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HYBRID VISCOSITY EXTRAGRADIENT METHOD FOR SYSTEMS OF VARIATIONAL INEQUALITIES, FIXED POINTS OF NONEXPANSIVE MAPPINGS, ZERO POINTS OF ACCRETIVE OPERATORS IN BANACH SPACES

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Abstract. In this paper, we introduce a hybrid viscosity extragradient method for a general system of variational inequalities with solutions being also common fixed points of a countable family of nonexpansive mappings and zeros point of an accretive operator in real smooth Banach spaces. Under quite appropriate assumptions, we obtain some strong convergence results which improve and develop the corresponding results in the literature.

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1. Introduction

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$, C be a nonempty closed convex subset of H and P_C be the metric projection from H onto C. Let $T : C \to H$ be a mapping. We denote by $F(T)$ the set of fixed points of T. The symbol $\mathbb{R}$ denotes the set of all real numbers, while $\mathbb{R}_+$ stands for the set of all nonnegative real numbers.

A mapping $A : C \to H$ is called monotone if $\langle Ax - Ay, x - y \rangle \geq 0$, for all $x, y \in C$. Also, $A : C \to H$ is called α-strongly monotone if there exists a constant $\alpha > 0$ such that

$$\langle Ax - Ay, x - y \rangle \geq \alpha \|x - y\|^2, \ \forall x, y \in C. \tag{1.1}$$

The variational inequality was first discussed by Lions and Stampacchia [14] which has emerged as an important tool in the study of a wide class of obstacle, unilateral, free, moving, equilibrium problems arising in several branches of pure and applied sciences in a unified and general framework. Several efficient methods for variational inequalities have received much attention in the literature, see e.g., [7]-[13], [18], [22], [26]-[30] and the references therein.

Let $B_1, B_2 : C \to H$ be two nonlinear mappings. In 2008, Ceng et al. [10] considered and studied the following problem of finding $(x^*, y^*) \in C \times C$ such that

$$\begin{cases} \langle \rho B_1 y^* + x^* - y^*, x - x^* \rangle \geq 0, \ \forall x \in C, \\ \langle \eta B_2 x^* + y^* - x^*, x - y^* \rangle \geq 0, \ \forall x \in C, \end{cases} \tag{1.2}$$

which is called a general system of variational inequalities (GSVI). In [10], problem (1.2) is transformed into a fixed point problem according to the following relation.

Lemma 1.1. ([10]) *For given $x^*, y^* \in C$, (x^*, y^*) is a solution of problem (1.2) if and only if $x^* \in \mathrm{GSVI}(C, B_1, B_2)$, where $\mathrm{GSVI}(C, B_1, B_2)$ is the fixed point set of the mapping $G := P_C(I - \rho B_1) P_C(I - \eta B_2)$, and $y^* = P_C(I - \eta B_2)x^*$.*

Utilizing Lemma 1.1, Ceng et al. [10] proposed a relaxed extragradient method for solving problem (1.2) and proved the strong convergence of the proposed method to a solution of problem (1.2).

Let E be a real Banach space with the dual E^* and $J : E \to 2^{E^*}$ be the normalized duality mapping from E into 2^{E^*} defined by

$$J(x) = \{\varphi \in E^* : \langle x, \varphi \rangle = \|x\| \|\varphi\|, \ \|\varphi\| = \|x\|\}, \ \forall x \in E,$$

where $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing between E and E^*. Recall that if E is smooth then J is single-valued. In the sequel, we shall denote by j the single-valued normalized duality mapping.

Let C be a nonempty closed convex subset of E. A self-mapping $f : C \to C$ is said to be k-Lipschitz on C if $k \in \mathbb{R}_+$ and $\|f(x) - f(y)\| \leq k\|x - y\|$ for all $x, y \in C$. If f is k-Lipschitz with $k < 1$, then f is called a k-contraction mapping or a contraction with coefficient k. A self-mapping $f : C \to C$ is said to be nonexpansive if it is Lipschitz with $k = 1$. Recall that a (possibly multi-valued) mapping A with domain $D(A)$ and range $R(A)$ in a real Banach space E is accretive if, for each $x_i \in D(A)$ and $y_i \in Ax_i$ $(i \in \{1, 2\})$ there exists a $j(x_1 - x_2) \in J(x_1 - x_2)$ such that

$$\langle y_1 - y_2, j(x_1 - x_2) \rangle \geq 0.$$

An accretive operator A satisfy the range condition if $\overline{D(A)} \subset R(I + rA)$, for all $r > 0$.

Within the period of past thirty years, a great deal of effort has gone into the iterative construction of zero points of accretive mappings, and of fixed points of pseudocontractive mappings, see, e.g., [1, 3, 5, 6, 7, 11, 15, 16, 21, 24, 25].

Let $B_1, B_2 : C \to E$ be two nonlinear mappings. The general system of variational inequalities (GSVI) is to find $(x^*, y^*) \in C \times C$ such that

$$\begin{cases} \langle \rho B_1 y^* + x^* - y^*, j(x - x^*) \rangle \geq 0, \ \forall x \in C, \\ \langle \eta B_2 x^* + y^* - x^*, j(x - y^*) \rangle \geq 0, \ \forall x \in C, \end{cases} \tag{1.3}$$

where ρ and η are two positive constants. In particular, if $B_1 = B_2 = B$, then problem (1.3) reduces to the following system of variational inequalities (SVI) in

Banach spaces: find $(x^*, y^*) \in C \times C$ such that

$$\begin{cases} \langle \rho By^* + x^* - y^*, j(x - x^*)\rangle \geq 0, \ \forall x \in C, \\ \langle \eta Bx^* + y^* - x^*, j(x - y^*)\rangle \geq 0, \ \forall x \in C. \end{cases} \tag{1.4}$$

Further, if $x^* = y^*$, then we obtain the following variational inequality (VI) in Banach spaces: find $x^* \in C$ such that

$$\langle Bx^*, j(x - x^*)\rangle \geq 0, \ \forall x \in C. \tag{1.5}$$

Whenever $E = H$ a Hilbert space, it is easy to see that the GSVI (1.3) reduces to the GSVI (1.2) and that the SVI (1.4) and VI (1.5) reduce to the following SVI (1.6) and VI (1.7), respectively, that is, find $(x^*, y^*) \in C \times C$ such that

$$\begin{cases} \langle \rho By^* + x^* - y^*, x - x^*\rangle \geq 0, \ \forall x \in C, \\ \langle \eta Bx^* + y^* - x^*, x - y^*\rangle \geq 0, \ \forall x \in C, \end{cases} \tag{1.6}$$

and find $x^* \in C$ such that

$$\langle Bx^*, x - x^*\rangle \geq 0, \ \forall x \in C. \tag{1.7}$$

In 2005, Verma [22] suggested a two-step projection method for solving GSVI (1.2). In 2013, in order to solve GSVI (1.3), Yao et al. [27] first extended Verma's two-step method from Hilbert spaces to Banach spaces.

The purpose of this paper is to solve the GSVI (1.3) with solutions being also common fixed points of a countable family $\{S_i\}_{i=0}^{\infty}$ of nonexpansive mappings and zero points of an accretive operator A in a strictly convex and 2-uniformly smooth Banach space E. By applying the equivalence between the GSVI (1.3) and the fixed point problem, we construct a hybrid viscosity extragradient method for finding a solution of the GSVI (1.3), which is also a common fixed point of $\{S_i\}_{i=0}^{\infty}$ and a zero point of A. Under very suitable conditions, we derive some strong convergence results, which improve and develop the corresponding results announced by Ceng et al. [10], Yao et al. [27] and Ceng and Wen [11].

2. Preliminaries

Let E be a real Banach space with the dual E^*. Let C be a nonempty closed convex subset of E. Recall that a mapping $T : C \to E$ is said to be

(a) accretive if, for each $x, y \in C$, there exists $j(x - y) \in J(x - y)$ such that

$$\langle Tx - Ty, j(x - y)\rangle \geq 0;$$

(b) α-strongly accretive if, for each $x, y \in C$, there exists $j(x-y) \in J(x-y)$ such that

$$\langle Tx - Ty, j(x - y)\rangle \geq \alpha\|x - y\|^2 \text{ for some } \alpha \in (0, 1);$$

(c) β-inverse-strongly accretive if, for each $x, y \in C$, there exists $j(x - y) \in J(x - y)$ such that

$$\langle Tx - Ty, j(x - y)\rangle \geq \beta\|Tx - Ty\|^2 \text{ for some } \beta > 0.$$

Let $U = \{x \in E : \|x\| = 1\}$ be the unit sphere of E. Then the Banach space E is said to be strictly convex if for any $x, y \in U$, $x \neq y \;\Rightarrow\; \|\frac{x+y}{2}\| < 1$. It is also said to be uniformly convex if for each $\epsilon \in (0, 2]$, there exists $\delta > 0$ such that for any $x, y \in U$, $\|x-y\| \geq \epsilon \;\Rightarrow\; \|\frac{x+y}{2}\| \leq 1-\delta$. A Banach space E is said to have a Gateaux differentiable norm if the limit

$$\lim_{t\to 0} \frac{\|x+ty\| - \|x\|}{t}$$

exists for each $x, y \in U$ and in this case we call E smooth; E is said to have a uniformly Fréchet differentiable norm if the above limit is attained uniformly for $x, y \in U$. In this case we say that E is uniformly smooth. The space E is also said to have a Fréchet differentiable norm if for each $x \in U$, the above limit is attained uniformly for $y \in U$ and in this case we call E strongly smooth. The modulus of smoothness of E is defined by

$$\varrho(\tau) = \sup\left\{\frac{1}{2}(\|x+y\| + \|x-y\|) - 1 : x, y \in E,\ \|x\| = 1,\ \|y\| = \tau\right\},$$

where $\varrho : [0, \infty) \to [0, \infty)$ is a function. It is known that E is uniformly smooth if and only if $\lim\limits_{\tau\to 0} \frac{\varrho(\tau)}{\tau} = 0$. Let q be a fixed real number with $1 < q \leq 2$. A Banach space E is said to be q-uniformly smooth if there exists a constant $\kappa > 0$ such that $\varrho(\tau) \leq \kappa\tau^q$ for all $\tau > 0$. Let q be a real number with $1 < q \leq 2$. E is q-uniformly smooth if and only if there exists a constant $c > 0$ such that

$$\|x+y\|^q + \|x-y\|^q \leq 2(\|x\|^q + \|cy\|^q),\ \forall x, y \in E.$$

The best constant c in the above inequality is called the q-uniformly smooth constant of E; see [4] for more details. Note that no Banach space is q-uniformly smooth for $q > 2$; see [20] for more details.

Proposition 2.1. ([3]) *Let C be a nonempty closed convex subset of a Banach space E. Let $S_0, S_1, \ldots$ be a sequence of mappings of C into itself. Suppose that*

$$\sum_{n=1}^{\infty} \sup\{\|S_n x - S_{n-1}x\| : x \in C\} < \infty.$$

Then for each $y \in C$, $\{S_n y\}$ converges strongly to some point of C. Moreover, let S be a mapping of C into itself defined by $Sy = \lim\limits_{n\to\infty} S_n y$ for all $y \in C$. Then

$$\lim_{n\to\infty} \sup\{\|Sx - S_n x\| : x \in C\} = 0.$$

Proposition 2.2. ([23]) *Let E be a 2-uniformly smooth Banach space. Then*

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, j(x)\rangle + 2\|cy\|^2,\ \forall x, y \in E,$$

where c is the 2-uniformly smooth constant of E.

In particular, if E is a Hilbert space, then the duality pairing $\langle\cdot,\cdot\rangle$ reduces to the inner product, $j = I$ the identity mapping of E, and $c = 1/\sqrt{2}$.

Let D be a subset of C and let Π be a mapping of C into D. Then Π is said to be sunny if

$$\Pi[\Pi(x) + t(x - \Pi(x))] = \Pi(x),$$

whenever $\Pi(x) + t(x - \Pi(x)) \in C$ for $x \in C$ and $t \geq 0$. A mapping Π of C into itself is called a retraction if $\Pi^2 = \Pi$. If a mapping Π of C into itself is a retraction, then $\Pi(z) = z$ for each $z \in R(\Pi)$, where $R(\Pi)$ is the range of Π. A subset D of C is called a sunny nonexpansive retract of C if there exists a sunny nonexpansive retraction from C onto D.

Proposition 2.3. ([16]) *Let C be a nonempty closed convex subset of a smooth Banach space E, D be a nonempty subset of C and Π be a retraction of C onto D. Then the following are equivalent:*

(i) *Π is sunny and nonexpansive;*
(ii) $\|\Pi(x) - \Pi(y)\|^2 \leq \langle x - y, j(\Pi(x) - \Pi(y))\rangle, \forall x, y \in C$;
(iii) $\langle x - \Pi(x), j(y - \Pi(x))\rangle \leq 0, \forall x \in C, y \in D$.

In order to prove our main result, we need to use the following lemmas.

Lemma 2.4. (Resolvent identity). *For $\lambda, \mu > 0$ and $x \in E$,*

$$J_\lambda x = J_\mu \left(\frac{\mu}{\lambda}x + \left(1 - \frac{\mu}{\lambda}\right) J_\lambda x\right).$$

The following lemma is an immediate consequence of the subdifferential inequality of the function $\frac{1}{2}\|\cdot\|^2$.

Lemma 2.5. *Let E be a real Banach space and J be the normalized duality mapping on E. Then for any given $x, y \in E$, the following inequality holds:*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, j(x + y)\rangle, \quad \forall j(x + y) \in J(x + y).$$

Lemma 2.6. ([2]) *Let C be a nonempty closed convex subset of a smooth Banach space E. Let Π_C be a sunny nonexpansive retraction from E onto C, and let $B : C \to E$ be an accretive mapping. Then for all $\lambda > 0$,*

$$\mathrm{VI}(C, B) = F(\Pi_C(I - \lambda B)),$$

where $\mathrm{VI}(C, B)$ denotes the set of solutions to problem (1.5).

By Lemmas 1.1 and 2.6 respectively, we immediately obtain the following results.

Lemma 2.7. *Let C be a nonempty closed convex subset of a smooth Banach space E and $B_1, B_2 : C \to E$ be two nonlinear mappings. Let Π_C be a sunny nonexpansive retraction from E onto C. For given $x^*, y^* \in C$, (x^*, y^*) is a solution of the GSVI (1.3) if and only if $x^* \in \mathrm{GSVI}(C, B_1, B_2)$ where $\mathrm{GSVI}(C, B_1, B_2)$ is the set of fixed points of the mapping $G := \Pi_C(I - \rho B_1)\Pi_C(I - \eta B_2)$ and $y^* = \Pi_C(I - \eta B_2)x^*$.*

Lemma 2.8. *Let C be a nonempty closed convex subset of a 2-uniformly smooth Banach space E. Let the mapping $A : C \to E$ be α-inverse-strongly accretive. Then,*

$$\|(I - \lambda A)x - (I - \lambda A)y\|^2 \leq \|x - y\|^2 + 2\lambda(c^2\lambda - \alpha)\|Ax - Ay\|^2.$$

In particular, if $0 \leq \lambda \leq \frac{\alpha}{c^2}$, then $I - \lambda A$ is nonexpansive.

Utilizing Lemma 2.8, we immediately obtain the following lemma.

Lemma 2.9 *Let C be a nonempty closed convex subset of a 2-uniformly smooth Banach space E. Let Π_C be a sunny nonexpansive retraction from E onto C. Let* *the mappings $B_1, B_2 : C \to E$ be α-inverse-strongly accretive and β-inverse-strongly accretive, respectively. Let the mapping $G : C \to C$ be defined as*

$$G := \Pi_C(I - \rho B_1)\Pi_C(I - \eta B_2).$$

If $0 \leq \rho \leq \frac{\alpha}{c^2}$ and $0 \leq \eta \leq \frac{\beta}{c^2}$, then $G : C \to C$ is nonexpansive.

Lemma 2.10. ([5]) *Let C be a nonempty closed convex subset of a strictly convex Banach space E. Let $\{T_n\}_{n=0}^{\infty}$ be a sequence of nonexpansive mappings on C. Suppose that $\bigcap_{n=0}^{\infty} F(T_n)$ is nonempty. Let $\{\lambda_n\}$ be a sequence of positive numbers with*

$$\sum_{n=0}^{\infty} \lambda_n = 1.$$

Then a mapping S on C defined by $Sx = \sum_{n=0}^{\infty} \lambda_n T_n x$ for $x \in C$ is defined well, nonexpansive and $F(S) = \bigcap_{n=0}^{\infty} F(T_n)$ holds.

Lemma 2.11. ([24]) *Let E be a uniformly smooth Banach space, C be a nonempty closed convex subset of E, $T : C \to C$ be a nonexpansive mapping with $F(T) \neq \emptyset$, and f be a fixed contraction mapping. For each $t \in (0, 1)$, let $z_t \in C$ be the unique fixed point of the contraction $C \ni z \mapsto tf(z) + (1 - t)Tz$ on C, that is,*

$$z_t = tf(z_t) + (1 - t)Tz_t.$$

Then $\{z_t\}$ converges strongly to a fixed point $x^* \in F(T)$, which solves the variational inequality

$$\langle (I - f)x^*, j(x^* - x)\rangle \leq 0,\ \forall x \in F(T).$$

Corollary 2.12. ([17]) *Let C be a nonempty closed convex subset of a uniformly smooth Banach space E and let $T : C \to C$ be a nonexpansive mapping. For each fixed $u \in C$ and every $t \in (0, 1)$, the unique fixed point $x_t \in C$ of the contraction $C \ni z \mapsto tu + (1 - t)Tz$ converges strongly to a fixed point of T as $t \to 0^+$.*

Lemma 2.13. ([19]) *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space E and $\{\beta_n\}$ be a sequence in $[0, 1]$ with $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$. Suppose that $x_{n+1} = (1 - \beta_n)y_n + \beta_n x_n$ for all $n \geq 0$ and $\limsup_{n\to\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0$, then $\lim_{n\to\infty} \|y_n - x_n\| = 0$.*

Lemma 2.14. ([24]) *Let $\{a_n\}_{n=0}^{\infty}$ be a sequence of nonnegative real numbers satisfying*

$$a_{n+1} \leq (1 - \lambda_n)a_n + \lambda_n\sigma_n, \quad \forall n \geq 0,$$

where $\{\lambda_n\}_{n=0}^{\infty}$ and $\{\sigma_n\}_{n=0}^{\infty}$ are real sequences satisfying

(i) $\{\lambda_n\}_{n=0}^{\infty} \subset (0, 1), \quad \sum_{n=0}^{\infty} \lambda_n = \infty;$

(ii) *either* $\limsup_{n\to\infty} \sigma_n \le 0$ *or* $\sum_{n=0}^{\infty} |\lambda_n \sigma_n| < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

3. Main results

We are now in a position to state and prove our main result.

Theorem 3.1. *Let C be a nonempty closed convex subset of a strictly convex and 2-uniformly smooth Banach space E. Let Π_C be a sunny nonexpansive retraction from E onto C. Let $A \subset E \times E$ be an accretive operator such that*

$$\overline{D(A)} \subset C \subset \bigcap_{r>0} R(I + rA).$$

Let $B_1, B_2 : C \to E$ be α-inverse-strongly accretive and β-inverse-strongly accretive, respectively. Let $f : C \to C$ be a contraction with coefficient $k \in [0, 1)$. Let $\{S_i\}_{i=0}^{\infty}$ be a countably family of nonexpansive mappings of C into itself such that

$$\Omega := \bigcap_{i=0}^{\infty} F(S_i) \cap \mathrm{GSVI}(C, B_1, B_2) \cap A^{-1}0 \neq \emptyset,$$

where $\mathrm{GSVI}(C, B_1, B_2)$ is the fixed point set of the mapping

$$G := \Pi_C(I - \rho B_1)\Pi_C(I - \eta B_2)$$

with $0 < \rho \le \frac{\alpha}{c^2}$ and $0 < \eta \le \frac{\beta}{c^2}$. For arbitrarily given $x_0 \in C$, compute the sequences $\{x_n\}$ and $\{y_n\}$ by

$$\begin{cases} y_n = \Pi_C(x_n - \eta B_2 x_n), \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + (1 - \alpha_n - \beta_n)[\lambda \Pi_C(y_n - \rho B_1 y_n) + \mu S_n x_n \\ \qquad\quad + (1 - \lambda - \mu) J_{r_n} x_n], n \ge 0, \end{cases} \tag{3.1}$$

where $\lambda, \mu \in (0, 1)$ are two constants with $\lambda + \mu < 1$, and $\{\alpha_n\}, \{\beta_n\}$ and $\{r_n\}$ are the positive sequences such that

(i) $\alpha_n + \beta_n \le 1$ *for all* $n \ge 0$*;*

(ii) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$*;*

(iii) $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$*;*

(iv) $\lim_{n\to\infty} |r_{n+1} - r_n| = 0$ *and* $r_n \ge \varepsilon > 0$ *for all* $n \ge 0$.

Assume that $\sum_{n=0}^{\infty} \sup_{x\in D} \|S_{n+1}x - S_n x\| < \infty$ *for any bounded subset D of C.*
Let $S : C \to C$ be a mapping defined by $Sx = \lim_{n\to\infty} S_n x$ for all $x \in C$, and suppose that $F(S) = \bigcap_{i=0}^{\infty} F(S_i)$. If $\{r_n\}$ is a monotone decreasing sequence such that $\lim_{n\to\infty} r_n = r$,

then $\{x_n\}$ and $\{y_n\}$ converge strongly to $x^* \in \Omega$ and y^*, respectively, where (x^*, y^*) solves the GSVI (1.3) and x^* solves the variational inequality

$$\langle (I-f)x^*, j(x^*-x)\rangle \le 0,\ \forall x \in \Omega.$$

Proof. Note that the mapping $G : C \to C$ is defined as $G := \Pi_C(I-\rho B_1)\Pi_C(I-\eta B_2)$, where $0 < \rho \le \frac{\alpha}{c^2}$ and $0 < \eta \le \frac{\beta}{c^2}$. So, by Lemma 2.9, we know that G is nonexpansive. It is easy to see that the two-step iterative scheme (3.1) can be rewritten as

$$\begin{aligned} x_{n+1} = {} & \alpha_n f(x_n) + \beta_n x_n + (1-\alpha_n-\beta_n)[\lambda G x_n + \mu S_n x_n \\ & + (1-\lambda-\mu)J_{r_n}x_n],\ \forall n \ge 0. \end{aligned} \tag{3.2}$$

Next, we divide the rest of the proof into several steps.

Step 1. We show that $\{x_n\}$ is bounded. Indeed, take a fixed $p \in \Omega$ arbitrarily. Then we get $Gp = p$, $S_n p = p$ and $J_{r_n}p = p$ for all $n \ge 0$. It is clear that

$$\begin{aligned} x_{n+1} - p &= \alpha_n f(x_n) + \beta_n x_n + (1-\alpha_n-\beta_n)[\lambda G x_n + \mu S_n x_n \\ &\quad + (1-\lambda-\mu)J_{r_n}x_n] - p \\ &= \alpha_n(f(x_n)-p) + \beta_n(x_n-p) + (1-\alpha_n-\beta_n)[\lambda(Gx_n-p) \\ &\quad + \mu(S_n x_n - p) + (1-\lambda-\mu)(J_{r_n}x_n - p)], \end{aligned}$$

which hence yields

$$\begin{aligned} \|x_{n+1}-p\| &\le \alpha_n\|f(x_n)-p\| + \beta_n\|x_n-p\| + (1-\alpha_n-\beta_n)\|\lambda(Gx_n-p) \\ &\quad + \mu(S_n x_n-p) + (1-\lambda-\mu)(J_{r_n}x_n-p)\| \\ &\le \alpha_n(k\|x_n-p\| + \|f(p)-p\|) + \beta_n\|x_n-p\| \\ &\quad + (1-\alpha_n-\beta_n)[\lambda\|x_n-p\| + \mu\|x_n-p\| + (1-\lambda-\mu)\|x_n-p\|] \\ &\le [1-(1-k)\alpha_n]\|x_n-p\| + \alpha_n\|f(p)-p\| \\ &\le \max\left\{\|x_n-p\|, \frac{1}{1-k}\|f(p)-p\|\right\}. \end{aligned} \tag{3.3}$$

By induction, we derive

$$\|x_n-p\| \le \max\left\{\|x_0-p\|, \frac{1}{1-k}\|f(p)-p\|\right\},\ \forall n \ge 0.$$

Thus, $\{x_n\}$ is bounded. Observe that

$$\|f(x_n)\| \le \|f(x_n)-f(p)\| + \|f(p)\| \le k\|x_n-p\| + \|f(p)\|,$$

and

$$\|S_n x_n\| \le \|S_n x_n - S_n p\| + \|S_n p\| \le \|x_n-p\| + \|p\|.$$

Similarly, by the nonexpansivity of J_{r_n}, $\Pi_C(I-\rho B_1)$, $\Pi_C(I-\eta B_2)$ and G, we know that $\{y_n\}$, $\{z_n\}$, $\{Gx_n\}$ and $\{J_{r_n}x_n\}$ all are bounded, where

$$z_n = \Pi_C(I-\rho B_1)y_n = Gx_n.$$

Step 2. We show that $\|x_{n+1}-x_n\| \to 0$ and $\|W_n x_n - x_n\| \to 0$ as $n \to \infty$, where

$$W_n x_n := \lambda G x_n + \mu S_n x_n + (1-\lambda-\mu)J_{r_n}x_n.$$

Indeed, we first claim that

$$\|J_{r_{n+1}}x_{n+1} - J_{r_n}x_n\| \le \|x_{n+1} - x_n\| + M_0|r_{n+1} - r_n|, \ \forall n \ge 0, \tag{3.4}$$

where

$$\sup_{n\ge 0}\left\{\frac{1}{\varepsilon}(\|J_{r_{n+1}}x_{n+1} - x_n\| + \|J_{r_n}x_n - x_{n+1}\|)\right\} \le M_0$$

for some $M_0 > 0$. As a matter of fact, if $r_n \le r_{n+1}$, using the resolvent identity in Lemma 2.4,

$$J_{r_{n+1}}x_{n+1} = J_{r_n}\left(\frac{r_n}{r_{n+1}}x_{n+1} + (1 - \frac{r_n}{r_{n+1}})J_{r_{n+1}}x_{n+1}\right),$$

we get

$$\begin{aligned}\|J_{r_{n+1}}x_{n+1} - J_{r_n}x_n\| &= \left\|J_{r_n}\left(\frac{r_n}{r_{n+1}}x_{n+1} + \left(1 - \frac{r_n}{r_{n+1}}\right)J_{r_{n+1}}x_{n+1}\right) - J_{r_n}x_n\right\| \\ &\le \frac{r_n}{r_{n+1}}\|x_{n+1} - x_n\| + \left(1 - \frac{r_n}{r_{n+1}}\right)\|J_{r_{n+1}}x_{n+1} - x_n\| \\ &\le \|x_{n+1} - x_n\| + \frac{r_{n+1} - r_n}{r_{n+1}}\|J_{r_{n+1}}x_{n+1} - x_n\| \\ &\le \|x_{n+1} - x_n\| + \frac{1}{\varepsilon}|r_{n+1} - r_n|\|J_{r_{n+1}}x_{n+1} - x_n\|.\end{aligned}$$

If $r_{n+1} \le r_n$, we derive in the similar way

$$\|J_{r_{n+1}}x_{n+1} - J_{r_n}x_n\| \le \|x_n - x_{n+1}\| + \frac{1}{\varepsilon}|r_n - r_{n+1}|\|J_{r_n}x_n - x_{n+1}\|.$$

Thus, combining the above cases, we know that (3.4) holds.

Now, putting $W_nx_n = \lambda Gx_n + \mu S_nx_n + (1-\lambda-\mu)J_{r_n}x_n$ for each $n \ge 0$, we obtain from (3.4) that

$$\begin{aligned}\|W_{n+1}x_{n+1} - W_nx_n\| &\le \lambda\|x_{n+1} - x_n\| + \mu(\|S_{n+1}x_{n+1} - S_{n+1}x_n\| \\ &\quad + \|S_{n+1}x_n - S_nx_n\|) + (1 - \lambda - \mu)\|J_{r_{n+1}}x_{n+1} - J_{r_n}x_n\| \\ &\le \lambda\|x_{n+1} - x_n\| + \mu(\|x_{n+1} - x_n\| + \|S_{n+1}x_n - S_nx_n\|) \\ &\quad + (1 - \lambda - \mu)(\|x_{n+1} - x_n\| + M_0|r_{n+1} - r_n|) \\ &= \|x_{n+1} - x_n\| + \mu\|S_{n+1}x_n - S_nx_n\| \\ &\quad + (1 - \lambda - \mu)M_0|r_{n+1} - r_n| \\ &\le \|x_{n+1} - x_n\| + \|S_{n+1}x_n - S_nx_n\| + M_0|r_{n+1} - r_n|.\end{aligned} \tag{3.5}$$

Setting

$$x_{n+1} = (1 - \beta_n)e_n + \beta_nx_n, \quad \forall n \ge 0, \tag{3.6}$$

we get

$$\begin{aligned} e_{n+1} - e_n &= \frac{\alpha_{n+1} f(x_{n+1}) + (1 - \alpha_{n+1} - \beta_{n+1}) W_{n+1} x_{n+1}}{1 - \beta_{n+1}} \\ &\quad - \frac{\alpha_n f(x_n) + (1 - \alpha_n - \beta_n) W_n x_n}{1 - \beta_n} \\ &= \frac{\alpha_{n+1}}{1 - \beta_{n+1}} (f(x_{n+1}) - W_{n+1} x_{n+1}) + W_{n+1} x_{n+1} \\ &\quad - \frac{\alpha_n}{1 - \beta_n} (f(x_n) - W_n x_n) - W_n x_n, \end{aligned}$$

and so it follows that

$$\begin{aligned} \|e_{n+1} - e_n\| \le \frac{\alpha_{n+1}}{1 - \beta_{n+1}} \|f(x_{n+1}) - W_{n+1} x_{n+1}\| + \frac{\alpha_n}{1 - \beta_n} \|f(x_n) - W_n x_n\| \\ + \|W_{n+1} x_{n+1} - W_n x_n\|, \end{aligned}$$

which together with (3.5) leads to

$$\begin{aligned} \|e_{n+1} - e_n\| - \|x_{n+1} - x_n\| \le \frac{\alpha_{n+1}}{1 - \beta_{n+1}} \|f(x_{n+1}) - W_{n+1} x_{n+1}\| + M_0 |r_{n+1} - r_n| \\ + \frac{\alpha_n}{1 - \beta_n} \|f(x_n) - W_n x_n\| + \|S_{n+1} x_n - S_n x_n\|. \end{aligned}$$

Since $\lim_{n\to\infty} \alpha_n = 0$ and $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$, we deduce from the boundedness of $\{f(x_n)\}$ and $\{W_n x_n\}$ that $\limsup_{n\to\infty} (\|e_{n+1} - e_n\| - \|x_{n+1} - x_n\|) \le 0$. So, from Lemma 2.13 it follows that

$$\lim_{n\to\infty} \|e_n - x_n\| = 0. \tag{3.7}$$

Noticing that (3.6), we have $\|x_{n+1} - x_n\| = (1 - \beta_n)\|e_n - x_n\|$.
Since $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$, we get from (3.7) that

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{3.8}$$

In addition,

$$x_{n+1} - x_n = \alpha_n (f(x_n) - W_n x_n) + (1 - \beta_n)(W_n x_n - x_n).$$

It follows that

$$(1 - \beta_n)\|W_n x_n - x_n\| \le \|x_{n+1} - x_n\| + \alpha_n \|f(x_n) - W_n x_n\|.$$

Since $\lim_{n\to\infty} \alpha_n = 0$ and $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$, we conclude from (3.8) that

$$\lim_{n\to\infty} \|W_n x_n - x_n\| = 0. \tag{3.9}$$

Step 3. We show that if $r_n \downarrow r$ as $n \to \infty$, then

$$\limsup_{n\to\infty} \langle x^* - f(x^*), j(x^* - x_n)\rangle \le 0, \ x^* \in \Omega, \tag{3.10}$$

where z_t is the fixed point of the mapping $z \mapsto tf(z) + (1-t)Wz$ with

$$W := \lambda G + \mu S + (1 - \lambda - \mu) J_r$$

for constants $\lambda, \mu \in (0,1)$ satisfying $\lambda + \mu < 1$, $x^* = \lim_{t\to 0^+} z_t$ and x^* solves the VI:

$$\langle (I-f)x^*, j(x^*-x)\rangle \le 0,\ \forall x \in \Omega.$$

Indeed, we define the mapping $Wx := \lambda Gx + \mu Sx + (1-\lambda-\mu)J_r x,\ \forall x \in C$, where $r_n \downarrow r$ as $n \to \infty$. Then by Lemma 2.9, we know that W is nonexpansive and

$$F(W) = F(S)\cap F(G)\cap F(J_r) = \bigcap_{i=0}^{\infty} F(S_i)\cap \mathrm{GSVI}(C,B_1,B_2)\cap A^{-1}0 \quad (=: \Omega).$$

We claim that

$$\lim_{n\to\infty} \|Wx_n - x_n\| = 0. \tag{3.11}$$

As a matter of fact, taking into account the resolvent identity in Proposition 2.4, we have

$$\begin{aligned}\|J_{r_n}x_n - J_r x_n\| &= \left\| J_r\left(\frac{r}{r_n}x_n + \left(1-\frac{r}{r_n}\right)J_{r_n}x_n\right) - J_r x_n\right\| \\ &\le \left(1-\frac{r}{r_n}\right)\|J_{r_n}x_n - x_n\|,\end{aligned}$$

which together with $r_n \downarrow r$ as $n \to \infty$ and the boundedness of $\{x_n\}, \{J_{r_n}x_n\}$, implies that

$$\lim_{n\to\infty} \|J_{r_n}x_n - J_r x_n\| = 0. \tag{3.12}$$

Utilizing Proposition 2.1, we get

$$\lim_{n\to\infty} \|S_n x_n - Sx_n\| = 0. \tag{3.13}$$

So, from (3.9), (3.12) and (3.13) it follows that

$$\begin{aligned}\|Wx_n - x_n\| &\le \|Wx_n - W_n x_n\| + \|W_n x_n - x_n\| \\ &\le \mu\|Sx_n - S_n x_n\| + (1-\lambda-\mu)\|J_r x_n - J_{r_n}x_n\| + \|W_n x_n - x_n\| \to 0,\end{aligned}$$

as $n \to \infty$. This means that (3.11) holds.

It is clear that the mapping $z \mapsto tf(z) + (1-t)Wz$ is a contraction of C into itself for each $t \in (0,1)$. So, z_t solves the fixed point equation $z_t = tf(z_t) + (1-t)Wz_t$. Then we have

$$z_t - x_n = (1-t)(Wz_t - x_n) + t(f(z_t) - x_n). \tag{3.14}$$

Thus, from Lemma 2.5 and (3.14), we obtain

$$\begin{aligned}\|z_t - x_n\|^2 &\le (1-t)^2\|Wz_t - x_n\|^2 + 2t\langle f(z_t) - x_n, j(z_t - x_n)\rangle \\ &\le (1-t)^2[\|Wz_t - Wx_n\| + \|Wx_n - x_n\|]^2 + 2t\langle f(z_t) - x_n, j(z_t - x_n)\rangle \\ &\le (1-t)^2[\|z_t - x_n\| + \|Wx_n - x_n\|]^2 + 2t\langle f(z_t) - x_n, j(z_t - x_n)\rangle \\ &= (1-t)^2[\|z_t - x_n\|^2 + 2\|z_t - x_n\|\|Wx_n - x_n\| + \|Wx_n - x_n\|^2] \\ &\quad + 2t\langle f(z_t) - x_n, j(z_t - x_n)\rangle,\end{aligned}$$

that is,

$$\begin{aligned}\|z_t - x_n\|^2 &\le (1-t)^2\|z_t - x_n\|^2 + \|Wx_n - x_n\|[2\|z_t - x_n\| + \|Wx_n - x_n\|] \\ &\quad + 2t\langle f(z_t) - z_t, j(z_t - x_n)\rangle + 2t\|z_t - x_n\|^2 \\ &= (1+t^2)\|z_t - x_n\|^2 + \|Wx_n - x_n\|[2\|z_t - x_n\| + \|Wx_n - x_n\|] \\ &\quad + 2t\langle f(z_t) - z_t, j(z_t - x_n)\rangle.\end{aligned}$$

It follows that

$$\langle z_t - f(z_t), j(z_t - x_n)\rangle \le \frac{t}{2}\|z_t - x_n\|^2 + \frac{1}{2t}\|Wx_n - x_n\|[2\|z_t - x_n\| + \|Wx_n - x_n\|]. \quad (3.15)$$

Letting $n \to \infty$ in (3.15) and noting that $\lim_{n\to\infty}\|x_n - Wx_n\| = 0$, we have

$$\limsup_{n\to\infty}\langle z_t - f(z_t), j(z_t - x_n)\rangle \le \frac{t}{2}M, \quad (3.16)$$

where M is a constant such that $\|z_t - x_n\|^2 \le M$ for all $n \ge 0$ and $t \in (0,1)$. Utilizing Lemma 2.11, we deduce that $\{z_t\}$ converges strongly to a fixed point

$$x^* \in F(W) = F(S) \cap F(G) \cap F(J_r) = \Omega,$$

which solves the variational inequality:

$$\langle (I - f)x^*, j(x^* - x)\rangle \le 0, \quad \forall x \in \Omega.$$

Since the duality mapping j is norm-to-norm uniformly continuous on bounded subsets of E, by letting $t \to 0^+$ in (3.16), we know that (3.10) holds.

Step 4. We show that $x_n \to x^*$ and $y_n \to y^*$ as $n \to \infty$, where (x^*, y^*) solves the GSVI (1.3). Indeed, utilizing Lemma 2.5, from (3.2) and the convexity of $\|\cdot\|^2$, we get

$$\begin{aligned}\|x_{n+1} - x^*\|^2 &= \|\alpha_n(f(x_n) - f(x^*)) + \beta_n(x_n - x^*) + (1-\alpha_n-\beta_n)[\lambda(Gx_n - x^*) \\ &\quad + \mu(S_nx_n - x^*) + (1-\lambda-\mu)(J_{r_n}x_n - x^*)] + \alpha_n(f(x^*) - x^*)\|^2 \\ &\le (1-\alpha_n-\beta_n)\|\lambda(Gx_n - x^*) + \mu(S_nx_n - x^*) \\ &\quad + (1-\lambda-\mu)(J_{r_n}x_n - x^*)\|^2 + 2\alpha_n\langle f(x^*) - x^*, j(x_{n+1} - x^*)\rangle \\ &\quad + \alpha_n\|f(x_n) - f(x^*)\|^2 + \beta_n\|x_n - x^*\|^2 \\ &\le (1-(1-k)\alpha_n)\|x_n - x^*\|^2 + 2\alpha_n\langle f(x^*) - x^*, j(x_{n+1} - x^*)\rangle.\end{aligned} \quad (3.17)$$

By (3.10) and $\sum_{n=0}^{\infty}\alpha_n = \infty$, we apply Lemma 2.14 to (3.17) to obtain $x_n \to x^*$.

Taking into account $x^* \in \Omega \subset \mathrm{GSVI}(C, B_1, B_2)$ and utilizing Lemma 2.7, we know that (x^*, y^*) solves the GSVI (1.3), where $y^* = \Pi_C(I - \eta B_2)x^*$. Since Π_C and $I - \eta B_2$ is nonexpansive mappings by Proposition 2.3 and Lemma 2.6, we have

$$\begin{aligned}\|y_n - y^*\| &= \|\Pi_C(I-\eta B_2)x_n - \Pi_C(I-\eta B_2)x^*\| \\ &\le \|(I-\eta B_2)x_n - (I-\eta B_2)x^*\| \\ &\le \|x_n - x^*\| \to 0 \text{ as } n \to \infty,\end{aligned}$$

that is, $y_n \to y^*$ as $n \to \infty$. This completes the proof.

Several consequences of the main result are now proposed.

Corollary 3.2. *Let C be a nonempty closed convex subset of a strictly convex and 2-uniformly smooth Banach space E. Let Π_C be a sunny nonexpansive retraction from E onto C. Let $A \subset E \times E$ be an accretive operator in E such that*

$$\overline{D(A)} \subset C \subset \bigcap_{r>0} R(I + rA),$$

let $B : C \to E$ be γ-inverse-strongly accretive, and let $f : C \to C$ be a contraction with coefficient $k \in [0, 1)$. Let $\{S_i\}_{i=0}^{\infty}$ be a countably family of nonexpansive mappings of C into itself such that

$$\Omega := \bigcap_{i=0}^{\infty} F(S_i) \cap F(G) \cap A^{-1}0 \neq \emptyset,$$

where $F(G)$ is the fixed point set of the mapping $G := \Pi_C(I - \rho B)\Pi_C(I - \eta B)$ with $0 < \rho \leq \frac{\gamma}{c^2}$ and $0 < \eta \leq \frac{\gamma}{c^2}$ for c the 2-uniformly smooth constant of E. For arbitrarily given $x_0 \in C$, compute the sequences $\{x_n\}$ and $\{y_n\}$ such that

$$\begin{cases} y_n = \Pi_C(x_n - \eta B x_n), \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + (1 - \alpha_n - \beta_n)[\lambda \Pi_C(y_n - \rho B y_n) + \mu S_n x_n \\ \qquad\qquad + (1 - \lambda - \mu) J_{r_n} x_n], n \geq 0, \end{cases}$$

where $\lambda, \mu \in (0, 1)$ are two constants with $\lambda + \mu < 1$, and $\{\alpha_n\}, \{\beta_n\}$ and $\{r_n\}$ are the positive sequences such that

(i) $\alpha_n + \beta_n \leq 1$ *for all* $n \geq 0$*;*

(ii) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$*;*

(iii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$*;*

(iv) $\lim_{n\to\infty} |r_{n+1} - r_n| = 0$ *and* $r_n \geq \varepsilon > 0$ *for all* $n \geq 0$.

Assume that $\sum_{n=0}^{\infty} \sup_{x\in D} \|S_{n+1}x - S_n x\| < \infty$ *for any bounded subset D of C, and let S be a mapping of C into itself defined by $Sx = \lim_{n\to\infty} S_n x$ for all $x \in C$, and suppose that*

$$F(S) = \bigcap_{i=0}^{\infty} F(S_i).$$

If $\{r_n\}$ is a monotone decreasing sequence such that $\lim_{n\to\infty} r_n = r$, then $\{x_n\}$ and $\{y_n\}$ converge strongly to $x^* \in \Omega$ and y^*, respectively, where (x^*, y^*) solves the SVI (1.4) and x^* solves the variational inequality: $\langle (I - f)x^*, j(x^* - x)\rangle \leq 0$, $\forall x \in \Omega$.

Corollary 3.3. *Let C be a nonempty closed convex subset of a strictly convex and 2-uniformly smooth Banach space E. Let Π_C be a sunny nonexpansive retraction from E onto C. Let $A \subset E \times E$ be an accretive operator in E such that*

$$\overline{D(A)} \subset C \subset \bigcap_{r>0} R(I + rA),$$

let $B : C \to E$ be γ-inverse-strongly accretive, and let $f : C \to C$ be a contraction with coefficient $k \in [0, 1)$. Let $\{S_i\}_{i=0}^{\infty}$ be a countably family of nonexpansive mappings of C into itself such that

$$\Omega := \bigcap_{i=0}^{\infty} F(S_i) \cap \mathrm{VI}(C, B) \cap A^{-1}0 \neq \emptyset,$$

where $\mathrm{VI}(C, B)$ is the solution set of the VI (1.5). For arbitrarily given $x_0 \in C$, compute the sequence $\{x_n\}$ such that

$$x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + (1-\alpha_n-\beta_n)[\lambda \Pi_C(x_n - \rho B x_n) + \mu S_n x_n + (1-\lambda-\mu) J_{r_n} x_n],$$

where $\lambda, \mu \in (0, 1)$ are two constants with $\lambda + \mu < 1$, $0 < \rho \leq \frac{\gamma}{c^2}$ for c the 2-uniformly smooth constant of E, and $\{\alpha_n\}, \{\beta_n\}$ and $\{r_n\}$ are the positive sequences such that

(i) $\alpha_n + \beta_n \leq 1$ *for all* $n \geq 0$;

(ii) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$;

(iii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;

(iv) $\lim_{n\to\infty} |r_{n+1} - r_n| = 0$ *and* $r_n \geq \varepsilon > 0$ *for all* $n \geq 0$.

Assume that $\sum_{n=0}^{\infty} \sup_{x\in D} \|S_{n+1}x - S_n x\| < \infty$ for any bounded subset D of C, and let S be a mapping of C into itself defined by $Sx = \lim_{n\to\infty} S_n x$ for all $x \in C$, and suppose that $F(S) = \bigcap_{i=0}^{\infty} F(S_i)$. If $\{r_n\}$ is a monotone decreasing sequence such that $\lim_{n\to\infty} r_n = r$, then $\{x_n\}$ converges strongly to $x^* \in \Omega$.

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References

[1] C.D. Alecsa, *Stability results and qualitative properties for Mann's algorithm via admissible perturbations technique*, Appl. Anal. Optim., **1**(2017), 327-344.

[2] K. Aoyama, H. Iiduka, W. Takahashi, *Weak convergence of an iterative sequence for accretive operators in Banach spaces*, Fixed Point Theory Appl., **2006** (2006), Article ID 35390, 13 pp.

[3] K. Aoyama, Y. Kimura, W. Takahashi, M. Toyoda, *Approximation of common fixed points of a countable family of nonexpansive mappings in a Banach space*, Nonlinear Anal., **67**(8)(2007), 2350-2360.

[4] K. Ball, E.A. Carlen, E.H. Lieb, *Sharp uniform convexity and smoothness inequalities for trace norms*, Invent. Math., **115**(1994), 463-482.

[5] R.E. Bruck, *Properties of fixed-point sets of nonexpansive mappings in Banach spaces*, Trans. Amer. Math. Soc., **179**(1973), 251-262.

[6] L.C. Ceng, Q.H. Ansari, S. Schaible, J.C. Yao, *Hybrid viscosity approximation method for zeros of m-accretive operators in Banach spaces*, Numer. Funct. Anal. Optim., **33**(2012), 142-165.

[7] L.C. Ceng, S.M. Guu, J.C. Yao, *Hybrid viscosity CQ method for finding a common solution of a variational inequality, a general system of variational inequalities, and a fixed point problem*, Fixed Point Theory Appl., **2013**(2013), Art. ID 313, 25 pp.

[8] L.C. Ceng, Y.C. Liou, C.F. Wen, *Systems of variational inequalities with hierarchical variational inequality constraints in Banach spaces*, J. Nonlinear Sci. Appl., **10**(2017), 3136-3154.

[9] L.C. Ceng, Y.C. Liou, J.C. Yao, Y.H. Yao, *Well-posedness for systems of time-dependent hemivariational inequalities in Banach spaces*, J. Nonlinear Sci. Appl., **10**(2017), 4318-4336.

[10] L.C. Ceng, C.Y. Wang, J.C. Yao, *Strong convergence theorems by a relaxed extragradient method for a general system of variational inequalities*, Math. Methods Oper. Res., **67**(2008), 375-390.

[11] L.C. Ceng, C.F. Wen, *System of variational inequalities and an accretive operator in Banach spaces*, Fixed Point Theory Appl., **2013**(2013), Art. ID 249, 37 pp.

[12] L.C. Ceng, C.F. Wen, Y.H. Yao, *Iteration approaches to hierarchical variational inequalities for infinite nonexpansive mappings and zero points of m-accretive operators*, J. Nonlinear Var. Anal., **1**(2017), 213-235.

[13] S. Kamimura, W. Takahashi, *Weak and strong convergence of solutions to accretive operator inclusions and applications*, Set-Valued Anal., **8**(2000), 361-374.

[14] J.L. Lions, G. Stampacchia, *Variational inequalities*, Comm. Pure Appl. Math., **20**(1967), 493-517.

[15] G. Marino, R. Zaccone, *On strong convergence of some midpoint type methods for nonexpansive mappings*, J. Nonlinear Var. Anal., **1**(2017), 159-174.

[16] S. Reich, *Weak convergence theorems for nonexpansive mappings in Banach spaces*, J. Math. Anal. Appl., **67**(1979), 274-276.

[17] S. Reich, *Strong convergence theorems for resolvents of accretive operators in Banach spaces*, J. Math. Anal. Appl., **75**(1980), 287-292.

[18] D.R. Sahu, J.C. Yao, *A generalized hybrid steepest descent method and applications*, J. Nonlinear Var. Anal., **1**(2017), 111-126.

[19] T. Suzuki, *Strong convergence of Krasnoselskii and Mann's type sequences for one-parameter nonexpansive semigroups without Bochner integrals*, J. Math. Anal. Appl., **305**(2005), 227-239.

[20] Y. Takahashi, K. Hashimoto, M. Kato, *On sharp uniform convexity, smoothness, and strong type, cotype inequalities*, J. Nonlinear Convex Anal., **3**(2002), 267-281.

[21] W. Takahashi, J.C. Yao, *A strong convergence theorem by the hybrid method for a new class of nonlinear operators in a Banach space and applications*, Appl. Anal. Optim., **1**(2017), 1-17.

[22] R.U. Verma, *General convergence analysis for two-step projection methods and applications to variational problems*, Appl. Math. Lett., **18**(2005), 1286-1292.

[23] H.K. Xu, *Inequalities in Banach spaces with applications*, Nonlinear Anal., **16**(1991), 1127-1138.

[24] H.K. Xu, *Viscosity approximation methods for nonexpansive mappings*, J. Math. Anal. Appl., **298**(2004), 279-291.

[25] Y.H. Yao, R.D. Chen, J.C. Yao, *Strong convergence and certain control conditions for modified Mann iteration*, Nonlinear Anal., **68**(2008), 1687-1693.

[26] Y.H. Yao, Y.C. Liou, S.M. Kang, *Approach to common elements of variational inequality problems and fixed point problems via a relaxed extragradient method*, Comput. Math. Appl., **59**(2010), 3472-3480.

[27] Y.H. Yao, Y.C. Liou, S.M. Kang, *Two-step projection methods for a system of variational inequality problems in Banach spaces*, J. Global Optim., **55**(2013), 801-811.

[28] Y.H. Yao, Y.C. Liou, J.C. Yao, *Iterative algorithms for the split variational inequality and fixed point problems under nonlinear transformations*, J. Nonlinear Sci. Appl., **10**(2017), 843-854.

[29] Y.H. Yao, N. Shahzad, *Strong convergence of a proximal point algorithm with general errors*, Optim. Lett., **6**(2012), 621-628.

[30] H. Zegeye, N. Shahzad, Y.H. Yao, *Minimum-norm solution of variational inequality and fixed point problem in Banach spaces*, Optimization, **64**(2015), 453-471.

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