use $Sol(SEP)$ to denote the solution set of the split equality problem, which was introduced and studied by Moudafi and Al-Shemas [17]; see also [16]. Obviously, if $B = I$ (identity mapping on H_2) and $H_3 = H_2$, then (4.1) reduces to (1.1).

By virtue of the product space techniques, we can convert the split equality problem to a split feasibility problem. To see this, set $M = C \times Q$ and define

$$G = [A, -B], \omega = [x, y]^T. \tag{4.2}$$

With these notations, we know that solving the the split equality problem is equivalent to finding a point $\omega \in M$ such that $G\omega = 0$.

Assume that the split equality problem is consistent, *i.e.*, $Sol(SEP) \neq \emptyset$. Then it is not difficult to see that $\omega \in M$ solves the the split equality problem iff it solves operator equation $G^*G\omega = 0$, where G^* is the adjoint operator of G. It is clear that $G^*G : H_1 \times H_2 \to H_1 \times H_2$ is $\frac{1}{\|G\|^2}$-inverse-strongly monotone. By Theorem 3.1, we deduce the following result immediately .

Theorem 4.1. *Let H_1, H_2 and H_3 be three Hilbert spaces. Let C be a nonempty closed and convex subset of H_1 and let Q be a nonempty closed and convex subset of H_2. Let $A : H_1 \to H_3$ and $B : H_2 \to H_3$ be two bounded linear operators. Let $S : C \times Q \to C \times Q$ be a nonexpansive mapping. Suppose $Sol(SEP) \cap Fix(S) \neq \emptyset$. Let $\{\omega_n\}$ be a sequence generated in the following algorithm:*

$$\omega_1 \in M, \quad \omega_{n+1} = \alpha_n S\omega_n + (1-\alpha_n) P_M\big(\beta_n \omega_n + (1-\beta_n)(\omega_n - \gamma_n G^* G \omega_n)\big), n \geq 1, \tag{4.3}$$

where $M = C \times Q$, G is defined in (4.2), $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are three real nonnegative sequences such that
(i) $0 < \alpha \leq \alpha_n \leq \alpha' < 1$,
(ii) $0 \leq \beta_n \leq \beta < 1$,

(iii) $0 < \gamma \le \gamma_n \le \gamma' \le \frac{2}{\|A\|^2}$,
where $\alpha, \alpha', \beta, \gamma$ *and* γ' *are four real numbers. Then* $\{\omega_n\}$ *converges weakly to some point in* $Sol(SEP) \cap Fix(S)$.

Remark 4.2. (4.3) can be expressed in terms of x_n and y_n, that is,

$$\begin{cases} x_1 \in C, y_1 \in Q, \\ x_{n+1} = \alpha_n S_1 x_n + (1-\alpha_n) P_C\big(\beta_n x_n + (1-\beta_n)(x_n - \gamma_n A^*(Ax_n - By_n))\big), \ n \ge 1, \\ y_{n+1} = \alpha_n S_2 x_n + (1-\alpha_n) P_Q\big(\beta_n y_n + (1-\beta_n)(y_n + \gamma_n B^*(Ax_n - By_n))\big), \ n \ge 1, \end{cases} \tag{4.4}$$

where $S_1 \times S_2 = S$.

Remark 4.3. Putting $B = I$ in (4.4), we have the following algorithm to solve split feasibility problem (1.1)

$$\begin{cases} x_1 \in C, y_1 \in Q, \\ x_{n+1} = \alpha_n S_1 x_n + (1-\alpha_n) P_C\big(\beta_n x_n + (1-\beta_n)(x_n - \gamma_n A^*(Ax_n - y_n))\big), \quad n \ge 1, \\ y_{n+1} = \alpha_n S_2 x_n + (1-\alpha_n) P_Q\big(\beta_n y_n + (1-\beta_n)(y_n + \gamma_n B^*(Ax_n - y_n))\big), \quad n \ge 1, \end{cases} \tag{4.5}$$

where $S_1 \times S_2 = S$.

If $S = I$ in Theorem 4.1, we have the following result on split equity problem (4.1).

Corollary 4.4. *Let* H_1, H_2 *and* H_3 *be three Hilbert spaces. Let* C *be a nonempty closed and convex subsets of* H_1 *and let* Q *be a nonempty closed and convex subsets of* H_2. *Let* $A : H_1 \to H_3$ *and* $B : H_2 \to H_3$ *be two bounded linear operators. Suppose that split equality problem is consistent. Let* $\{\omega_n\}$ *be a sequence generated in the following algorithm:*

$$\omega_1 \in M, \quad \omega_{n+1} = \alpha_n \omega_n + (1-\alpha_n) P_M\big(\beta_n \omega_n + (1-\beta_n)(\omega_n - \gamma_n G^* G \omega_n)\big), n \ge 1, \tag{4.6}$$

where $M = C \times Q$, G *is defined in (4.2),* $\{\alpha_n\}$, $\{\beta_n\}$ *and* $\{\gamma_n\}$ *are three real nonnegative sequences such that*

(i) $0 < \alpha \le \alpha_n \le \alpha' < 1$,
(ii) $0 \le \beta_n \le \beta < 1$,
(iii) $0 < \gamma \le \gamma_n \le \gamma' \le \frac{2}{\|A\|^2}$,

where $\alpha, \alpha', \beta, \gamma$ *and* γ' *are four real numbers. Then* $\{\omega_n\}$ *converges weakly to some point in* $Sol(SEP)$.

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