Fixed Point Theory, 20(2019), No. 1, 113-134
DOI: 10.24193/fpt-ro.2019.1.07
http://www.math.ubbcluj.ro/~nodeacj/sfptcj.html

SYSTEMS OF VARIATIONAL INEQUALITIES WITH HIERARCHICAL VARIATIONAL INEQUALITY CONSTRAINTS FOR LIPSCHITZIAN PSEUDOCONTRACTIONS

LU-CHUAN CENG*, ADRIAN PETRUŞEL**, JEN-CHIH YAO*** AND YONGHONG YAO****

*Department of Mathematics, Shanghai Normal University
Shanghai 200234, China
E-mail: zenglc@hotmail.com

**Department of Mathematics, Babeş-Bolyai University
Kogălniceanu Str., no. 1, 400084 Cluj-Napoca, Romania
E-mail: petrusel@math.ubbcluj.ro

***Center for General Education, China Medical University
Taichung 40402, Taiwan
and
Department of Applied Mathematics, National Sun Yat-sen University
Kaohsiung, Taiwan 804
E-mail: yaojc@mail.cmu.edu.tw

****Corresponding author. Department of Mathematics
Tianjin Polytechnic University
Tianjin 300387, China
and
School of Mathematics and Information Science, North Minzu University
Yinchuan, 750021, China
E-mail: yaoyonghong@aliyun.com

Abstract. In this paper, we consider the problem of solving a general system of variational inequalities (GSVI) with a hierarchical variational inequality (HVI) constraint for countably many uniformly Lipschitzian pseudocontractive mappings and an accretive operator in a real Banach space. We propose an implicit composite extragradient-like method based on the Mann iteration method, the viscosity approximation method and the Korpelevich extragradient method. Convergence results for the proposed iteration method are also established under some suitable assumptions.

Key Words and Phrases: Implicit composite extragradient-like method, general system of variational inequalities, fixed point, accretive operator, uniform convexity, uniform smoothness.
2010 Mathematics Subject Classification: 49J30, 47H09, 47H10, 47J20.

---

*This research was partially supported by the Innovation Program of Shanghai Municipal Education Commission (15ZZ068), Ph.D. Program Foundation of Ministry of Education of China (20123127110002) and Program for Outstanding Academic Leaders in Shanghai City (15XD1503100)
**This research was partially supported by the grant MOST 106-2923-E-039-001-MY3.

1. Introduction

Let C be a nonempty closed convex subset of a real Hilbert space H with inner product $\langle \cdot, \cdot \rangle$ and induced norm $\| \cdot \|$, respectively. Let P_C be the metric projection from H onto C. Assume that $T : C \to H$ is a mapping on C and that $F(T)$ is the set of fixed points of T, i.e., $F(T) = \{x \in C : x = Tx\}$. We denote by $\mathbb{R}$ the set of all real numbers.

The mapping $T : C \to H$ is called:
(i) monotone if $\langle Tx - Ty, x - y \rangle \geq 0 \ \forall x, y \in C$;
(ii) α-strongly monotone if there exists $\alpha > 0$ such that

$$\langle Tx - Ty, x - y \rangle \geq \alpha \|x - y\|^2 \ \forall x, y \in C;$$

(iii) β-inverse-strongly monotone if there exists $\beta > 0$ such that

$$\langle Tx - Ty, x - y \rangle \geq \beta \|Tx - Ty\|^2, \quad \forall x, y \in C. \tag{1.1}$$

It is clear that each inverse-strongly monotone mapping is monotone and Lipschitz continuous and that each strongly monotone and Lipschitz continuous mapping is inverse-strongly monotone but the converse is not true.

Variational inequality theory has emerged as an important tool in the study of a wide class of obstacle, unilateral, free, moving, equilibrium problems arising in several branches of pure and applied sciences in a unified and general framework. The literature on the variational inequalities is vast. Various efficient methods were developed by many authors and new iterative algorithms for solving other relevant problems were proposed, see e.g., [13], [11], [18], [22], [30], [29], [33], [34]. and the references therein. However, iterative algorithms for solving variational inequalities is still an important and interesting topic.

Let $B_1, B_2 : C \to H$ be two nonlinear mappings. Recently, Ceng et al. [13], considered and studied the problem of finding $(x^*, y^*) \in C \times C$ such that

$$\begin{cases} \langle \rho B_1 y^* + x^* - y^*, x - x^* \rangle \geq 0, & \forall x \in C, \\ \langle \eta B_2 x^* + y^* - x^*, x - y^* \rangle \geq 0, & \forall x \in C, \end{cases} \tag{1.2}$$

which is called a general system of variational inequalities (GSVI), where ρ and η are two positive constants. In [13], problem (1.2) is transformed into a fixed point problem in the following way.

Lemma 1.1. (See [13]). *For given $x^*, y^* \in C$, (x^*, y^*) is a solution of problem (1.2) if and only if $x^* \in \mathrm{GSVI}(C, B_1, B_2)$, where $\mathrm{GSVI}(C, B_1, B_2)$ is the fixed point set of the mapping*

$$G := P_C(I - \rho B_1) P_C(I - \eta B_2),$$

and $y^* = P_C(I - \eta B_2)x^*$.

Utilizing the equivalence between the problem (1.2) and the fixed-point problem, Ceng et al. [13] proposed a relaxed extragradient method for solving problem (1.2) and proved the strong convergence of the proposed method to a solution of problem (1.2).

On the other hand, let E be a real Banach space whose dual space is denoted by E^*. The normalized duality mapping $J : E \to 2^{E^*}$ is defined by

$$J(x) = \{\varphi \in E^* : \langle x, \varphi\rangle = \|x\|^2 = \|\varphi\|^2\}, \quad \forall x \in E,$$

where $\langle \cdot, \cdot\rangle$ denotes the generalized duality pairing between E and E^*. Recall that if E is smooth then J is single-valued. In the sequel, we denote by j the single-valued normalized duality mapping.

Let E be a smooth Banach space. Let $B_1, B_2 : C \to E$ be two nonlinear mappings. The general system of variational inequalities (GSVI) is to find $(x^*, y^*) \in C \times C$ such that

$$\begin{cases} \langle \rho B_1 y^* + x^* - y^*, j(x - x^*)\rangle \geq 0, & \forall x \in C, \\ \langle \eta B_2 x^* + y^* - x^*, j(x - y^*)\rangle \geq 0, & \forall x \in C, \end{cases} \tag{1.3}$$

where ρ and η are two positive constants. In particular, if $E = H$ a Hilbert space, it is easy to see that the GSVI (1.3) reduces to the GSVI (1.2).

In 2005, Verma [30] proved strong convergence of a two-step projection method for solving GSVI (1.2). This method proposed in [30] contains several known as well as new projection schemes as special cases, while some have been applied to the problems arising, especially from complementarity problems, convex quadratic programming and other variational problems, see [8, 12, 14, 29, 35, 36, 37] and the references therein. Many authors have studied the problems of finding a common element of the set of fixed points of nonlinear mappings and of the set of solutions to variational inequalities by iterative methods.

Furthermore, within the period of past 30 years, a great deal of effort has gone into the existence of zeros of accretive mappings or fixed points of pseudocontractive mappings (including nonexpansive mappings) and iterative construction of zeros of accretive mappings and of fixed points of pseudocontractive mappings, see, e.g., [4, 11, 14, 16, 17, 19, 26, 32]. In 2011, Ceng et al. [11] introduced an implicit viscosity approximation method for computing approximate fixed points of a pseudocontractive mapping T, deriving strong convergence theorems to a fixed point of T.

Motivated and inspired by the above researches, we introduce an implicit composite extragradient-like method for solving the general system of variational inequalities (GSVI) (1.3) with a hierarchical variational inequality (HVI) constraint for countably many uniformly Lipschitzian pseudocontractive self-mappings and an accretive operator. Under quite mild assumptions, we prove some convergence results for the proposed iteration method in a real Banach space. Our results improve, extend and develop the corresponding ones in the literature, see, [9, 11, 13, 14, 30].

2. Preliminaries

Let E be a real Banach space with the dual E^*. Throughout this paper we write $x_n \rightharpoonup x$ (respectively, $x_n \to x$) to indicate that the sequence $\{x_n\}$ converges weakly (respectively, strongly) to x. Let C be a nonempty closed convex subset of E. Recall that a mapping $T : C \to E$ is said to be:

(a) accretive if, for each $x, y \in C$, there exists $j(x - y) \in J(x - y)$ such that

$$\langle Tx - Ty, j(x - y)\rangle \geq 0,$$

where J is the normalized duality mapping;

(b) α-strongly accretive if, for each $x, y \in C$, there exists $j(x-y) \in J(x-y)$ such that

$$\langle Tx - Ty, j(x-y)\rangle \geq \alpha\|x-y\|^2 \quad \text{for some } \alpha \in (0,1);$$

(c) λ-strictly pseudocontractive if, for each $x, y \in C$, there exists $j(x-y) \in J(x-y)$ such that

$$\langle Tx - Ty, j(x-y)\rangle \leq \|x-y\|^2 - \lambda\|(I-T)x - (I-T)y\|^2, \quad \text{for some } \lambda \in (0,1).$$

(d) pseudocontractive if, for each $x, y \in C$, there exists $j(x-y) \in J(x-y)$ such that

$$\langle Tx - Ty, j(x-y)\rangle \leq \|x-y\|^2.$$

Next, we introduce a concept, which will be used in the proof of our main results.

Definition 2.1. Let C be a nonempty closed convex subset of a real Banach space E. Let $\{T_n\}_{n=0}^{\infty}$ be a sequence of continuous pseudocontractive self-mappings on C. Then $\{T_n\}_{n=0}^{\infty}$ is said to be a countable family of ℓ-uniformly Lipschitzian pseudocontractive self-mappings on C if there exists a constant $\ell > 0$ such that each T_n is ℓ-Lipschitz continuous.

Example 2.1. Let $\{T_n\}_{n=0}^{\infty}$ be a sequence of λ-strictly pseudocontractive self-mappings on C. Then $\{T_n\}_{n=0}^{\infty}$ is a countable family of ℓ-uniformly Lipschitzian pseudocontractive self-mappings on C with $\ell = 1 + \frac{1}{\lambda}$.

Some necessary notions and results are presented now.

Proposition 2.1. (See [4]). *Let C be a nonempty closed convex subset of a Banach space E. Let $T_0, T_1, \ldots$ be a sequence of mappings of C into itself. Suppose that*

$$\sum_{n=0}^{\infty} \sup\{\|T_n x - T_{n-1}x\| : x \in C\} < \infty.$$

Then, for each $y \in C$, $\{T_n y\}$ converges strongly to some point of C. Moreover, let T be a mapping of C into itself defined by $Ty = \lim_{n\to\infty} T_n y$ for all $y \in C$. Then $\lim_{n\to\infty} \sup\{\|Tx - T_n x\| : x \in C\} = 0$.

Let D be a subset of C and let Π be a mapping of C into D. Then Π is said to be sunny if $\Pi[\Pi(x) + t(x - \Pi(x))] = \Pi(x)$, whenever $\Pi(x) + t(x - \Pi(x)) \in C$ for $x \in C$ and $t \geq 0$. A mapping Π of C into itself is called a retraction if $\Pi^2 = \Pi$. If a mapping Π of C into itself is a retraction, then $\Pi(z) = z$ for each $z \in R(\Pi)$, where $R(\Pi)$ is the range of Π. A subset D of C is called a sunny nonexpansive retract of C if there exists a sunny nonexpansive retraction from C onto D.

Proposition 2.2. (See [26]). *Let C be a nonempty closed convex subset of a smooth Banach space E, D be a nonempty subset of C and Π be a retraction of C onto D. Then the following are equivalent:*

(i) Π is sunny and nonexpansive;

(ii) $\|\Pi(x) - \Pi(y)\|^2 \leq \langle x - y, j(\Pi(x) - \Pi(y))\rangle, \forall x, y \in C$;

(iii) $\langle x - \Pi(x), j(y - \Pi(x))\rangle \leq 0, \forall x \in C, y \in D$.

It is well known that if E is a Hilbert space, then a sunny nonexpansive retraction Π_C of E onto C coincides with the metric projection of E onto C. Let C be a nonempty closed convex subset of a uniformly convex and uniformly smooth Banach space E and let T be a nonexpansive mapping of C into itself with the fixed point set $F(T) \neq \emptyset$. Then the set $F(T)$ is a sunny nonexpansive retract of C.

Recall that a (possibly multivalued) operator $A \subset E \times E$ with domain $D(A)$ and range $R(A)$ in a real Banach space E is accretive if, for each $x_i \in D(A)$ and $y_i \in Ax_i$ $(i = 1, 2)$, there exists $j(x_1 - x_2) \in J(x_1 - x_2)$ such that $\langle y_1 - y_2, j(x_1 - x_2)\rangle \geq 0$. An accretive operator A is said to satisfy the range condition if $\overline{D(A)} \subset R(I + rA)$ for all $r > 0$. An accretive operator A is m-accretive if $R(I + rA) = E$ for each $r > 0$. If A is an accretive operator which satisfies the range condition, then we can define, for each $r > 0$ a mapping $J_r : R(I + rA) \to D(A)$ by $J_r = (I + rA)^{-1}$, which is called the resolvent of A. It is well known that J_r is nonexpansive and $F(J_r) = A^{-1}0$ for all $r > 0$. Hence,

$$F(J_r) = A^{-1}0 = \{x \in D(A) : 0 \in Ax\}.$$

If $A^{-1}0 \neq \emptyset$, then the inclusion $0 \in Ax$ is solvable. The following resolvent identity is well known. More details on accretive operators can be found in [15].

Proposition 2.3. *(Resolvent identity). For $\lambda, \mu > 0$ and $x \in E$,*

$$J_\lambda x = J_\mu \left(\frac{\mu}{\lambda}x + \left(1 - \frac{\mu}{\lambda}\right) J_\lambda x\right).$$

Proposition 2.4. (See [22]). *Let E be a smooth and uniformly convex Banach space, and let $r > 0$. Then there exists a strictly increasing, continuous, and convex function $g : [0, 2r] \to \mathbf{R}$, $g(0) = 0$ such that*

$$g(\|x - y\|) \leq \|x\|^2 - 2\langle x, j(y)\rangle + \|y\|^2, \quad \forall x, y \in B_r,$$

where $B_r = \{x \in E : \|x\| \leq r\}$.

Proposition 2.5. (See [31]). *Given a number $r > 0$. A real Banach space E is uniformly convex if and only if there exists a continuous strictly increasing function $g : [0, \infty) \to [0, \infty)$, $g(0) = 0$ such that*

$$\|\lambda x + (1 - \lambda)y\|^2 \leq \lambda\|x\|^2 + (1 - \lambda)\|y\|^2 - \lambda(1 - \lambda)g(\|x - y\|)$$

for all $\lambda \in [0, 1]$ and $x, y \in E$ such that $\|x\| \leq r$ and $\|y\| \leq r$.

In order to prove our main results, we need to use some lemmas in the sequel. The following lemma is an immediate consequence of the subdifferential inequality of the function $\frac{1}{2}\|\cdot\|^2$.

Lemma 2.6. *Let E be a real Banach space and J be the normalized duality mapping on E. Then for any given $x, y \in E$, the following inequality holds:*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, j(x + y)\rangle, \quad \forall j(x + y) \in J(x + y).$$

Lemma 2.7. (See [14], Lemma 3.2). *Let C be a nonempty closed convex subset of a smooth Banach space E. Let Π_C be a sunny nonexpansive retraction from E onto*

C, and let the mapping $B_i : C \to E$ be λ_i-strictly pseudocontractive and ζ_i-strongly accretive with $\lambda_i + \zeta_i \geq 1$ for $i \in \{1, 2\}$. Let $G : C \to C$ be the mapping defined by

$$G := \varPi_C(I - \rho B_1)\varPi_C(I - \eta B_2).$$

If

$$1 - \frac{\lambda_1}{1+\lambda_1}\left(1 - \sqrt{\frac{1-\zeta_1}{\lambda_1}}\right) \leq \rho \leq 1 \text{ and } 1 - \frac{\lambda_2}{1+\lambda_2}\left(1 - \sqrt{\frac{1-\zeta_2}{\lambda_2}}\right) \leq \eta \leq 1,$$

then $G : C \to C$ is nonexpansive.

Lemma 2.8. (See [5], Lemma 3). *Let C be a nonempty closed convex subset of a strictly convex Banach space E. Let $\{T_n\}_{n=0}^{\infty}$ be a sequence of nonexpansive mappings on C. Suppose that $\bigcap_{n=0}^{\infty} F(T_n)$ is nonempty. Let $\{\lambda_n\}$ be a sequence of positive numbers with $\sum_{n=0}^{\infty} \lambda_n = 1$. Then a mapping S on C defined by $Sx = \sum_{n=0}^{\infty} \lambda_n T_n x$ for $x \in C$ is defined well, nonexpansive and $F(S) = \bigcap_{n=0}^{\infty} F(T_n)$ holds.*

Lemma 2.9. (See [32]). *Let E be a uniformly smooth Banach space, C be a nonempty closed convex subset of E, $T : C \to C$ be a nonexpansive mapping with $F(T) \neq \emptyset$, and $f \in \varXi_C$ (the collection of all contractive self-mappings on C). Then the net $\{x_t\}$ defined by $x_t = tf(x_t) + (1-t)Tx_t$, $\forall t \in (0, 1)$, converges strongly to a point in $F(T)$. If we define a mapping $Q : \varXi_C \to F(T)$ by*

$$Q(f) := s - \lim_{t \to 0} x_t, \forall f \in \varXi_C,$$

then $Q(f)$ solves the VI: $\langle (I - f)Q(f), j(Q(f) - x)\rangle \leq 0, \quad \forall x \in F(T)$.

Recall that a gauge is a continuous strictly increasing function $\varphi : [0, \infty) \to [0, \infty)$ such that $\varphi(0) = 0$ and $\varphi(t) \to \infty$ as $t \to \infty$. Associated to the gauge φ is the duality map $J_\varphi : E \to 2^{E^*}$ defined by

$$J_\varphi(x) = \{\xi \in E^* : \langle x, \xi\rangle = \|x\|\varphi(\|x\|), \|\xi\| = \varphi(\|x\|)\}, \quad \forall x \in E.$$

We say that a Banach space E has a weakly continuous duality map if there exists a gauge φ for which the duality map J_φ is single-valued and weak-to-weak* sequentially continuous. It is known that l^p has a weakly continuous duality map with gauge $\varphi(t) = t^{p-1}$ for all $1 < p < \infty$. Set

$$\varPhi(t) = \int_0^t \varphi(s)ds, \quad \forall t \geq 0.$$

Then $J_\varphi(x) = \partial \varPhi(\|x\|)$ for all $x \in E$, where ∂ denotes the subdifferential in the sense of convex analysis; see [28] for more details.

The first part of the following lemma is an immediate consequence of the subdifferential inequality, and the proof of the second part can be found in [23]. In what follows, we denote by j_φ the single-valued duality map J_φ.

Proposition 2.10. *Assume that E has a weakly continuous duality map j_φ with gauge φ.*

(i) For all $x, y \in E$, the following inequality holds:

$$\Phi(\|x+y\|) \leq \Phi(\|x\|) + \langle y, j_\varphi(x+y)\rangle.$$

(ii) Assume that a sequence $\{x_n\}$ in E is weakly convergent to a point x. Then the following identity holds:

$$\limsup_{n\to\infty} \Phi(\|x_n - y\|) = \limsup_{n\to\infty} \Phi(\|x_n - x\|) + \Phi(\|y - x\|), \quad \forall y \in E.$$

Lemma 2.11. (See [17], Theorem 3.1). *Let E be a reflexive Banach space and have a weakly continuous duality map j_φ with gauge φ, let C be a nonempty closed convex subset of E, let $T : C \to C$ be a nonexpansive mapping with $F(T) \neq \emptyset$, and let $f \in \Xi_C$. Then the net $\{x_t\}$ defined by $x_t = tf(x_t) + (1-t)Tx_t$, $\forall t \in (0,1)$, converges strongly to a point in $F(T)$ as $t \to 0^+$. Define $Q : \Xi_C \to F(T)$ by*

$$Q(f) := s - \lim_{t\to 0^+} x_t.$$

Then $Q(f)$ solves the VI

$$\langle (I-f)Q(f), j_\varphi(Q(f)-x)\rangle \leq 0, \quad \forall x \in F(T).$$

Lemma 2.12. (See [32], Lemma 2.1). *Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying the condition*

$$a_{n+1} \leq (1-\mu_n)a_n + \mu_n\nu_n, \quad \forall n \geq 0,$$

where $\{\mu_n\}$ and $\{\nu_n\}$ are sequences of real numbers such that

(i) $\{\mu_n\} \subset [0,1]$ and $\sum_{n=0}^{\infty} \mu_n = \infty$, or equivalently,

$$\prod_{n=0}^{\infty}(1-\mu_n) := \lim_{n\to\infty}\prod_{i=0}^{n}(1-\mu_i) = 0;$$

(ii) $\limsup_{n\to\infty} \nu_n \leq 0$, or $\sum_{n=0}^{\infty} \mu_n|\nu_n| < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.13. (See [27], Lemma 2). *Let $\{x_n\}$ and $\{z_n\}$ be bounded sequences in a Banach space E, and let $\{\beta_n\}$ be a sequence of nonnegative numbers in $[0,1]$ with $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$. Suppose that $x_{n+1} = \beta_n x_n + (1-\beta_n)z_n$ for all integers $n \geq 0$ and $\limsup_{n\to\infty}(\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|) \leq 0$. Then $\lim_{n\to\infty} \|x_n - z_n\| = 0$.*

Lemma 2.14. (See [7]). *Let C be a nonempty closed convex subset of a real smooth Banach space E, and let $F : C \to E$ be a mapping.*

(i) If $F : C \to E$ is δ-strongly accretive and ζ-strictly pseudocontractive with $\delta + \zeta \geq 1$, then $I - F$ is nonexpansive, and F is Lipschitz continuous with constant $1 + \frac{1}{\zeta}$;

(ii) If $F : C \to E$ is δ-strongly accretive and ζ-strictly pseudocontractive with $\delta + \zeta \geq 1$, then for any fixed $\alpha \in (0, 1)$, $I - \alpha F$ is a contraction with coefficient $1 - \alpha\left(1 - \sqrt{\frac{1-\delta}{\zeta}}\right)$.

3. Formulations and Main Results

Throughout the paper, unless otherwise specified, C is assumed to be a nonempty closed convex of a uniformly convex Banach space E. In addition, we assume that E is either uniformly smooth or has a weakly continuous duality map j_φ (with φ a gauge function) and Π_C is assumed to be a sunny nonexpansive retraction from E onto C. Let $B_1, B_2 : C \to H$ be two nonlinear mappings. The general system of variational inequalities (GSVI) means to find $(x^*, y^*) \in C \times C$ such that

$$\begin{cases} \langle \rho B_1 y^* + x^* - y^*, j(x - x^*)\rangle \geq 0, & \forall x \in C, \\ \langle \eta B_2 x^* + y^* - x^*, j(x - y^*)\rangle \geq 0, & \forall x \in C, \end{cases}$$

where ρ and η are positive constants. We denote the solution set of this problem by $S(C, B_1, B_2)$. If $\mathrm{GSVI(C, B_1, B_2)}$ is the fixed point set of the mapping

$$G := \mathit{\Pi}_C(I - \rho B_1)\mathit{\Pi}_C(I - \eta B_2),$$

then we have that $(x^*, y^*) \in S(C, B_1, B_2)$ if and only if $x^* \in \mathrm{GSVI}(C, B_1, B_2)$ and $y^* = \mathit{\Pi}_C(I - \eta B_2)x^*$.

Let $A \subset E \times E$ be an accretive operator such that $D(A) \subset C$. The problem of finding the zeros of the accretive operator A means to find $x^* \in D(A)$ such that

$$x^* \in A^{-1}(0).$$

Let $\{T_n\}_{n=0}^{\infty}$ be a countable family of ℓ-uniformly Lipschitzian pseudocontractive self-mappings on C. Then the problem of finding a common fixed point for this family means to find $x^* \in C$ such that

$$x^* \in \bigcap_{n=0}^{\infty} F(T_n).$$

In this paper, we consider the problem of finding a common solution of above mentioned three problems, that is, to find $x^* \in C$ such that

$$x^* \in \bigcap_{n=0}^{\infty} F(T_n) \cap A^{-1}(0) \cap GSVI(C; B_1; B_2).$$

We are now in a position to state and prove the main results in this paper.

Theorem 3.1. *Let C be a nonempty closed convex subset of a uniformly convex Banach space E. Assume, in addition, that E either is uniformly smooth or has a weakly continuous duality map j_φ with gauge φ. Let $\mathit{\Pi}_C$ be a sunny nonexpansive retraction from E onto C. Let $A \subset E \times E$ be an accretive operator in E such that*

$$\overline{D(A)} \subset C \subset \bigcap_{r>0} R(I + rA).$$

Let the mapping $B_i : C \to E$ be ζ_i-strictly pseudocontractive and ν_i-strongly accretive with $\zeta_i + \nu_i \geq 1$ for $i \in \{1, 2\}$. Let $f \in \mathit{\Xi}_C$ with a contractive coefficient $k \in (0, 1)$, and *let $F : C \to E$ be δ-strongly accretive and ζ-strictly pseudocontractive with $\delta + \zeta \geq 1$. Let $\{T_n\}_{n=0}^{\infty}$ be a countable family of ℓ-uniformly Lipschitzian pseudocontractive self-mappings on C such that*

$$\Omega := \bigcap_{n=0}^{\infty} F(T_n) \cap \mathrm{GSVI}(C, B_1, B_2) \cap A^{-1}0 \neq \emptyset$$

where $\mathrm{GSVI}(C, B_1, B_2)$ is the fixed point set of the mapping

$$G := \Pi_C(I - \rho B_1)\Pi_C(I - \eta B_2)$$

with $1 - \frac{\zeta_1}{1+\zeta_1}\left(1 - \sqrt{\frac{1-\nu_1}{\zeta_1}}\right) \leq \rho \leq 1$ and $1 - \frac{\zeta_2}{1+\zeta_2}\left(1 - \sqrt{\frac{1-\nu_2}{\zeta_2}}\right) \leq \eta \leq 1$. For an arbitrary $x_0 \in C$, let $\{x_n\}$ be generated by

$$\begin{cases} z_n = \sigma_n x_n + (1 - \sigma_n)T_n z_n, \\ y_n = \beta_n f(x_n) + (1 - \beta_n)[\mu I + (1 - \mu)\Pi_C(I - \alpha_n F)]\Pi_C(I - \rho B_1)\Pi_C(I - \eta B_2)z_n, \\ x_{n+1} = \gamma_n x_n + (1 - \gamma_n)[\lambda y_n + (1 - \lambda)J_{r_n} y_n], \quad \forall n \geq 0, \end{cases} \tag{3.1}$$

where $\lambda, \mu \in (0, 1)$, $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ and $\{\sigma_n\}$ are the sequences in $(0, 1)$ and $\{r_n\}$ is a sequence in $(0, \infty)$. Suppose that the following conditions hold:

(i) $\lim_{n\to\infty} \beta_n = 0$ and $\sum_{n=0}^{\infty} \beta_n = \infty$;

(ii) $\lim_{n\to\infty} |\sigma_{n+1} - \sigma_n| = 0$ and $\lim_{n\to\infty} \alpha_n/\beta_n = 0$;

(iii) $0 < \liminf_{n\to\infty} \gamma_n \leq \limsup_{n\to\infty} \gamma_n < 1$ and $0 < \liminf_{n\to\infty} \sigma_n \leq \limsup_{n\to\infty} \sigma_n < 1$;

(iv) $\lim_{n\to\infty} |r_{n+1} - r_n| = 0$ and $r_n \geq \varepsilon > 0$ $\forall n \geq 0$.

Assume that $\sum_{n=1}^{\infty} \sup_{x\in D} \|T_n x - T_{n-1}x\| < \infty$ for any bounded subset D of C, and let T be a mapping of C into itself defined by $Tx = \lim_{n\to\infty} T_n x$ for all $x \in C$, and suppose that $F(T) = \bigcap_{n=0}^{\infty} F(T_n)$. Then the following statements hold:

(a) if E is uniformly smooth, then $\{x_n\}$ converges strongly to $x^* \in \Omega$, which solves the VI: $\langle (I - f)x^*, j(x^* - x)\rangle \leq 0$, $\forall x \in \Omega$;

(b) if E has a weakly continuous duality map j_φ with gauge φ, then $\{x_n\}$ converges strongly to $x^* \in \Omega$, provided $\|x_n - y_n\| = o(\beta_n)$, which solves the VI:

$$\langle (I - f)x^*, j_\varphi(x^* - x)\rangle \leq 0, \ \forall x \in \Omega.$$

Proof. First of all, taking into account $0 < \liminf_{n\to\infty} \sigma_n \leq \limsup_{n\to\infty} \sigma_n < 1$, we may assume, without loss of generality, that $\{\sigma_n\} \subset [a, b] \subset (0, 1)$ for some $a, b \in (0, 1)$. Note that the mapping $G : C \to C$ is defined as $G := \Pi_C(I - \rho B_1)\Pi_C(I - \eta B_2)$, where $1 - \frac{\zeta_1}{1+\zeta_1}\left(1 - \sqrt{\frac{1-\nu_1}{\zeta_1}}\right) \leq \rho \leq 1$ and $1 - \frac{\zeta_2}{1+\zeta_2}\left(1 - \sqrt{\frac{1-\nu_2}{\zeta_2}}\right) \leq \eta \leq 1$. So, by Lemma 2.7, we obtain that G is nonexpansive. It is easy to see that for each $n \geq 0$ there exists a unique element $z_n \in C$ such that

$$z_n = \sigma_n x_n + (1 - \sigma_n)T_n z_n. \tag{3.2}$$

So, it can be readily seen that the implicit composite iterative scheme (3.1) can be rewritten as

$$\begin{cases} z_n = \sigma_n x_n + (1-\sigma_n)T_n z_n, \\ y_n = \beta_n f(x_n) + (1-\beta_n)[\mu G z_n + (1-\mu)\Pi_C(I-\alpha_n F)Gz_n], \\ x_{n+1} = \gamma_n x_n + (1-\gamma_n)[\lambda y_n + (1-\lambda)J_{r_n} y_n], \quad \forall n \geq 0, \end{cases} \tag{3.3}$$

Next, we divide the rest of the proof into several steps.

Step 1. We claim that $\{x_n\}, \{y_n\}, \{z_n\}, \{Gz_n\}, \{F(Gz_n)\}, \{J_{r_n} y_n\}$ and $\{T_n z_n\}$ are bounded. Indeed, take an element $p \in \Omega = \bigcap_{n=0}^{\infty} F(T_n) \cap \mathrm{GSVI}(C, B_1, B_2) \cap A^{-1}0$ arbitrarily. Then we have $T_n p = p$, $Gp = p$ and $J_{r_n} p = p$. Since each $T_n : C \to C$ is a pseudocontraction mapping, it follows that

$$\|z_n - p\|^2 \leq \sigma_n \|x_n - p\| \|z_n - p\| + (1-\sigma_n)\|z_n - p\|^2,$$

which hence yields

$$\|z_n - p\| \leq \|x_n - p\|, \quad \forall n \geq 0. \tag{3.4}$$

Because $\lim_{n\to\infty} \alpha_n/\beta_n = 0$, we may assume without loss of generality that $\alpha_n \leq \beta_n$ for all $n \geq 0$. Noting that $F : C \to E$ is δ-strongly accretive and ζ-strictly pseudocontractive with $\delta + \zeta \geq 1$, we deduce from (3.3), (3.4) and Lemma 2.14 that

$$\begin{aligned} \|y_n - p\| &\leq \beta_n(\|f(x_n) - f(p)\| + \|f(p) - p\|) \\ &+(1-\beta_n)\{\mu\|Gz_n - p\| + (1-\mu)[\|(I-\alpha_n F)Gz_n - (I-\alpha_n F)p\| + \alpha_n\|F(p)\|]\} \\ &\leq \beta_n k\|x_n - p\| + \beta_n\|f(p) - p\| \\ &+(1-\beta_n)\{\mu\|Gz_n - p\| + (1-\mu)[(1-\alpha_n\tau)\|Gz_n - p\| + \alpha_n\|F(p)\|]\} \\ &\leq (1-(1-k)\beta_n)\|x_n - p\| + (1-k)\beta_n \frac{\|f(p) - p\| + \|F(p)\|}{1-k} \\ &\leq \max\left\{\|x_n - p\|, \frac{\|f(p) - p\| + \|F(p)\|}{1-k}\right\}, \end{aligned} \tag{3.5}$$

where $\tau := 1 - \sqrt{\frac{1-\delta}{\zeta}} \in [0,1)$. Since J_{r_n} is nonexpansive, from (3.3) and (3.5) it follows that

$$\begin{aligned} \|x_{n+1} - p\| &\leq \gamma_n\|x_n - p\| + (1-\gamma_n)\|y_n - p\| \\ &\leq \max\left\{\|x_n - p\|, \frac{\|f(p) - p\| + \|F(p)\|}{1-k}\right\}, \ \forall\ n \geq 0. \end{aligned}$$

By induction, we conclude that

$$\|x_n - p\| \leq \max\left\{\|x_0 - p\|, \frac{\|f(p) - p\| + \|F(p)\|}{1-k}\right\}, \quad \forall n \geq 0. \tag{3.6}$$

It then follows that $\{x_n\}$ is bounded, and so are the sequences $\{y_n\}$, $\{z_n\}$, $\{Gz_n\}$, $\{F(Gz_n)\}$ and $\{J_{r_n} y_n\}$ (due to (3.4), (3.5) and the Lipschitz continuity of G, J_{r_n} and F). Since $\{T_n\}$ is ℓ-uniformly Lipschitzian on C, we know that $\{T_n z_n\}$ is bounded.

Step 2. We claim that $\|x_{n+1} - x_n\| \to 0$ as $n \to \infty$. Indeed, writing

$$x_{n+1} = \gamma_n x_n + (1 - \gamma_n)\hat{z}_n,$$

where $\hat{z}_n := \lambda y_n + (1 - \lambda)J_{r_n} y_n, \ \forall n \geq 0$, we get

$$\hat{z}_{n+1} - \hat{z}_n = \lambda(y_{n+1} - y_n) + (1 - \lambda)(J_{r_{n+1}} y_{n+1} - J_{r_n} y_n). \tag{3.7}$$

From (3.3) we have

$$y_n = \beta_n f(x_n) + (1 - \beta_n)W_n z_n,$$

where

$$W_n := \mu G + (1 - \mu)\Pi_C(I - \alpha_n F)G, \ \forall \ n \geq 0.$$

Simple calculations show that

$$\begin{aligned} y_n - y_{n-1} \quad &= \beta_n(f(x_n) - f(x_{n-1})) + (\beta_n - \beta_{n-1})(f(x_{n-1}) - W_{n-1}z_{n-1}) \\ &+(1 - \beta_n)(W_n z_n - W_{n-1}z_{n-1}). \end{aligned} \tag{3.8}$$

Also, simple calculations show that

$$z_n - z_{n-1} = \sigma_n(x_n - x_{n-1}) + (1 - \sigma_n)(T_n z_n - T_{n-1}z_{n-1}) + (\sigma_n - \sigma_{n-1})(x_{n-1} - T_{n-1}z_{n-1}),$$

which hence yields

$$\begin{aligned} \|z_n - z_{n-1}\|^2 \leq \sigma_n\|x_n - x_{n-1}\|\|z_n - z_{n-1}\| + (1 - \sigma_n)[\|T_n z_n - T_{n-1}z_n\|\|z_n - z_{n-1}\| \\ + \|z_n - z_{n-1}\|^2] + |\sigma_n - \sigma_{n-1}|\|x_{n-1} - T_{n-1}z_{n-1}\|\|z_n - z_{n-1}\|. \end{aligned}$$

So it follows that

$$\|z_n - z_{n-1}\| \leq \|x_n - x_{n-1}\| + \frac{1}{a}\|T_n z_n - T_{n-1}z_n\| + |\sigma_n - \sigma_{n-1}|\frac{\|x_{n-1} - T_{n-1}z_{n-1}\|}{a}. \tag{3.9}$$

Putting $D = \{z_n : n \geq 0\}$, we know that D is a bounded subset of C. Then by the assumption we get

$$\sum_{n=1}^{\infty} \sup_{x \in D} \|T_n x - T_{n-1}x\| < \infty.$$

Noting that $\|T_n z_n - T_{n-1}z_n\| \leq \sup_{x\in D} \|T_n x - T_{n-1}x\|, \ \forall n \geq 1$, we have

$$\sum_{n=1}^{\infty} \|T_n z_n - T_{n-1}z_n\| < \infty. \tag{3.10}$$

Furthermore, if $r_{n-1} \leq r_n$, using

$$J_{r_n} y_n = J_{r_{n-1}} \left(\frac{r_{n-1}}{r_n} y_n + (1 - \frac{r_{n-1}}{r_n}) J_{r_n} y_n \right)$$

(due to Proposition 2.3), we get

$$\|J_{r_n} y_n - J_{r_{n-1}} y_{n-1}\| \leq \|y_n - y_{n-1}\| + \frac{1}{\varepsilon}|r_n - r_{n-1}|\|J_{r_n} y_n - y_{n-1}\|.$$

If $r_n \leq r_{n-1}$, we similarly get

$$\|J_{r_n} y_n - J_{r_{n-1}} y_{n-1}\| \leq \|y_{n-1} - y_n\| + \frac{1}{\varepsilon}|r_{n-1} - r_n|\|J_{r_{n-1}} y_{n-1} - y_n\|.$$

Thus, combining the above cases, we obtain

$$\|J_{r_n}y_n - J_{r_{n-1}}y_{n-1}\| \leq \|y_n - y_{n-1}\| + M_0|r_n - r_{n-1}|, \quad \forall n \geq 1, \tag{3.11}$$

where

$$\sup_{n\geq 1}\left\{\frac{1}{\varepsilon}(\|J_{r_n}y_n - y_{n-1}\| + \|J_{r_{n-1}}y_{n-1} - y_n\|)\right\} \leq M_0$$

for some $M_0 > 0$. Utilizing Lemma 2.14, we deduce from (3.8) and (3.9) that for all $n \geq 1$

$$\begin{aligned}
\|y_n - y_{n-1}\| &\leq \beta_n\|f(x_n) - f(x_{n-1})\| + |\beta_n - \beta_{n-1}|\|f(x_{n-1}) - W_{n-1}z_{n-1}\| \\
&+(1-\beta_n)[\mu\|Gz_n - Gz_{n-1}\| + (1-\mu)\|\varPi_C(I-\alpha_nF)Gz_n - \varPi_C(I-\alpha_{n-1}F)Gz_{n-1}\|] \\
&\leq \beta_nk\|x_n - x_{n-1}\| + |\beta_n - \beta_{n-1}|\|f(x_{n-1}) - W_{n-1}z_{n-1}\| + (1-\beta_n)[\|x_n - x_{n-1}\| \\
&+\frac{1}{a}\|T_nz_n - T_{n-1}z_n\| + |\sigma_n - \sigma_{n-1}|\frac{\|x_{n-1} - T_{n-1}z_{n-1}\|}{a}] + |\alpha_n - \alpha_{n-1}|\|F(Gz_{n-1})\| \\
&\leq \|x_n - x_{n-1}\| + M(|\alpha_n - \alpha_{n-1}| + |\beta_n - \beta_{n-1}| \\
&\quad +|\sigma_n - \sigma_{n-1}| + \|T_nz_n - T_{n-1}z_n\|),
\end{aligned} \tag{3.12}$$

where $\tau = 1 - \sqrt{\frac{1-\delta}{\zeta}}$ and

$$\sup_{n\geq 0}\left\{\|f(x_n)\| + \|Gz_n\| + \|\varPi_C(I-\alpha_nF)Gz_n\| + \frac{1}{a} + \frac{\|x_n - T_nz_n\|}{a} + \|F(Gz_n)\|\right\} \leq M$$

for some $M > 0$. Thus, from (3.7), (3.11) and (3.12) we deduce that

$$\begin{aligned}
\|\hat{z}_{n+1} - \hat{z}_n\| &\leq \|x_{n+1} - x_n\| + M(|\alpha_{n+1} - \alpha_n| + |\beta_{n+1} - \beta_n| + |\sigma_{n+1} - \sigma_n| \\
&\quad + \|T_{n+1}z_{n+1} - T_nz_{n+1}\|) + M_0|r_{n+1} - r_n|,
\end{aligned}$$

which implies that

$$\begin{aligned}
\|\hat{z}_{n+1} - \hat{z}_n\| - \|x_{n+1} - x_n\| &\leq M(|\alpha_{n+1} - \alpha_n| + |\beta_{n+1} - \beta_n| + |\sigma_{n+1} - \sigma_n| \\
&\quad +\|T_{n+1}z_{n+1} - T_nz_{n+1}\|) + M_0|r_{n+1} - r_n|.
\end{aligned} \tag{3.13}$$

From (3.10) and conditions (i), (ii), (iv) we get $\limsup_{n\to\infty}(\|\hat{z}_{n+1} - \hat{z}_n\| - \|x_{n+1} - x_n\|) \leq 0$. It follows from Lemma 2.13 and condition (iii) that $\lim_{n\to\infty}\|\hat{z}_n - x_n\| = 0$.
Hence we get

$$\lim_{n\to\infty}\|x_{n+1} - x_n\| = \lim_{n\to\infty}(1-\gamma_n)\|\hat{z}_n - x_n\| = 0. \tag{3.14}$$

Step 3. We claim that $\|x_n - Gx_n\| \to 0$, $\|J_{r_n}x_n - x_n\| \to 0$ and $\|T_nx_n - x_n\| \to 0$ as $n \to \infty$. Indeed, we put $u_n := \varPi_C(I - \alpha_nF)Gz_n$. According to Proposition 2.2 (iii) we have

$$\langle (I-\alpha_nF)Gz_n - \varPi_C(I-\alpha_nF)Gz_n, j(p - u_n)\rangle \leq 0,$$

which hence leads to

$$\begin{aligned}
\|u_n - p\|^2 &= \langle \varPi_C(I-\alpha_nF)Gz_n - (I-\alpha_nF)Gz_n, j(u_n - p)\rangle \\
&+ \langle (I-\alpha_nF)Gz_n - p, j(u_n - p)\rangle \\
&\leq \frac{1}{2}\|Gz_n - p\|^2 + \frac{1}{2}\|u_n - p\|^2 + \alpha_n\|F(p)\|\|p - u_n\|.
\end{aligned}$$

It immediately follows that

$$\|u_n - p\|^2 \le \|Gz_n - p\|^2 + 2\alpha_n \|F(p)\|\|p - u_n\|. \tag{3.15}$$

Utilizing Proposition 2.5, from (3.3) and (3.4) we get

$$\begin{aligned}
\|y_n - p\|^2 &\le \beta_n k\|x_n - p\|^2 + (1-\beta_n)[\mu\|Gz_n - p\|^2 + (1-\mu)\|u_n - p\|^2] \\
&\quad + 2\beta_n\langle f(p) - p, j(y_n - p)\rangle \\
&\le [1-(1-k)\beta_n]\|x_n - p\|^2 + 2\alpha_n\|F(p)\|\|p - u_n\| + 2\beta_n\langle f(p) - p, j(y_n - p)\rangle \\
&\le \|x_n - p\|^2 + 2\alpha_n\|F(p)\|\|p - u_n\| + 2\beta_n\|f(p) - p\|\|y_n - p\|, \qquad (3.16)
\end{aligned}$$

which together with Proposition 2.5, implies that

$$\begin{aligned}
\|x_{n+1} - p\|^2 &\le \gamma_n\|x_n - p\|^2 + (1-\gamma_n)[\lambda\|y_n - p\|^2 + (1-\lambda)\|J_{r_n}y_n - p\|^2 \\
&\quad - \lambda(1-\lambda)g_1(\|y_n - J_{r_n}y_n\|)] \\
&\le \gamma_n\|x_n - p\|^2 + (1-\gamma_n)[\|x_n - p\|^2 + 2\alpha_n\|F(p)\|\|p - u_n\| \\
&\quad + 2\beta_n\|f(p) - p\|\|y_n - p\| - \lambda(1-\lambda)g_1(\|y_n - J_{r_n}y_n\|)] \\
&\le \|x_n - p\|^2 + 2\alpha_n\|F(p)\|\|p - u_n\| + 2\beta_n\|f(p) - p\|\|y_n - p\| \\
&\quad - (1-\gamma_n)\lambda(1-\lambda)g_1(\|y_n - J_{r_n}y_n\|).
\end{aligned}$$

So it follows that

$$\begin{aligned}
(1-\gamma_n)\lambda(1-\lambda)g_1(\|y_n - J_{r_n}y_n\|) &\le (\|x_n - p\| + \|x_{n+1} - p\|)\|x_n - x_{n+1}\| \\
&\quad + 2\alpha_n\|F(p)\|\|p - u_n\| + 2\beta_n\|f(p) - p\|\|y_n - p\|.
\end{aligned}$$

Since $\lambda \in (0,1)$, $\liminf_{n\to\infty}(1-\gamma_n) > 0$, $\lim_{n\to\infty}\beta_n = 0$ and $\lim_{n\to\infty}\alpha_n = 0$, from (3.14) and the boundedness of $\{x_n\}, \{y_n\}, \{u_n\}$ we conclude that $\lim_{n\to\infty} g_1(\|y_n - J_{r_n}y_n\|) = 0$. Utilizing the properties of g_1, we get

$$\lim_{n\to\infty}\|y_n - J_{r_n}y_n\| = 0. \tag{3.17}$$

Note that $x_{n+1} - x_n = (1-\gamma_n)(y_n - x_n) + (1-\gamma_n)(1-\lambda)(J_{r_n}y_n - y_n)$. It immediately follows that

$$\begin{aligned}
(1-\gamma_n)\|y_n - x_n\| &\le \|x_{n+1} - x_n\| + (1-\gamma_n)(1-\lambda)\|J_{r_n}y_n - y_n\| \\
&\le \|x_{n+1} - x_n\| + \|J_{r_n}y_n - y_n\|.
\end{aligned}$$

Thus, from (3.14), (3.17) and $\liminf_{n\to\infty}(1-\gamma_n) > 0$, we have

$$\lim_{n\to\infty}\|y_n - x_n\| = 0. \tag{3.18}$$

Also, according to (3.2) we have

$$\begin{aligned}
\|z_n - p\|^2 &= \sigma_n\langle x_n - p, j(z_n - p)\rangle + (1-\sigma_n)\langle T_n z_n - p, j(z_n - p)\rangle \\
&\le \sigma_n\langle x_n - p, j(z_n - p)\rangle + (1-\sigma_n)\|z_n - p\|^2,
\end{aligned}$$

which together with Proposition 2.4, yields

$$\|z_n - p\|^2 \le \langle x_n - p, j(z_n - p)\rangle \le \frac{1}{2}[\|x_n - p\|^2 + \|z_n - p\|^2 - g_2(\|x_n - z_n\|)].$$

This immediately implies that $\|z_n - p\|^2 \le \|x_n - p\|^2 - g_2(\|x_n - z_n\|)$, which together with (3.16), yields

$$\begin{aligned}\|y_n - p\|^2 &\le \beta_n k\|x_n - p\|^2 + (1-\beta_n)[\|z_n - p\|^2 + 2\alpha_n\|F(p)\|\|p - u_n\|] \\ &\quad + 2\beta_n\langle f(p) - p, j(y_n - p)\rangle \\ &\le \|x_n - p\|^2 - (1-\beta_n)g_2(\|x_n - z_n\|) + 2\alpha_n\|F(p)\|\|p - u_n\| \\ &\quad + 2\beta_n\|f(p) - p\|\|y_n - p\|,\end{aligned}$$

which hence yields

$$\begin{aligned}(1-\beta_n)g_2(\|x_n - z_n\|) &\le (\|x_n - p\| + \|y_n - p\|)\|x_n - y_n\| + 2\alpha_n\|F(p)\|\|p - u_n\| \\ &\quad + 2\beta_n\|f(p) - p\|\|y_n - p\|.\end{aligned}$$

Since $\lim_{n\to\infty}\beta_n = 0$ and $\lim_{n\to\infty}\alpha_n = 0$, from (3.18) and the boundedness of $\{x_n\}$, $\{y_n\}$, $\{u_n\}$ we conclude that $\lim_{n\to\infty} g_2(\|x_n - z_n\|) = 0$. Utilizing the properties of g_2, we get

$$\lim_{n\to\infty}\|x_n - z_n\| = 0. \tag{3.19}$$

Also, observe that

$$y_n - x_n = \beta_n(f(x_n) - x_n) + (1-\beta_n)(Gz_n - x_n) + (1-\beta_n)(1-\mu)(u_n - Gz_n). \tag{3.20}$$

It immediately follows that

$$\begin{aligned}(1-\beta_n)\|Gz_n - x_n\| &\le \|y_n - x_n\| + \beta_n\|f(x_n) - x_n\| + (1-\beta_n)(1-\mu)\|u_n - Gz_n\| \\ &\le \|y_n - x_n\| + \beta_n\|f(x_n) - x_n\| + \alpha_n\|F(Gz_n)\|.\end{aligned}$$

Since $\lim_{n\to\infty}\beta_n = 0$ and $\lim_{n\to\infty}\alpha_n = 0$, from (3.18) and the boundedness of $\{x_n\}$, $\{f(x_n)\}$, $\{F(Gz_n)\}$ we conclude that $\lim_{n\to\infty}\|Gz_n - x_n\| = 0$. Meantime, it is clear that

$$\|x_n - Gx_n\| \le \|x_n - Gz_n\| + \|z_n - x_n\|.$$

From (3.19) we have

$$\lim_{n\to\infty}\|x_n - Gx_n\| = 0. \tag{3.21}$$

Further, since

$$\|x_n - J_{r_n}x_n\| \le \|x_n - y_n\| + \|y_n - J_{r_n}x_n\| \le 2\|x_n - y_n\| + \|y_n - J_{r_n}y_n\|,$$

from (3.17) and (3.18) we have

$$\lim_{n\to\infty}\|x_n - J_{r_n}x_n\| = 0. \tag{3.22}$$

In addition, combining (3.2) with (3.19), implies that

$$\|T_n z_n - z_n\| = \frac{\sigma_n}{1-\sigma_n}\|x_n - z_n\| \le \frac{b}{1-b}\|x_n - z_n\| \to 0 \quad (n\to\infty). \tag{3.23}$$

Since $\{T_n\}_{n=0}^{\infty}$ is ℓ-uniformly Lipschitzian on C, we deduce from (3.19) and (3.23) that

$$\begin{aligned}\|T_n x_n - x_n\| &\le \|T_n x_n - T_n z_n\| + \|T_n z_n - z_n\| + \|z_n - x_n\| \\ &= (\ell+1)\|x_n - z_n\| + \|T_n z_n - z_n\| \to 0 \quad (n\to\infty).\end{aligned} \tag{3.24}$$

Step 4. We claim that $\|x_n - \overline{T}x_n\| \to 0$ and $\|x_n - J_r x_n\| \to 0$ as $n \to \infty$ where $\overline{T} := (2I - T)^{-1}$ and $r \in (0, \varepsilon)$. Indeed, it is clear that $T : C \to C$ is pseudocontractive and ℓ-Lipschitzian. Utilizing the boundedness of $\{x_n\}$ and putting $D = \overline{\mathrm{conv}}\{x_n : n \geq 0\}$ (the closed convex hull of the set $\{x_n : n \geq 0\}$), from the assumption we have

$$\sum_{n=1}^{\infty} \sup_{x \in D} \|T_n x - T_{n-1} x\| < \infty.$$

Hence, by Proposition 2.1 we get $\lim_{n\to\infty} \sup_{x \in D} \|T_n x - Tx\| = 0$, which immediately yields

$$\lim_{n\to\infty} \|T_n x_n - Tx_n\| = 0. \tag{3.25}$$

Thus, combining (3.24) with (3.25) we have

$$\|x_n - Tx_n\| \leq \|x_n - T_n x_n\| + \|T_n x_n - Tx_n\| \to 0 \quad (n \to \infty). \tag{3.26}$$

Next, let us show that if we define $\overline{T} := (2I - T)^{-1}$, then $\overline{T} : C \to C$ is nonexpansive,

$$F(\overline{T}) = F(T) = \bigcap_{n=0}^{\infty} F(T_n) \text{ and } \lim_{n\to\infty} \|x_n - \overline{T}x_n\| = 0.$$

Indeed, put $\overline{T} := (2I - T)^{-1}$, where I is the identity mapping of E. Then it is known that $\overline{T}$ is nonexpansive and the fixed point set

$$F(\overline{T}) = F(T) = \bigcap_{n=0}^{\infty} F(T_n),$$

as a consequence of Theorem 6 of [25]. From (3.26) it follows that

$$\begin{aligned} \|x_n - \overline{T}x_n\| &= \|\overline{T}\overline{T}^{-1} x_n - \overline{T}x_n\| \leq \|\overline{T}^{-1} x_n - x_n\| \\ &= \|(2I - T)x_n - x_n\| = \|x_n - Tx_n\| \to 0 \quad (n \to \infty). \end{aligned} \tag{3.27}$$

In addition, let us show that $\lim_{n\to\infty} \|J_r x_n - x_n\| = 0$ for any given $r \in (0, \varepsilon)$. In fact, by Proposition 2.3, we get

$$\|J_{r_n} x_n - J_r x_n\| = \left\| J_r \left(\frac{r}{r_n} x_n + \left(1 - \frac{r}{r_n}\right) J_{r_n} x_n \right) - J_r x_n \right\| \leq \|x_n - J_{r_n} x_n\|,$$

which together with (3.22), implies that

$$\|x_n - J_r x_n\| \leq \|x_n - J_{r_n} x_n\| + \|J_{r_n} x_n - J_r x_n\| \leq 2\|x_n - J_{r_n} x_n\| \to 0 \quad (n \to \infty). \tag{3.28}$$

Step 5. We claim that $\{x_n\}$ converges strongly to $x^* \in \Omega$. Indeed, we define a mapping

$$Wx := \theta_1 \overline{T}x + \theta_2 Gx + (1 - \theta_1 - \theta_2) J_r x, \ \forall\, x \in C,$$

where $\theta_1, \theta_2 \in (0, 1)$ are two constants with $\theta_1 + \theta_2 < 1$. Then from Lemma 2.8 it is easy to see that W is nonexpansive and

$$F(W) = F(\overline{T}) \cap F(G) \cap F(J_r) = \bigcap_{n=1}^{\infty} F(T_n) \cap \mathrm{GSVI}(C, B_1, B_2) \cap A^{-1}0 (=: \Omega) \neq \emptyset. \tag{3.29}$$

Since

$$\|x_n - Wx_n\| \le \theta_1 \|\overline{T}x_n - x_n\| + \theta_2 \|Gx_n - x_n\| + (1-\theta_1-\theta_2)\|J_r x_n - x_n\|,$$

from (3.21), (3.27) and (3.28), we get $\lim_{n\to\infty} \|x_n - Wx_n\| = 0$.

In the following, we discuss two cases.

(i) First, suppose that E is uniformly smooth. Let x_t be the unique fixed point of the contraction mapping T_t given by

$$T_t x = tf(x) + (1-t)Wx, \ \forall\, t \in (0,1).$$

By Lemma 2.9, we can define $x^* := s - \lim_{t\to 0^+} x_t$, and $x^* \in F(W) = \Omega$ solves the VI:

$$\langle x^* - f(x^*), j(x^* - x)\rangle \le 0, \ \forall\, x \in \Omega.$$

Let us show that

$$\limsup_{n\to\infty} \langle f(x^*) - x^*, j(x_n - x^*)\rangle \le 0. \tag{3.30}$$

Note that $x_t - x_n = t(f(x_t) - x_n) + (1-t)(Wx_t - x_n)$. Applying Lemma 2.6, we derive

$$\begin{aligned}\|x_t - x_n\|^2 &\le (1-t)^2(\|Wx_t - Wx_n\| + \|Wx_n - x_n\|)^2 \\ &\quad + 2t\langle f(x_t) - x_t, j(x_t - x_n)\rangle + 2t\|x_t - x_n\|^2 \\ &\le (1-t)^2\|x_t - x_n\|^2 + a_n(t) \\ &\quad + 2t\langle f(x_t) - x_t, j(x_t - x_n)\rangle + 2t\|x_t - x_n\|^2,\end{aligned}$$

where

$$a_n(t) = \|Wx_n - x_n\|(2\|x_t - x_n\| + \|Wx_n - x_n\|) \to 0$$

as $n \to \infty$. The last inequality implies

$$\langle x_t - f(x_t), j(x_t - x_n)\rangle \le \frac{t}{2}\|x_t - x_n\|^2 + \frac{1}{2t}a_n(t).$$

It follows that

$$\limsup_{n\to\infty} \langle x_t - f(x_t), j(x_t - x_n)\rangle \le M_1 \frac{t}{2}, \tag{3.31}$$

where $M_1 > 0$ is a constant such that $M_1 \ge \|x_t - x_n\|^2$ for all $n \ge 0$ and small enough $t \in (0,1)$. Taking the limsup as $t \to 0^+$ in (3.31) and noticing the fact that the two limits are interchangeable due to the fact that the duality map $j(\cdot)$ is norm-to-norm uniformly continuous on any bounded subset of E, we get (3.30).

Now, let us show that $x_n \to x^*$ as $n \to \infty$. Indeed, from (3.18) we get

$$\|(y_n - x^*) - (x_n - x^*)\| \to 0 \ (n \to \infty).$$

Since $j(\cdot)$ is norm-to-norm uniformly continuous on any bounded subset of E, we deduce from (3.30) that

$$\limsup_{n\to\infty} \langle f(x^*) - x^*, j(y_n - x^*)\rangle = \limsup_{n\to\infty} \langle f(x^*) - x^*, j(x_n - x^*)\rangle \le 0. \tag{3.32}$$

Since

$$x_{n+1} = \gamma_n x_n + (1-\gamma_n)[\lambda y_n + (1-\lambda)J_{r_n} y_n],$$

putting $p = x^* \in \Omega$ we get from (3.16)

$$\begin{aligned}\|y_n - x^*\|^2 &\le \beta_n k\|x_n - x^*\|^2 + (1-\beta_n)[\|x_n - x^*\|^2 + 2\alpha_n\|F(x^*)\|\|x^* - u_n\|]\\ &\quad +2\beta_n\langle f(x^*) - x^*, j(y_n - x^*)\rangle\\ &\le [1-(1-k)\beta_n]\|x_n - x^*\|^2 + 2\alpha_n\|F(x^*)\|\|x^* - u_n\|\\ &\quad +2\beta_n\langle f(x^*) - x^*, j(y_n - x^*)\rangle, \end{aligned} \tag{3.33}$$

which hence implies that

$$\begin{aligned}\|x_{n+1} - x^*\|^2 &\le \gamma_n\|x_n - x^*\|^2 + (1-\gamma_n)\|y_n - x^*\|^2\\ &\le \gamma_n\|x_n - x^*\|^2 + (1-\gamma_n)\{[1-(1-k)\beta_n]\|x_n - x^*\|^2\\ &\quad +2\alpha_n\|F(x^*)\|\|x^* - u_n\| + 2\beta_n\langle f(x^*) - x^*, j(y_n - x^*)\rangle\}\\ &= [1-(1-k)\beta_n(1-\gamma_n)]\|x_n - x^*\|^2 + (1-k)\beta_n(1-\gamma_n)\\ &\quad \cdot\left[\frac{2\alpha_n\|F(x^*)\|\|x^* - u_n\|}{(1-k)\beta_n} + \frac{2\langle f(x^*) - x^*, j(y_n - x^*)\rangle}{1-k}\right]. \end{aligned} \tag{3.34}$$

Since $\sum_{n=0}^{\infty}\beta_n = \infty$ and $\liminf_{n\to\infty}(1-\gamma_n) > 0$, we know that

$$\sum_{n=0}^{\infty}(1-k)\beta_n(1-\gamma_n) = \infty.$$

In addition, from (3.32) and $\lim_{n\to\infty}\alpha_n/\beta_n = 0$, we have

$$\limsup_{n\to\infty}\left(\frac{2\alpha_n\|F(x^*)\|\|x^* - u_n\|}{(1-k)\beta_n} + \frac{2\langle f(x^*) - x^*, j(y_n - x^*)\rangle}{1-k}\right) \le 0. \tag{3.35}$$

Consequently, applying Lemma 2.12 to (3.34), we obtain that $x_n \to x^*$ as $n \to \infty$.

(ii) Second, suppose that E has a weakly continuous duality map j_φ with gauge φ. Let x_t be the unique fixed point of the contraction T_t given by

$$T_t x = tf(x) + (1-t)Wx,\ \forall\ t \in (0,1).$$

By Lemma 2.11, we can define

$$x^* := s - \lim_{t\to 0^+} x_t,$$

and $x^* \in F(W) = \Omega$ solves the VI:

$$\langle x^* - f(x^*), j_\varphi(x^* - x)\rangle \le 0, \quad \forall x \in \Omega. \tag{3.36}$$

Let us show that

$$\limsup_{n\to\infty}\langle f(x^*) - x^*, j_\varphi(x_n - x^*)\rangle \le 0. \tag{3.37}$$

We take a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle f(x^*) - x^*, j_\varphi(x_n - x^*)\rangle = \lim_{i\to\infty}\langle f(x^*) - x^*, j_\varphi(x_{n_i} - x^*)\rangle. \tag{3.38}$$

Since E is reflexive and $\{x_n\}$ is bounded, we may further assume that $x_{n_i} \rightharpoonup \bar{x}$ for some $\bar{x} \in C$. Since $j_\varphi(\cdot)$ is weakly continuous, utilizing Proposition 2.10 (ii), we have

$$\limsup_{i\to\infty}\Phi(\|x_{n_i} - x\|) = \limsup_{i\to\infty}\Phi(\|x_{n_i} - \bar{x}\|) + \Phi(\|x - \bar{x}\|), \quad \forall x \in E.$$

Put $\Gamma(x) = \limsup_{i\to\infty} \Phi(\|x_{n_i} - x\|),\ \forall\, x \in E$. It follows that

$$\Gamma(x) = \Gamma(\bar{x}) + \Phi(\|x - \bar{x}\|),\ \forall\, x \in E.$$

From $\|x_n - Wx_n\| \to 0$ $(n \to \infty)$, we have

$$\begin{aligned}\Gamma(W\bar{x}) &= \limsup_{i\to\infty} \Phi(\|x_{n_i} - W\bar{x}\|) = \limsup_{i\to\infty} \Phi(\|Wx_{n_i} - W\bar{x}\|)\\ &\le \limsup_{i\to\infty} \Phi(\|x_{n_i} - \bar{x}\|) = \Gamma(\bar{x}).\end{aligned} \tag{3.39}$$

Furthermore, observing that $\Gamma(W\bar{x}) = \Gamma(\bar{x}) + \Phi(\|W\bar{x} - \bar{x}\|)$, which together with (3.39), yields $\Phi(\|W\bar{x} - \bar{x}\|) \le 0$. Hence $W\bar{x} = \bar{x}$ and $\bar{x} \in F(W) = \Omega$. Thus, from (3.36) and (3.38), it is easy to see that

$$\limsup_{n\to\infty} \langle f(x^*) - x^*, j_\varphi(x_n - x^*)\rangle = \langle f(x^*) - x^*, j_\varphi(\bar{x} - x^*)\rangle \le 0. \tag{3.40}$$

Therefore, we deduce that (3.37) holds.

Next, let us show that $x_n \to x^*$ as $n \to \infty$. Indeed, putting $p = x^*$ and utilizing Lemma 2.14 we obtain from (3.3) and (3.4) that $\|z_n - x^*\| \le \|x_n - x^*\|$ and

$$\begin{aligned}\|x_n - x^*\|\varphi(\|x_n - x^*\|) &= \langle x_n - y_n, j_\varphi(x_n - x^*)\rangle + \langle y_n - x^*, j_\varphi(x_n - x^*)\rangle\\ &= \langle x_n - y_n, j_\varphi(x_n - x^*)\rangle + \beta_n\langle f(x_n) - f(x^*), j_\varphi(x_n - x^*)\rangle\\ +(1-\beta_n)\langle \mu(Gz_n - x^*) &+ (1-\mu)(\Pi_C(I - \alpha_n F)Gz_n - \Pi_C(I - \alpha_n F)x^*), j_\varphi(x_n - x^*)\rangle\\ +(1-\beta_n)(1-\mu)\langle \Pi_C(I &- \alpha_n F)x^* - x^*, j_\varphi(x_n - x^*)\rangle + \beta_n\langle f(x^*) - x^*, j_\varphi(x_n - x^*)\rangle\\ &\le \|x_n - y_n\|\varphi(\|x_n - x^*\|) + [1 - (1-k)\beta_n]\|x_n - x^*\|\varphi(\|x_n - x^*\|)\\ &\quad +\alpha_n\|F(x^*)\|\varphi(\|x_n - x^*\|) + \beta_n\langle f(x^*) - x^*, j_\varphi(x_n - x^*)\rangle.\end{aligned} \tag{3.40}$$

This immediately leads to

$$\begin{aligned}\|x_n - x^*\|\varphi(\|x_n - x^*\|) \le \frac{\|x_n - y_n\|}{\beta_n}\cdot\frac{\varphi(\|x_n - x^*\|)}{1-k} + \frac{\alpha_n}{\beta_n}\cdot\frac{\|F(x^*)\|\varphi(\|x_n - x^*\|)}{1-k}\\ + \frac{1}{1-k}\langle (f(x^*) - x^*), j_\varphi(x_n - x^*)\rangle.\end{aligned}$$

Since $\lim_{n\to\infty} \beta_n = 0$, $\lim_{n\to\infty} \frac{\alpha_n}{\beta_n} = 0$ and $\lim_{n\to\infty} \frac{\|x_n - y_n\|}{\beta_n} = 0$, from (3.37) and the boundedness of $\{\varphi(\|x_n - x^*\|)\}$ we get $\limsup_{n\to\infty} \|x_n - x^*\|\varphi(\|x_n - x^*\|) = 0$, which hence implies that $\|x_n - x^*\| \to 0$ $(n \to \infty)$, i.e., $x_n \to x^*$ $(n \to \infty)$. This completes the proof. $\square$

By a similar approach we can also prove the following result.

Theorem 3.2. *Let C be a nonempty closed convex subset of a Hilbert space H. Let $A \subset H \times H$ be a monotone operator in H such that*

$$\overline{D(A)} \subset C \subset \bigcap_{r>0} R(I + rA).$$

Let the mappings $B_1, B_2 : C \to H$ be α-inverse strongly monotone and β-inverse strongly monotone, respectively. Let $f \in \Xi_C$ with a contraction coefficient $k \in (0,1)$, and let $F : C \to H$ be δ-strongly monotone and ζ-strictly pseudocontractive with

$\delta+\zeta \geq 1$. Let $\{T_n\}_{n=0}^{\infty}$ be a countable family of ℓ-uniformly Lipschitzian pseudocontractive self-mappings on C such that

$$\Omega := \bigcap_{n=0}^{\infty} F(T_n) \cap \mathrm{GSVI}(C, B_1, B_2) \cap A^{-1}0 \neq \emptyset$$

where $\mathrm{GSVI}(C, B_1, B_2)$ is the fixed point set of the mapping

$$G := P_C(I-\rho B_1)P_C(I-\eta B_2)$$

with $0<\rho<2\alpha$ and $0<\eta<2\beta$. For an arbitrary $x_0 \in C$, let $\{x_n\}$ be generated by

$$\begin{cases} z_n = \sigma_n x_n + (1-\sigma_n)T_n z_n, \\ y_n = \beta_n f(x_n) + (1-\beta_n)[\mu I + (1-\mu)P_C(I-\alpha_n F)]P_C(I-\rho B_1)P_C(I-\eta B_2)z_n, \\ x_{n+1} = \gamma_n x_n + (1-\gamma_n)[\lambda y_n + (1-\lambda)J_{r_n} y_n], \quad \forall n \geq 0, \end{cases} \tag{3.41}$$

where $\lambda, \mu \in (0,1)$, $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ and $\{\sigma_n\}$ are the sequences in $(0,1)$ and $\{r_n\}$ is a sequence in $(0,\infty)$. Suppose that the following conditions hold:

(i) $\sum_{n=0}^{\infty}(\alpha_n+\beta_n) < \infty$ *and* $r_n \geq \varepsilon > 0 \ \forall n \geq 0$*;*

(ii) $0 < \liminf_{n\to\infty} \gamma_n \leq \limsup_{n\to\infty} \gamma_n < 1$*;*

(iii) $0 < \liminf_{n\to\infty} \sigma_n \leq \limsup_{n\to\infty} \sigma_n < 1$.

Assume that $\sum_{n=1}^{\infty} \sup_{x\in D} \|T_n x - T_{n-1}x\| < \infty$ for any bounded subset D of C, and let T be a mapping of C into itself defined by $Tx = \lim_{n\to\infty} T_n x$ for all $x \in C$, and suppose that $F(T) = \bigcap_{n=0}^{\infty} F(T_n)$. Then the sequence $\{x_n\}$ defined by (3.41) converges weakly to a point in Ω.

The following example (which follows by Goebel-Kirk fixed point theorem in [21] and Lemma 2.7) shows that the non-emptiness assumption of the fixed point set $\mathrm{GSVI}(C, B_1, B_2)$ of the mapping $G := \Pi_C(I-\rho B_1)\Pi_C(I-\eta B_2)$ is verified.

Example 3.1. Let C be a nonempty closed convex subset of a smooth and uniformly convex Banach space E. Let $B_1, B_2 : C \to E$ be two mappings. Let ρ and η be two constants. Assume that (a) the mapping B_1 is λ_1-strictly pseudocontractive and ζ_1-strongly accretive with $\lambda_1+\zeta_1 \geq 1$, (b) the mapping B_2 is λ_2-strictly pseudocontractive and ζ_2-strongly accretive with $\lambda_2+\zeta_2 \geq 1$, and (c) the constants ρ and η satisfy the conditions $1-\frac{\lambda_1}{1+\lambda_1}\left(1-\sqrt{\frac{1-\zeta_1}{\lambda_1}}\right) \leq \rho \leq 1$ and $1-\frac{\lambda_2}{1+\lambda_2}\left(1-\sqrt{\frac{1-\zeta_2}{\lambda_2}}\right) \leq \eta \leq 1$. Let the mapping $G : C \to C$ be defined as $G := \Pi_C(I-\rho B_1)\Pi_C(I-\eta B_2)$. If the set C is bounded, then the solution set of the GSVI (1.3) is nonempty.

References

[1] Q.H. Ansari, A. Rehan, C.-F. Wen, *Split hierarchical variational inequality problems and fixed point problems*, J. Ineq. Appl., 2015 (2015), Article ID 793.

[2] Q.H. Ansari, J.-C. Yao, *A fixed point theorem and its applications to the system of variational inequalities*, Bull. Austral. Math. Soc., **59**(1999), 433-442.

[3] Q.H. Ansari, J.-C. Yao, *Systems of generalized variational inequalities and their applications*, Applicable Anal., **76**(2000), 203-217.

[4] K. Aoyama, Y. Kimura, W. Takahashi, M. Toyoda, *Approximation of common fixed points of a countable family of nonexpansive mappings in a Banach space*, Nonlinear Anal., **67**(2007), 2350-2360.

[5] R.E. Bruck Jr., *Properties of fixed-point sets of nonexpansive mappings in Banach spaces*, Trans. Amer. Math. Soc., **179**(1973), 251-262.

[6] L.-C. Ceng, S. Al-Mezel, Q.H. Ansari, *Implicit and explicit iterative methods for systems of variational inequalities and zeros of accretive operators*, Appl. Abstract Anal., 2013 (2013), Article ID 631382, 19 pages.

[7] L.C. Ceng, Q.H. Ansari, J.C. Yao, *Mann-type steepest-descent and modified hybrid steepest-descent methods for variational inequalities in Banach spaces*, Numer. Funct. Anal. Optim., **29**(2008), 987-1033.

[8] L.C. Ceng, H. Gupta, Q.H. Ansari, *Implicit and explicit algorithms for a system of nonlinear variational inequalities in Banach spaces*, J. Nonlinear Convex Anal., **16**(2015), 965-984.

[9] L.C. Ceng, A. Latif, J.C. Yao, *On solutions of a system of variational inequalities and fixed point problems in Banach spaces*, Fixed Point Theory Appl., 2013, 2013:176, 34 pp.

[10] L.C. Ceng, Y.C. Liou, C.F. Wen, *Systems of variational inequalities with hierarchical variational inequality constraints in Banach spaces*, J. Nonlinear Sci. Appl., **10**(2017), 3136-3154.

[11] L.C. Ceng, A. Petruşel, M.M. Wong, S.J. Yu, *Strong convergence of implicit viscosity approximation methods for pseudocontractive mappings in Banach spaces*, Optimization, **60**(2011), 659-670.

[12] L.C. Ceng, S. Plubtieng, M.M. Wong, J.C. Yao, *System of variational inequalities with constraints of mixed equilibria, variational inequalities, and convex minimization and fixed point problems*, J. Nonlinear Convex Anal., **16**(2015) 385-421.

[13] L.C. Ceng, C.Y. Wang, J.C. Yao, *Strong convergence theorems by a relaxed extragradient method for a general system of variational inequalities*, Math. Methods Oper. Res., **67**(2008), 375-390.

[14] L.C. Ceng, C.F. Wen, *Three-step Mann iterations for a general system of variational inequalities and an infinite family of nonexpansive mappings in Banach spaces*, J. Ineq. Appl. 2013, 2013:539, 27 pp.

[15] L.C. Ceng, C.F. Wen, *System of variational inequalities and an accretive operator in Banach spaces*, Fixed Point Theory Appl., 2013, 2013:249, 37 pp.

[16] L.C. Ceng, H.K. Xu, *Strong convergence of a hybrid viscosity approximation method with perturbed mappings for nonexpansive and accretive operators*, Taiwanese J. Math., **11**(2007), 661-682.

[17] R.D. Chen, Z.C. Zhu, *Viscosity approximation of fixed points for nonexpansive and m-accretive operators*, Fixed Point Theory Appl., 2006, Article ID 81325, 10 pp.

[18] S.Y. Cho, X. Qin, J.C. Yao, Y. Yao, *Viscosity approximation splitting methods for monotone and nonexpansive operators in Hilbert spaces*, J. Nonlinear Convex Anal., **19**(2018), 251-264.

[19] K. Deimling, *Zeros of accretive operators*, Manuscripta Math., **13**(1974), 365-374.

[20] F. Facchinei, J.S. Pang, *Finite-Dimensional Variational Inequalities and Complementarity Problems I*, Springer-Verlag, New York, 2003.

[21] K. Goebel, W.A. Kirk, *A fixed point theorem for asymptotically nonexpansive mappings*, Proc. Amer. Math. Soc., **35**(1972) 171-174.

[22] S. Kamimura, W. Takahashi, *Strong convergence of a proximal-type algorithm in a Banach space*, SIAM J. Optim., **13**(2002) 938-945.

[23] T.C. Lim, H.K. Xu, *Fixed point theorems for asymptotically nonexpansive mappings*, Nonlinear Anal., **22**(1994), 1345-1355.

[24] L.-J. Lin, Q.H. Ansari, *Collective fixed points and maximal elements with applications to abstract economies*, J. Math. Anal. Appl., **296**(2004), 455-472.
[25] R.H. Martin Jr., *Differential equations on closed subsets of a Banach space*, Trans. Amer. Math. Soc., **179**(1973), 399-414.
[26] S. Reich, *Weak convergence theorems for nonexpansive mappings in Banach spaces*, J. Math. Anal. Appl., **67**(1979), 274-276.
[27] T. Suzuki, *Strong convergence of Krasnoselskii and Mann's type sequences for one-parameter nonexpansive semigroups without Bochner integrals*, J. Math. Anal. Appl., **305**(2005), 227-239.
[28] W. Takahashi, *Nonlinear Functional Analysis*, Yokohama Publisher, Yokohama, 2000.
[29] W. Takahashi, C.F. Wen, J.C. Yao, *Split common fixed point problems and hierarchical variational inequality problems in Hilbert spaces*, J. Nonlinear Convex Anal., **18**(2017), 925-935.
[30] R.U. Verma, *General convergence analysis for two-step projection methods and applications to variational problems*, Appl. Math. Lett., **18**(2005), 1286-1292.
[31] H.K. Xu, *Inequalities in Banach spaces with applications*, Nonlinear Anal., **16**(1991), 1127-1138.
[32] H.K. Xu, *Viscosity approximation methods for nonexpansive mappings*, J. Math. Anal. Appl., **298**(2004), 279-291.
[33] Y. Yao, Y.C. Liou, J.C. Yao, *Iterative algorithms for the split variational inequality and fixed point problems under nonlinear transformations*, J. Nonlinear Sci. Appl., **10**(2017), 843-854.
[34] Y. Yao, A. Petrusel, X. Qin, *An improved algorithm based on Korpelevich's method for variational inequalities in Banach spaces*, J. Nonlinear Convex Anal., **19**(2018), 397-405.
[35] Y. Yao, M. Postolache, Y.C. Liou, Z. Yao, *Construction algorithms for a class of monotone variational inequalities*, Optim. Lett., **10**(2016), 1519-1528.
[36] Y. Yao, X. Qin, J.-C. Yao, *Projection methods for firmly type nonexpansive operators*, J. Nonlinear Convex Anal., **19**(2018), 407-415.
[37] H. Zegeye, N. Shahzad, Y. Yao, *Minimum-norm solution of variational inequality and fixed point problem in Banach spaces*, Optimization, **64**(2015), 453-471.

Received: December 17, 2017; Accepted: February 9, 2018.