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DHAGE ITERATION METHOD FOR APPROXIMATING SOLUTIONS OF NONLINEAR DIFFERENTIAL EQUATIONS WITH MAXIMA

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Abstract. In this paper we study the initial value problem of first order nonlinear differential equations with maxima and discuss the existence and approximation of the solutions. The main result relies on the Dhage iteration method embodied in a recent hybrid fixed point theorem of Dhage (2014) in a partially ordered normed linear space. At the end, we give an example to illustrate the hypotheses and applicability of the abstract results of this paper.

Key Words and Phrases: Differential equations with maxima, Dhage iteration method, hybrid fixed point theorem, approximation of solutions.

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1. Introduction

The study of fixed point theorems for the contraction mappings in partially ordered metric space is initiated by Ran and Reurings [20] which is further continued by Nieto and Rodringuez-Lopez [14] and by Petruşel and Rus [19] and applied to boundary value problems of nonlinear first order ordinary differential equations and matrix equations for proving the existence results under certain monotonic conditions. Similarly, the study of hybrid fixed point theorems in a partially ordered metric space is initiated by Dhage [3, 4, 5] with applications to nonlinear differential and integral equation under weaker mixed conditions of nonlinearities. See Dhage [6, 7] and the references therein. In this paper we investigate the existence of approximate solutions of hybrid differential equations with maxima using the Dhage iteration method embodied in a hybrid fixed point theorem in a partially ordered spaces. We claim that the results of this paper are new to the theory of nonlinear differential equations with maxima.

Given a closed and bounded interval $J = [a, b]$ of the real line $\mathbb{R}$ for some $b > a \geq 0$, we consider the following hybrid differential equation (in short HDE)

$$\begin{cases} x'(t) = f(t, x(t)) + g\left(t, \max\limits_{a \leq \xi \leq t} x(\xi)\right), \\ x(a) = \alpha_0 \in \mathbb{R}, \end{cases} \tag{1.1}$$

for all $t \in J = [a, b]$ and $f, g : J \times \mathbb{R} \to \mathbb{R}$ are continuous functions.

By a *solution* of equation (1.1) we mean a differentiable function $x \in C(J, \mathbb{R})$ that satisfies equation (1.1), where $C(J, \mathbb{R})$ is the space of continuous real-valued functions defined on J.

Differential equations with maxima are often met in the applications, for instance in the theory of automatic control. Numerous results on existence and uniqueness, asymptotic stability as well as numerical solutions have been obtained. To name a few, we refer the reader to [1, 15, 16, 17, 18] and the references therein. The HDE (1.1) is a linear perturbation of first type of nonlinear differential equations. The details of different types of perturbation appears in Dhage [2]. The special cases of the HDE (1.1) in the forms

$$\begin{cases} x'(t) = f(t, x(t)), \ t \in J, \\ x(a) = \alpha_0, \end{cases} \tag{1.2}$$

and

$$\begin{cases} x'(t) = g\left(t, \max\limits_{a \leq \xi \leq t} x(\xi)\right), \ t \in J, \\ x(a) = \alpha_0, \end{cases} \tag{1.3}$$

have already been discussed in the literature for different aspects of the solutions using usual Picard iteration method. See Bainov and Hristova [1] and the references therein for the details. In this paper we discuss the HDE (1.1) for existence and approximation of solutions via a new approach based upon the Dhage iteration method. In consequence, we obtain the existence and approximation results for HDEs (1.2) and (1.3) as special cases which are also new to the literature.

In the following section we give some preliminaries and the key tool that will be used for proving the main result of this paper.

2. Preliminaries

Throughout this paper, unless otherwise mentioned, let $(E, \preceq, \|\cdot\|)$ denote a partially ordered normed linear space. Two elements x and y in E are said to be **comparable** if either the relation $x \preceq y$ or $y \preceq x$ holds. A non-empty subset C of E is called a **chain** or **totally ordered** if all the elements of C are comparable. It is known that E is **regular** if $\{x_n\}$ is a nondecreasing (resp. nonincreasing) sequence in E such that $x_n \to x^*$ as $n \to \infty$, then $x_n \preceq x^*$ (resp. $x_n \succeq x^*$) for all $n \in \mathbb{N}$. The conditions guaranteeing the regularity of E may be found in Heikkilä and Lakshmikantham [13] and the references therein.

We need the following definitions (see Dhage [3, 4, 5] and the references therein) in what follows.

Definition 2.1. A mapping $\mathcal{T} : E \to E$ is called **isotone** or **nondecreasing** if it preserves the order relation $\preceq$, that is, if $x \preceq y$ implies $\mathcal{T}x \preceq \mathcal{T}y$ for all $x, y \in E$. Similarly, $\mathcal{T}$ is called **nonincreasing** if $x \preceq y$ implies $\mathcal{T}x \succeq \mathcal{T}y$ for all $x, y \in E$. Finally, $\mathcal{T}$ is called **monotonic** or simply **monotone** if it is either nondecreasing or nonincreasing on E.

Definition 2.2. A mapping $\mathcal{T} : E \to E$ is called **partially continuous** at a point $a \in E$ if for $\epsilon > 0$ there exists a $\delta > 0$ such that $\|\mathcal{T}x - \mathcal{T}a\| < \epsilon$ whenever x is comparable to a and $\|x - a\| < \delta$. $\mathcal{T}$ called partially continuous on E if it is partially continuous at every point of it. It is clear that if $\mathcal{T}$ is partially continuous on E, then it is continuous on every chain C contained in E.

Definition 2.3. A non-empty subset S of the partially ordered Banach space E is called **partially bounded** if every chain C in S is bounded. An operator $\mathcal{T}$ on a partially normed linear space E into itself is called **partially bounded** if $\mathcal{T}(E)$ is a partially bounded subset of E. $\mathcal{T}$ is called **uniformly partially bounded** if all chains C in $\mathcal{T}(E)$ are bounded by a unique constant.

Definition 2.4. A non-empty subset S of the partially ordered Banach space E is called **partially compact** if every chain C in S is a relatively compact subset of E. A mapping $\mathcal{T} : E \to E$ is called **partially compact** if $\mathcal{T}(E)$ is a partially relatively compact subset of E. $\mathcal{T}$ is called **uniformly partially compact** if $\mathcal{T}$ is a uniformly partially bounded and partially compact operator on E. $\mathcal{T}$ is called **partially totally bounded** if for any bounded subset S of E, $\mathcal{T}(S)$ is a partially relatively compact subset of E. If $\mathcal{T}$ is partially continuous and partially totally bounded, then it is called **partially completely continuous** on E.

Remark 2.1. Suppose that $\mathcal{T}$ is a nondecreasing operator on E into itself. Then $\mathcal{T}$ is a partially bounded or partially compact if $\mathcal{T}(C)$ is a bounded or compact subset of E for each chain C in E.

Definition 2.5. The order relation $\preceq$ and the metric d on a non-empty set E are said to be **$\mathcal{D}$-compatible** if $\{x_n\}$ is a monotone sequence, that is, monotone nondecreasing or monotone nonincreasing sequence in E and if a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ converges to x^* implies that the original sequence $\{x_n\}$ converges to x^*. Similarly, given a partially ordered normed linear space $(E, \preceq, \|\cdot\|)$, the order relation $\preceq$ and the norm $\|\cdot\|$ are said to be compatible if $\preceq$ and the metric d defined through the norm $\|\cdot\|$ are compatible. A subset S of E is called **Janhavi** if the order relation $\preceq$ and the metric d or the norm $\|\cdot\|$ are compatible in it. In particular, if $S = E$, then E is called a **Janhavi metric** or **Janhavi Banach space**.

Clearly, the set $\mathbb{R}$ of real numbers with usual order relation $\leq$ and the norm defined by the absolute value function $|\cdot|$ has this property. Similarly, the finite dimensional Euclidean space $\mathbb{R}^n$ with usual componentwise order relation and the standard norm possesses the compatibility property and so is a **Janhavi Banach space**.

Definition 2.6. An upper semi-continuous and monotone nondecreasing function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ is called a $\mathcal{D}$-function provided $\psi(0) = 0$. An operator $\mathcal{T} : E \to E$ is called partially nonlinear $\mathcal{D}$-contraction if there exists a $\mathcal{D}$-function ψ such that

$$\|\mathcal{T}x - \mathcal{T}y\| \le \psi\big(\|x - y\|\big) \tag{2.1}$$

for all comparable elements $x, y \in E$, where $0 < \psi(r) < r$ for $r > 0$. In particular, if $\psi(r) = k\,r$, $k > 0$, $\mathcal{T}$ is called a partial Lipschitz operator with a Lischitz constant k and moreover, if $0 < k < 1$, $\mathcal{T}$ is called a partial linear contraction on E with a contraction constant k.

The **Dhage iteration method** or Dhage iteration principle embodied in the following applicable hybrid fixed point theorem of Dhage [4] in a partially ordered normed linear space is used as a key tool for our work contained in this paper. The details of a Dhage iteration method or principle is given in Dhage [6, 7, 8], Dhage *et al.* [11, 12] and the references therein.

Theorem 2.1 (Dhage [4])**.** *Let $\big(E, \preceq, \|\cdot\|\big)$ be a regular partially ordered complete normed linear space such that every compact chain C of E is Janhavi. Let $\mathcal{A}, \mathcal{B} : E \to E$ be two nondecreasing operators such that*

(a) *$\mathcal{A}$ is partially bounded and partially nonlinear $\mathcal{D}$-contraction,*
(b) *$\mathcal{B}$ is partially continuous and partially compact, and*
(c) *there exists an element $x_0 \in E$ such that $x_0 \preceq \mathcal{A}x_0 + \mathcal{B}x_0$ or $x_0 \succeq \mathcal{A}x_0 + \mathcal{B}x_0$.*

Then the operator equation $\mathcal{A}x + \mathcal{B}x = x$ has a solution x^* in E and the sequence $\{x_n\}$ of successive iterations defined by $x_{n+1} = \mathcal{A}x_n + \mathcal{B}x_n$, $n = 0, 1, \ldots,$ converges monotonically to x^*.

Remark 2.2. The condition that every compact chain of E is Janhavi holds if every partially compact subset of E possesses the compatibility property with respect to the order relation $\preceq$ and the norm $\|\cdot\|$ in it.

Remark 2.3. We remark that hypothesis (a) of Theorem 2.1 implies that the operator $\mathcal{A}$ is partially continuous and consequently both the operators $\mathcal{A}$ and $\mathcal{B}$ in the theorem are partially continuous on E. The regularity of E in above Theorem 2.1 may be replaced with a stronger continuity condition of the operators $\mathcal{A}$ and $\mathcal{B}$ on E which is a result proved in Dhage [3, 4].

3. Main result

In this section, we prove an existence and approximation result for the HDE (1.1) on a closed and bounded interval $J = [a, b]$ under mixed partial Lipschitz and partial compactness type conditions on the nonlinearities involved in it. We place the HDE (1.1) in the function space $C(J, \mathbb{R})$ of continuous real-valued functions defined on J. We define a norm $\|\cdot\|$ and the order relation $\le$ in $C(J, \mathbb{R})$ by

$$\|x\| = \sup_{t \in J} |x(t)| \tag{3.1}$$

and

$$x \le y \iff x(t) \le y(t) \quad \text{for all} \quad t \in J. \tag{3.2}$$

Clearly, $C(J, \mathbb{R})$ is a Banach space with respect to above supremum norm and also partially ordered w.r.t. the above partially order relation $\le$. It is known that the partially ordered Banach space $C(J,\mathbb{R})$ is regular and lattice so that every pair of elements of E has a lower and an upper bound in it. The following useful lemma concerning the Janhavi subsets of $C(J,\mathbb{R})$ follows immediately from the Arzelá-Ascoli theorem for compactness.

Lemma 3.1. *Let* $\big(C(J,\mathbb{R}),\leq,\|\cdot\|\big)$ *be a partially ordered Banach space with the norm* $\|\cdot\|$ *and the order relation* $\leq$ *defined by (3.1) and (3.2) respectively. Then every partially compact subset of* $C(J,\mathbb{R})$ *is Janhavi.*

Proof. The proof of the lemma is well-known and appears in the papers of Dhage [6, 7], Dhage and Dhage [9, 10], Dhage *et al.* [12] and so we omit the details. □

We need the following definition in what follows.

Definition 3.1. A differentiable function $u \in C(J,\mathbb{R})$ is said to be a lower solution of the equation (1.1) if it satisfies

$$\begin{cases} u'(t) \leq f(t,u(t)) + g\left(t, \max\limits_{a\leq\xi\leq t} u(\xi)\right), \\ u(a) \leq \alpha_0, \end{cases} \tag{$*$}$$

for all $t \in J$. Similarly, a differentiable function $v \in C(J,\mathbb{R})$ is called an upper solution of the HDE (1.1) if the above inequality is satisfied with reverse sign.

We consider the following set of assumptions in what follows:

(H_1) There exist constants $\lambda > 0, \mu > 0$ with $\lambda \geq \mu$ such that

$$0 \leq [f(t,x) + \lambda x] - [f(t,y) + \lambda y] \leq \mu(x-y)$$

for all $t \in J$ and $x, y \in \mathbb{R}$, $x \geq y$.

(H_2) There exists a constant $M > 0$ such that $|g(t,x)| \leq M$, for all $t \in J,\ x \in \mathbb{R}$;

(H_3) $g(t,x)$ is nondecreasing in x for each $t \in J$.

(H_4) HDE (1.1) has a lower solution $u \in C(J,\mathbb{R})$.

Now we consider the following HDE

$$\begin{cases} x'(t) + \lambda x(t) = \widetilde{f}(t,x(t)) + g\left(t, \max\limits_{a\leq\xi\leq t} u(\xi)\right), \\ x(a) = \alpha_0, \end{cases} \tag{3.3}$$

for all $t \in J = [a,b]$, where $\widetilde{f}, g \ :\ J\times\mathbb{R} \to \mathbb{R}$ and $\widetilde{f}(t,x) = f(t,x) + \lambda x,\ \lambda > 0$.

Remark 3.1. A differentiable function $u \in C(J,\mathbb{R})$ is a solution of the equation (3.3) if and only if it is a solution of the equation (1.1) defined on J.

We also consider the following condition in what follows.

(H_5) There exists a constant $K > 0$ such that $\left|\widetilde{f}(t,x)\right| \leq K$, for all $t \in J$ and $x \in \mathbb{R}$;

Lemma 3.2. *Suppose that the hypotheses $(H_2), (H_3)$ and (H_5) hold. Then a function $x \in C(J, \mathbb{R})$ is a solution of the HDE (3.3) if and only if it is a solution of the nonlinear integral equation*

$$\begin{aligned} x(t) &= \alpha_0 e^{-\lambda t} + e^{-\lambda t} \int_a^t e^{\lambda s} \widetilde{f}(s, x(s))\, ds \\ &\quad + e^{-\lambda t} \int_a^t e^{\lambda s} g\left(s, \max_{a \le \xi \le s} x(\xi)\right) ds, \end{aligned} \tag{3.4}$$

for all $t \in J$.

Theorem 3.1. *Suppose that hypotheses $(H_1) - (H_5)$ hold. Then the HDE (1.1) has a solution x^* defined on J and the sequence $\{x_n\}$ of successive approximations defined by*

$$\begin{aligned} x_0 &= u, \\ x_{n+1}(t) &= \alpha_0 e^{-\lambda t} + e^{-\lambda t} \int_a^t e^{\lambda s} \widetilde{f}(s, x_n(s))\, ds \\ &\quad + e^{-\lambda t} \int_a^t e^{\lambda s} g\left(s, \max_{a \le \xi \le s} x_n(\xi)\right) ds, \end{aligned} \tag{3.5}$$

for all $t \in J$, converges monotonically to x^*.

Proof. Set $E = C(J, \mathbb{R})$. Then, in view of Lemma 3.1, every compact chain C in E possesses the compatibility property with respect to the norm $\|\cdot\|$ and the order relation $\le$ so that every compact chain C is Janhavi in E.

Define two operators $\mathcal{A}$ and $\mathcal{B}$ on E by

$$\mathcal{A}x(t) = \alpha_0 e^{-\lambda t} + e^{-\lambda t} \int_a^t e^{\lambda s} \widetilde{f}(s, x(s))\, ds,\ t \in J, \tag{3.6}$$

and

$$\mathcal{B}x(t) = e^{-\lambda t} \int_a^t e^{\lambda s} g\left(s, \max_{a \le \xi \le s} x(\xi)\right) ds,\ t \in J. \tag{3.7}$$

From the continuity of the integral, it follows that $\mathcal{A}$ and $\mathcal{B}$ define the operators $\mathcal{A}, \mathcal{B} : E \to E$. Applying Lemma 3.2, the HDE (1.1) is equivalent to the operator equation

$$\mathcal{A}x(t) + \mathcal{B}x(t) = x(t),\ t \in J.$$

Now, we show that the operators $\mathcal{A}$ and $\mathcal{B}$ satisfy all the conditions of Theorem 2.1 in a series of following steps.

Step I. *$\mathcal{A}$ and $\mathcal{B}$ are nondecreasing on E.*

Let $x, y \in E$ be such that $x \ge y$. Then by hypothesis (H_1), we get

$$\begin{aligned} \mathcal{A}x(t) &= \alpha_0 e^{-\lambda t} + e^{-\lambda t} \int_a^t e^{\lambda s} \widetilde{f}(s, x(s))\, ds \\ &\ge \alpha_0 e^{-\lambda t} + e^{-\lambda t} \int_a^t e^{\lambda s} \widetilde{f}(s, y(s))\, ds \\ &= \mathcal{A}y(t), \end{aligned}$$

for all $t \in J$.

Next, we show that the operator $\mathcal{B}$ is also nondecreasing on E. Let $x, y \in E$ be such that $x \geq y$. Then $x(t) \geq y(t)$ for all $t \in J$. Since y is continuous on $[a, t]$, there exists a $\xi^* \in [a, t]$ such that

$$y(\xi^*) = \max_{a \leq \xi \leq t} y(\xi).$$

By definition of $\leq$, one has $x(\xi^*) \geq y(\xi^*)$. Consequently, we obtain

$$\max_{a \leq \xi \leq t} x(\xi) \geq x(\xi^*) \geq y(\xi^*) = \max_{a \leq \xi \leq t} y(\xi).$$

Now, using hypothesis (H_3), it can be shown that the operator $\mathcal{B}$ is also nondecreasing on E.

Step II. $\mathcal{A}$ *is partially bounded and partially contraction on* E.

Let $x \in E$ be arbitrary. Then by (H_5) we have

$$\begin{aligned} |\mathcal{A}x(t)| &\leq \left|\alpha_0 e^{-\lambda t}\right| + e^{-\lambda t} \int_a^t e^{-\lambda s} \left|\widetilde{f}(s, x(s))\right| ds \\ &\leq |\alpha_0| + K \int_a^b e^{\lambda s}\, ds \\ &\leq |\alpha_0| + e^{\lambda a} K(b-a), \end{aligned}$$

for all $t \in J$. Taking the supremum over t, we obtain

$$\|\mathcal{A}x(t)\| \leq |\alpha_0| + e^{\lambda a} K(b-a),$$

so $\mathcal{A}$ is a bounded operator on E. This implies that $\mathcal{A}$ is partially bounded on E.

Let $x, y \in E$ be such that $x \geq y$. Then by (H_1) we have

$$\begin{aligned} |\mathcal{A}x(t) - \mathcal{A}y(t)| &\leq \left| e^{-\lambda t} \int_a^t e^{\lambda s} \left[\widetilde{f}(s, x(s)) - \widetilde{f}(s, y(s))\right] ds \right| \\ &\leq e^{-\lambda t} \int_a^t e^{\lambda s} \mu \left|x(s) - y(s)\right| ds \\ &\leq e^{-\lambda t} \int_a^t e^{\lambda s} \lambda \left|x(s) - y(s)\right| ds \\ &\leq e^{-\lambda t} \int_a^t \frac{d}{ds} e^{\lambda s} \left\|x - y\right\| ds \\ &\leq (1 - e^{-\lambda a}) \left\|x - y\right\|, \end{aligned}$$

for all $t \in J$. Taking the supremum over t, we obtain

$$\|\mathcal{A}x - \mathcal{A}y\| \leq L \left\|x - y\right\|,$$

for all $x, y \in E$ with $x \geq y$. Hence $\mathcal{A}$ is a partially contraction on E and which also implies that $\mathcal{A}$ is partially continuous on E.

Step III. $\mathcal{B}$ *is partially continuous on* E.

Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in a chain C such that $x_n \to x$, for all $n \in \mathbb{N}$. Then

$$\begin{aligned}\lim_{n\to\infty} \mathcal{B}x_n(t) &= \lim_{n\to\infty} e^{-\lambda t} \int_a^t e^{\lambda s} g\left(s, \max_{a\le\xi\le s} x_n(\xi)\right) ds \\ &= e^{-\lambda t} \int_a^t e^{\lambda s} \left[\lim_{n\to\infty} g\left(s, \max_{a\le\xi\le s} x_n(\xi)\right)\right] ds \\ &= e^{-\lambda t} \int_a^t e^{\lambda s} g\left(s, \max_{a\le\xi\le s} x(\xi)\right) ds \\ &= \mathcal{B}x(t),\end{aligned}$$

for all $t \in J$. This shows that $\mathcal{B}x_n$ converges to $\mathcal{B}x$ pointwise on J.

Now we show that $\{\mathcal{B}x_n\}_{n\in\mathbb{N}}$ is an equicontinuous sequence of functions in E. Let $t_1, t_2 \in J$ with $t_1 < t_2$. We have

$$\begin{aligned}|\mathcal{B}x_n(t_2) - \mathcal{B}x_n(t_1)| &= \Bigg| e^{-\lambda t_2} \int_a^{t_2} e^{\lambda s} g\left(s, \max_{a\le\xi\le s} x_n(\xi)\right) ds \\ &\quad - e^{-\lambda t_1} \int_a^{t_1} e^{\lambda s} g\left(s, \max_{a\le\xi\le s} x_n(\xi)\right) ds \Bigg| \\ &\le \left| \left(e^{-\lambda t_2} - e^{-\lambda t_1}\right) \int_a^{t_1} e^{\lambda s} g\left(s, \max_{a\le\xi\le s} x_n(\xi)\right) ds \right| \\ &\quad + \left| e^{-\lambda t_2} \int_{t_1}^{t_2} e^{\lambda s} g\left(s, \max_{a\le\xi\le s} x_n(\xi)\right) ds \right| \\ &\to 0 \quad \text{as} \quad t_2 \to t_1,\end{aligned}$$

uniformly for all $n \in \mathbb{N}$. This shows that the convergence $\mathcal{B}x_n \to \mathcal{B}x$ is uniform and hence $\mathcal{B}$ is partially continuous on E.

Step IV. *$\mathcal{B}$ is partially compact operator on E.*

Let C be an arbitrary chain in E. We show that $\mathcal{B}(C)$ is uniformly bounded and equicontinuous set in E. First we show that $\mathcal{B}(C)$ is uniformly bounded. Let $y \in \mathcal{B}(C)$ be any element. Then there is an element $x \in C$ such that $y = \mathcal{B}x$. By hypothesis (H_2)

$$\begin{aligned}|y(t)| &= |\mathcal{B}x(t)| \\ &= \left| e^{-\lambda t} \int_a^t e^{\lambda s} g\left(s, \max_{a\le\xi\le s} x(\xi)\right) ds \right| \\ &\le \int_a^t e^{\lambda s} \left| g\left(s, \max_{a\le\xi\le s} x(\xi)\right) \right| ds \\ &\le \int_a^b e^{\lambda b} M \, ds \\ &\le e^{\lambda b} M(b-a) = r,\end{aligned}$$

for all $t \in J$. Taking the supremum over t we obtain $\|y\| \le \|\mathcal{B}x\| \le r$, for all $y \in \mathcal{B}(C)$. Hence $\mathcal{B}(C)$ is uniformly bounded subset of E. Next we show that $\mathcal{B}(C)$

is an equicontinuous set in E. Let $t_1, t_2 \in J$, with $t_1 < t_2$. Then, for any $y \in \mathcal{B}(C)$, one has

$$\begin{aligned}
\big|y(t_2) - y(t_1)\big| &= |\mathcal{B}x(t_2) - \mathcal{B}x(t_1)| \\
&= \left| e^{-\lambda t_2} \int_a^{t_2} e^{\lambda s} g\left(s, \max_{a \le \xi \le s} x(\xi)\right) ds - e^{-\lambda t_1} \int_a^{t_1} e^{\lambda s} g\left(s, \max_{a \le \xi \le s} x(\xi)\right) ds \right| \\
&\le \left| \left(e^{-\lambda t_2} - e^{-\lambda t_1}\right) \int_a^{t_1} e^{\lambda s} g\left(s, \max_{a \le \xi \le s} x(\xi)\right) ds \right| \\
&+ \left| e^{-\lambda t_2} \int_{t_1}^{t_2} e^{\lambda s} g\left(s, \max_{a \le \xi \le s} x(\xi)\right) ds \right| \\
&\to 0 \quad \text{as} \quad t_1 \to t_2
\end{aligned}$$

uniformly for all $y \in \mathcal{B}(C)$. This shows that $\mathcal{B}(C)$ is an equicontinuous subset of E. So $\mathcal{B}(C)$ is a uniformly bounded and equicontinuous set of functions in E and hence it is compact in view of Arzelá-Ascoli theorem. Consequently $\mathcal{B} : E \to E$ is a partially compact operator of E into itself.

Step V. *u satisfies the inequality $u \le \mathcal{A}u + \mathcal{B}u$.*

By hypothesis (H_4) the equation (1.1) has a lower solution u defined on J. Then we have

$$\begin{cases} u'(t) \le f(t, u(t)) + g\left(t, \max_{a \le \xi \le t} u(\xi)\right), \ t \in J, \\ u(a) \le \alpha_0. \end{cases} \tag{3.8}$$

Adding $\lambda u(t)$ on both sides of the first inequality in (3.8), we obtain

$$u'(t) + \lambda u(t) \le f(t, u(t)) + \lambda u(t) + g\left(t, \max_{a \le \xi \le t} u(\xi)\right), \ t \in J.$$

Again, multiplying the above inequality by $e^{\lambda t}$,

$$\left(e^{\lambda t} u(t)\right)' \le e^{\lambda t} \widetilde{f}(t, u(t)) + e^{\lambda t} g\left(t, \max_{a \le \xi \le t} u(\xi)\right). \tag{3.9}$$

A direct integration of (3.9) from a to t yields

$$\begin{aligned} u(t) \le \alpha_0 e^{-\lambda t} &+ e^{-\lambda t} \int_a^t e^{\lambda s} \widetilde{f}(s, u(s)) ds \\ &+ e^{-\lambda t} \int_a^t e^{\lambda s} g\left(s, \max_{a \le \xi \le s} u(\xi)\right) ds, \end{aligned} \tag{3.10}$$

for $t \in J$. From definitions of the operators $\mathcal{A}$ and $\mathcal{B}$ it follows that

$$u(t) \le \mathcal{A}u(t) + \mathcal{B}u(t),$$

for all $t \in J$. Hence $u \le \mathcal{A}u + \mathcal{B}u$. Thus $\mathcal{A}$ and $\mathcal{B}$ satisfy all the conditions of Theorem 2.1 and we apply it to conclude that the operator equation $\mathcal{A}x + \mathcal{B}x = x$ has a solution. Consequently the integral equation and the equation (1.1) has a solution x^* defined on J. Furthermore, the sequence $\{x_n\}_{n=0}^{\infty}$ of successive approximations defined by (3.5) converges monotonically to x^*. This completes the proof. □

Remark 3.2. The conclusion of Theorem 3.1 also remains true if we replace the hypothesis (H_4) with the following one.

(H_4') The HDE (1.1) has an upper solution $v \in C(J, \mathbb{R})$.

Remark 3.3. We note that if the HDE (1.1) has a lower solution u as well as an upper solution v such that $u \leq v$, then under the given conditions of Theorem 3.1 it has corresponding solutions x_* and x^* and these solutions satisfy $x_* \leq x^*$. Hence they are the minimal and maximal solutions of the HDE (1.1) in the vector segment $[u, v]$ of the Banach space $E = C(J, \mathbb{R})$, where the vector segment $[u, v]$ is a set of elements in $C(J, \mathbb{R})$ defined by

$$[u, v] = \{x \in C(J, \mathbb{R}) \mid u \leq x \leq v\}.$$

This is because the order relation $\leq$ defined by (3.2) is equivalent to the order relation defined by the order cone $\mathcal{K} = \{x \in C(J, \mathbb{R}) \mid x \geq \theta\}$ which is a closed set in $C(J, \mathbb{R})$.

In the following we illustrate our hypotheses and the main abstract result for the validity of conclusion.

Example 3.1. We consider the following HDE

$$\begin{cases} x'(t) = \arctan x(t) - x(t) + \tanh\left(\max\limits_{0 \leq \xi \leq t} x(\xi)\right), \ t \in J = [0, 1], \\ x(0) = 1. \end{cases} \tag{3.11}$$

Here $f(t, x) = \arctan x(t) - x(t)$ and $g(t, x) = \tanh x$. The functions f and g are continuous on $J \times \mathbb{R}$. Next, we have

$$0 \leq \arctan x(t) - \arctan y(t) \leq \frac{1}{\xi^2 + 1}(x - y),$$

for all $x, y \in \mathbb{R}, x > \xi > y$. Therefore $\lambda = 1 > \frac{1}{\xi^2+1} = \mu$. Hence the function f satisfies the hypothesis (H_1). Moreover, the function $\widetilde{f}(t, x) = \arctan x(t)$ is bounded on $J \times \mathbb{R}$ with bound $K = \pi/2$, so that the hypothesis (H_5) is satisfied. The function g is bounded on $J \times \mathbb{R}$ by $M = 1$, so (H_2) holds. The function $g(t, x)$ is increasing in x for each $t \in J$, so the hypothesis (H_3) is satisfied. The HDE (3.11) has a lower solution $u(t) = -2t + 1$, $t \in [0, 1]$. Thus all hypothesis of Theorem 3.1 are satisfied and hence the HDE (3.11) has a solution x^* defined on J and the sequence $\{x_n\}_{n=0}^{\infty}$ defined by

$$\begin{aligned} x_0 &= -2t + 1, \\ x_{n+1}(t) &= e^{-t} + e^{-t} \int_0^t e^s \arctan x_n(s)\, ds \\ &\quad + e^{-t} \int_0^t e^s \tanh\left(\max_{0 \leq \xi \leq s} x_n(\xi)\right) ds \end{aligned}$$

for each $t \in J$, converges monotonically to x^*.

Remark 3.4. Finally while concluding, we mention that the study of this paper may be extended with appropriate modifications to the nonlinear hybrid differential equation with maxima,

$$\begin{cases} x'(t) = f\left(t, x(t), \max\limits_{a\leq\xi\leq t} x(\xi)\right) + g\left(t, x(t), \max\limits_{a\leq\xi\leq t} x(\xi)\right), \\ x(a) = \alpha_0 \in \mathbb{R}, \end{cases} \tag{3.12}$$

for all $t \in J = [a, b]$, where $f, g \ : \ J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are continuous functions. When $g \equiv 0$, the differential equation (3.12) reduces to the nonlinear differential equations with maxima,

$$\begin{cases} x'(t) = f\left(t, x(t), \max\limits_{a\leq\xi\leq t} x(\xi)\right), \ \ t \in J, \\ x(a) = \alpha_0 \in \mathbb{R}, \end{cases} \tag{3.13}$$

which is studied in Otrocol and Rus [17] for existence and uniqueness theorem via Picard iterations under strong Lipschitz condition. Therefore, the obtained results for differential equation (3.12) with maxima via Dhage iteration method will include the existence and approximation results for the differential equation with maxima (3.13) under weak partial Lipschitz condition.

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