Fixed Point Theory, 19(2018), No. 2, 475-486
DOI: 10.24193/fpt-ro.2018.2.38
http://www.math.ubbcluj.ro/~nodeacj/sfptcj.html

AN APPLICATION OF A FIXED-POINT THEOREM TO NEUMANN PROBLEMS ON THE SIERPINSKI FRACTAL

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Dedicated to prof. dr. Ioan A. Rus on the occasion of his 80^{th} birthday.

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Abstract. We prepare first the background for the study of Neumann problems on the Sierpinski fractal in the N-dimensional Euclidean space. Hereafter we apply the Leray-Schauder continuation principle to prove the existence of at least one solution of certain Neumann problems on this fractal.
Key Words and Phrases: Sierpinski gasket, harmonic extension procedure, Leray-Schauder continuation principle, Neumann problem on the Sierpinski gasket.
2010 Mathematics Subject Classification: 28A80, 35J05, 35J25, 35J60, 46E20, 47H10, 49J52.

1. Introduction

Since its publication in 1934, the celebrated Leray-Schauder continuation principle has considerably influenced contemporary mathematics. There is a tremendous literature related to the consequences and extensions of this principle as well as to its applications to nonlinear PDEs (see, for instance, [9]). In the present note, we propose an application to elliptic problems on the Sierpinski fractal. As far we know, this is the first application of the Leray-Schauder principle to PDEs on fractals. In the mean time, the paper represents a contribution to the study of PDEs on fractals, a domain which has emerged in the last three decades (motivated by numerous problems in physics, mechanics, chemistry, and biology).

The Sierpinski fractal in the N-dimensional Euclidean space, known as the Sierpinski gasket (SG for short), plays an important role in fractal theory, since it is a typical example for a post-critically-finite fractal. For the latter, J. Kigami developed in [7] (see also [8]) an appropriate framework that allows the study of PDEs on them. Kigami's theory actually goes back to his pioneering paper [6], where he founded it in the case of the SG. Having the definition of the Laplacian on the unit interval of $\mathbb{R}$ as a model, Kigami gave in [6] a pointwise formula for the Laplacian on the SG.

In section 2.1 of [10], the Laplacian on the SG is introduced via a weak formulation, from which afterwards there is derived Kigami's pointwise formula. Moreover, based on [6], in section 2.3 of [10], there is introduced the normal derivative on the intrinsic boundary of the SG, and there is proved the Gauss-Green formula. In this approach, there is assumed that the Laplacian takes values in the space of real-valued continuous functions on the SG, but it is pointed out in Exercise 2.3.9 of [10] that the Gauss-Green formula is still valid if one assumes that the Laplacian takes values in an L^2-space on the SG. The first aim of this paper is to prove in Theorem 3.4 below the Gauss-Green formula in this general setting. Actually, this will be performed in a broader context, that prepares the framework for the study of elliptic problems with Neumann boundary conditions on the SG. It should be pointed out that an analogous framework was developed in [5] for elliptic problems with zero Dirichlet boundary conditions on the SG, and that this framework has been widely used subsequently for the study of Dirichlet problems on the SG by means of variational methods. In this sense, we refer to the papers mentioned in the introduction of [4].

In section 2 of this paper we briefly recall all concepts needed for the definition of the Sobolev-type spaces $H^1(SG)$ and $H^1_0(SG)$. A basic aspect (also with respect to the introduction of the Laplacian and for the study of Neumann problems on the SG) is to endow $H^1(SG)$ with the inner product $\langle\cdot,\cdot\rangle_{H^1}$ defined in assertion 5° of Theorem 2.2 below. This inner product has already been used in [3]. In section 3 there is introduced, by means of methods of functional analysis, the Laplacian on the SG. Moreover, there are prepared all notions and results in order to prove the Gauss-Green formula. Finally, section 4 is devoted to the study of Neumann problems on the SG. Theorem 4.4, based on the Leray-Schauder continuation principle, gives a sufficient condition for the existence of at least one solution of a certain type of Neumann problem on the SG.

The paper can be considered as a starting point for the study of Neumann problems on the SG, or also on the more general post-critically-finite fractals, by means of variational methods.

Notations. We denote by $\mathbb{N}$ the set of natural numbers $\{0, 1, 2, \dots\}$, by $\mathbb{N}^* := \mathbb{N}\setminus\{0\}$ the set of positive naturals, and by $|\cdot|$ the Euclidean norm on the spaces $\mathbb{R}^N$, $N \in \mathbb{N}^*$. The spaces $\mathbb{R}^N$ are endowed with the topology induced by $|\cdot|$. If Y is a nonempty subset of a set X and if $f\colon X \to \mathbb{R}$ is a function, then $f|Y$ stays for the restriction of f to Y.

2. Preliminaries

Throughout the paper, the letter V stands for the Sierpinski fractal, the so-called *Sierpinski gasket* (SG for short), in $\mathbb{R}^{N-1}$, where $N \geq 3$ is a fixed natural number. There are two different approaches that lead to V, starting from given points $p_1, \dots, p_N \in \mathbb{R}^{N-1}$ with $|p_i - p_j| = 1$ for $i \neq j$, and from the similarities $F_i\colon \mathbb{R}^{N-1} \to \mathbb{R}^{N-1}$, defined by

$$F_i(x) = \frac{1}{2}\,x + \frac{1}{2}\,p_i,$$

for $i \in \{1, \ldots, N\}$. While in the first approach the set V appears as the unique nonempty compact subset of $\mathbb{R}^{N-1}$ satisfying the equality

$$V = \bigcup_{i=1}^{N} F_i(V), \tag{2.1}$$

in the second one V is obtained as the closure of the set $V_* := \bigcup_{m\in\mathbb{N}} V_m$, where

$$V_0 := \{p_1, \ldots, p_N\} \text{ and } V_m := \bigcup_{i=1}^{N} F_i(V_{m-1}), \text{ for } m \in \mathbb{N}^*. \tag{2.2}$$

In what follows V is considered to be endowed with the relative topology induced from the Euclidean topology on $\mathbb{R}^{N-1}$. The set V_0 is called the *intrinsic boundary* of the SG.

For every $m \in \mathbb{N}^*$ denote by $\mathfrak{W}_m := \{1, \ldots, N\}^m$. For $w = (w_1, \ldots, w_m) \in \mathfrak{W}_m$ put $F_w := F_{w_1} \circ \cdots \circ F_{w_m}$. The equality (2.1) clearly yields

$$V = \bigcup_{w\in\mathfrak{W}_m} F_w(V). \tag{2.3}$$

Equation (2.3) is the *level m decomposition of* V, and each $F_w(V)$, $w \in \mathfrak{W}_m$, is called a *cell of level* m, or, for short, an m-cell. We refer to V as the 0-cell.

The Sobolev type spaces $H^1(V)$ and $H^1_0(V)$ on the Sierpinski gasket are obtained as subsets of the spaces $C(V)$ and $C_0(V)$, respectively, where $C(V)$ is the space of real-valued continuous functions on V, and

$$C_0(V) := \{u \in C(V) \mid u|_{V_0} = 0\}$$

is the space of continuous functions on V which vanish on the intrinsic boundary. Both spaces $C(V)$ and $C_0(V)$ are endowed with the usual supremum norm $\|\cdot\|_{\sup}$. The basic ingredient for defining the Sobolev type spaces on the SG is a certain energy form which involves difference quotients and presents some analogy to the Dirichlet energy associated with the Laplace operator on domains in $\mathbb{R}^N$. In order to define this energy form, we follow both [10, section 1.3], where these aspects are presented for $N = 3$, and [2], where they are treated for arbitrary values $N \geq 3$. For this, consider first the sets V_m, $m \in \mathbb{N}$, defined in (2.2). Let $m \in \mathbb{N}$. For x, $y \in V_m$ set $x \underset{m}{\sim} y$ if there is a cell of level m containing both x and y. Now, for functions u, $v\colon V_m \to \mathbb{R}$ we define the m-energy $W_m(u)$ by

$$W_m(u) := \frac{1}{2}\left(\frac{N+2}{N}\right)^m \sum_{\substack{x,y\in V_m \\ x \underset{m}{\sim} y}} (u(x) - u(y))^2, \tag{2.4}$$

and the semi-inner product $\langle u, v\rangle_m$ by

$$\langle u, v\rangle_m := \left(\frac{N+2}{N}\right)^m \sum_{\substack{x,y\in V_m \\ x \underset{m}{\sim} y}} (u(x) - u(y))(v(x) - v(y)). \tag{2.5}$$

Note that $W_m(u) = \frac{1}{2}\langle u, u\rangle_m$. Recall that V_* is the union of the sets V_m, $m \in \mathbb{N}$. For functions u, $v\colon V_* \to \mathbb{R}$ or u, $v\colon V \to \mathbb{R}$ we denote simply by $W_m(u)$ the corresponding energy of the restrictions of u to V_m, and we do similarly for $\langle u, v\rangle_m$. According to [2, Corollary 3.3] (see also [10, pp. 14, 15]), for a function $u\colon V_* \to \mathbb{R}$, the sequence $(W_m(u))_{m\in\mathbb{N}}$ is increasing. Thus it makes sense to define its energy $W(u)$ by

$$W(u) := \lim_{m\to\infty} W_m(u) \in [0, \infty]. \tag{2.6}$$

It can be shown (see the explanations on [10, p. 19] in the case $N = 3$, respectively [2, Theorem 4.4] for arbitrary $N \geq 3$) that functions of finite energy are Hölder continuous, hence uniformly continuous. In particular, functions of finite energy admit a unique continuous extension to V, and by identifying uniformly continuous functions on V_* with their continuous extensions on V, it thus makes sense to say that W is defined on the space $C(V)$. Define now

$$H^1(V) := \{u \in C(V) \mid W(u) < \infty\} \quad \text{and} \quad H^1_0(V) := H^1(V) \cap C_0(V).$$

Using the polarization identity, it can be proved that for every u, $v \in H^1(V)$ the sequence $(\langle u, v\rangle_m)_{m\in\mathbb{N}}$ is convergent, and that the function $\langle\cdot,\cdot\rangle\colon H^1(V)\times H^1(V) \to \mathbb{R}$ given by

$$\langle u, v\rangle = \lim_{m\to\infty} \langle u, v\rangle_m \quad \text{for all } u,\, v \in H^1(V), \tag{2.7}$$

is a semi-inner product satisfying $W(u) = \frac{1}{2}\langle u, u\rangle$ $(u \in H^1(V))$.

Without entering into details (for which we refer to [10, section 1.3], in case $N = 3$, and to [2, section 3], for arbitrary N), we mention that the definition of W_m in (2.4) is strongly related to the *harmonic extension procedure* and to the notion of *harmonic function on the SG*. Given $m \in \mathbb{N}$ and $u\colon V_m \to \mathbb{R}$, it can be shown that there is a unique extension $\widetilde{u}\colon V_{m+1} \to \mathbb{R}$ of u to V_{m+1}, called the *harmonic extension of* u, that minimizes the $m+1$-energy W_{m+1} for all extensions of u to V_{m+1}. Moreover, the equality $W_m(u) = W_{m+1}(\widetilde{u})$ holds. The harmonic extension procedure can be described recursively in the following manner: If $u\colon V_0 \to \mathbb{R}$, then the values of the harmonic extension $\tilde{u}\colon V_1 \to \mathbb{R}$ of u at the points in $V_1 \setminus V_0$ are computed as

$$\tilde{u}(F_i(p_j)) = \frac{u(p_i) + u(p_j) + \sum_{k=1}^{N} u(p_k)}{N+2}, \ \forall\, i,\, j \in \{1, \dots, N\} \text{ with } i \neq j. \tag{2.8}$$

If $m \in \mathbb{N}^*$ and $u\colon V_m \to \mathbb{R}$, then the harmonic extension $\tilde{u}\colon V_{m+1} \to \mathbb{R}$ of u is obtained such that, for all $w \in \mathfrak{W}_m$, the map $\tilde{u} \circ F_w\colon V_1 \to \mathbb{R}$ is the harmonic extension of $u \circ F_w\colon V_0 \to \mathbb{R}$. Note that harmonic extension is a linear transformation. More precisely, if $m \in \mathbb{N}$, u, $v\colon V_m \to \mathbb{R}$, α, $\beta \in \mathbb{R}$ then the following equality holds for the harmonic extensions of u, v and $\alpha u + \beta v$ to V_{m+1}

$$(\widetilde{\alpha u + \beta v}) = \alpha\widetilde{u} + \beta\widetilde{v}. \tag{2.9}$$

The harmonic extension procedure can be started at an arbitrary level, in the sense that, given $m \in \mathbb{N}$ and $f\colon V_m \to \mathbb{R}$, this map is then extended harmonically to V_k for each $k > m$. Denoting by $\tilde{f}\colon V_* \to \mathbb{R}$ the map obtained this stepwise way, it follows from the harmonic extension procedure that $W(\tilde{f}) = W_m(f)$. Thus $\tilde{f}$ has finite energy, hence it admits a unique continuous extension to V. This leads to the following definition.

Definition 2.1. Let $m \in \mathbb{N}$. A function $h\colon V \to \mathbb{R}$ is called a *harmonic function of level* m if h is obtained by specifying the values of h on V_m arbitrarily, then extending harmonically to V_k for each $k > m$, and finally extending continuously to V. Denote by $\mathcal{H}_m(V)$ the set of all harmonic functions of level m. The elements of $\mathcal{H}_0(V)$ are simply called *harmonic functions*.

Note that $\mathcal{H}_m(V) \subseteq H^1(V)$, for every $m \in \mathbb{N}$. For $m \in \mathbb{N}$ and $x \in V_m$ denote by $\varphi_m^x \in \mathcal{H}_m(V)$ the (unique) map satisfying the property

$$\varphi_m^x(y) = \begin{cases} 0, & y \in V_m \setminus \{x\} \\ 1, & y = x. \end{cases} \tag{2.10}$$

Let $f \in C(V)$. For every $m \in \mathbb{N}$, denote by $f_m \in \mathcal{H}_m(V)$ the harmonic function of level m obtained from the restriction of f to V_m. Using (2.9), we obtain the following equality

$$f_m = \sum_{x \in V_m} f(x) \cdot \varphi_m^x, \text{ for all } m \in \mathbb{N}. \tag{2.11}$$

We summarize the main results concerning the concepts that have been introduced in this section in the following theorem (see [2, Theorem 4.4, Theorem 4.6], [3, Theorem 2.1] and [10, sections 1.3 and 1.4]).

Theorem 2.2. *The following assertions hold:*

1° *Let $m \in \mathbb{N}$ and let u, $v\colon V_m \to \mathbb{R}$. If $\widetilde{u}\colon V_{m+1} \to \mathbb{R}$ is the harmonic extension of u, and if $v'\colon V_{m+1} \to \mathbb{R}$ is an arbitrary extension of v, then $\langle \widetilde{u}, v' \rangle_{m+1} = \langle u, v \rangle_m$.*

2° *Let $f \in C(V)$ and let $(f_m)_{m\in\mathbb{N}}$ be the sequence of functions introduced in (2.11). Then $\lim_{m\to\infty} ||f_m - f||_{\sup} = 0$.*

3° *The spaces $H^1(V)$ and $H_0^1(V)$ are linear, dense subspaces of $(C(V), \|\cdot\|_{\sup})$ and $(C_0(V), \|\cdot\|_{\sup})$, respectively.*

4° *For $\alpha := \frac{\ln \frac{N+2}{N}}{2\ln 2}$ there exists a constant $c > 0$ such that for every $u \in H^1(V)$ and every x, $y \in V$*

$$|u(x) - u(y)| \le c|x - y|^\alpha \sqrt{W(u)}. \tag{2.12}$$

5° *For any fixed $p \in V$*

$$\langle u, v \rangle_{H^1} := u(p)v(p) + \langle u, v \rangle \quad (u,\, v \in H^1(V))$$

defines an inner product on $H^1(V)$.

6° *The spaces $(H^1(V), \langle\cdot,\cdot\rangle_{H^1})$ and $(H_0^1(V), \langle\cdot,\cdot\rangle)$ are real Hilbert spaces.*

7° *The embedding of $(H^1(V), \|\cdot\|_{H^1})$ into $(C(V), \|\cdot\|_{\sup})$ is compact.*

3. The Gauss-Green formula

In [6], Kigami worked with the normalized d-dimensional Hausdorff measure on the SG, where d is the Hausdorff dimension of the SG. There is pointed out in section 2.1 of [10] that the theory for defining the Laplacian on the SG works also if one considers other measures (satisfying certain properties) on the SG. (Such more general measures were used by Kigami in [7] for the study of post-critically-finite self-similar sets.) In what follows we will adopt this general setting.

We retain the notations from the previous section, and consider a nonzero, finite Borel measure μ on V with the property that every nonempty, relatively open subset of V has strictly positive μ-measure, and that the intrinsic boundary V_0 is a μ-null set. Then, by [1, Lemma 26.2 and Theorem 29.14], $(C_0(V), ||\cdot||_{\sup})$ and $(C(V), ||\cdot||_{\sup})$ are densely and continuously embedded into $(L^2_\mu(V), \|\cdot\|_{L^2_\mu})$. Fix now $p \in V_0$. By Theorem 2.2, $(H^1(V), \|\cdot\|_{H^1})$ is then densely and continuously embedded in $(L^2_\mu(V), \|\cdot\|_{L^2_\mu})$, too. By the Riesz representation theorem, there exist a dense subset $D_\mu(V)$ of $H^1(V)$ and a linear, bijective, self-adjoint operator $\Delta_\mu \colon D_\mu(V) \to L^2_\mu(V)$, called the *Laplacian* on the SG, such that

$$\langle u, v\rangle + u(p)v(p) = -\int_V v \cdot \Delta_\mu u \,\mathrm{d}\mu, \ \forall u \in D_\mu(V), \ \forall v \in H^1(V). \tag{3.1}$$

It follows that

$$\langle u, v\rangle = -\int_V v \cdot \Delta_\mu u \,\mathrm{d}\mu, \ \forall u \in D_\mu(V), \ \forall v \in H^1_0(V). \tag{3.2}$$

For $m \in \mathbb{N}^*$ and $u \colon V_m \to \mathbb{R}$, the *m-Laplacian* of u, denoted by $\Delta_m u$, is the map $\Delta_m u \colon V_m \setminus V_0 \to \mathbb{R}$ given by

$$\Delta_m u(x) = \sum_{\substack{y \in V_m \\ y \underset{m}{\sim} x}} (u(y) - u(x)), \ \forall x \in V_m \setminus V_0. \tag{3.3}$$

Fix now $m \in \mathbb{N}^*$ and u, $v \colon V_m \to \mathbb{R}$. Note that (2.5) can be written as

$$\begin{aligned}\langle u, v\rangle_m = &-\left(\frac{N+2}{N}\right)^m \sum_{x \in V_m \setminus V_0} v(x) \cdot \Delta_m u(x) \\ &+ \left(\frac{N+2}{N}\right)^m \sum_{x \in V_0} v(x) \left(\sum_{\substack{y \in V_m \\ y \underset{m}{\sim} x}} (u(x) - u(y)) \right).\end{aligned} \tag{3.4}$$

We start by proving a result needed for one of the main theorems of this section.

Lemma 3.1. *Let $x \in V_0$ and $f \in L^2_\mu(V)$. If φ^x_m, $m \in \mathbb{N}$, are the maps defined in* (2.10)*, then* $\lim_{m\to\infty} \int_V \varphi^x_m \cdot f \,\mathrm{d}\mu = 0$.

Proof. Using the Hölder inequality, we get for every $m \in \mathbb{N}$

$$\left| \int_V \varphi^x_m \cdot f \,\mathrm{d}\mu \right| \leq ||\varphi^x_m||_{L^2_\mu} \cdot ||f||_{L^2_\mu}. \tag{3.5}$$

Let $i \in \{1, \dots, N\}$ be so that $x = p_i$. For every $m \in \mathbb{N}^*$ denote by $w^m_x := (i, i, \dots, i) \in \mathfrak{W}_m$ and by $V^m_x := F_{w^m_x}(V)$. Fix an arbitrary $m \in \mathbb{N}^*$. Since, by definition, the restriction of φ^x_m to $F_w(V)$, $w \in \mathfrak{W}_m \setminus \{w^m_x\}$, is the zero function, it follows that

$$\operatorname{supp} \varphi^x_m \subseteq V^m_x.$$

On the other hand, the harmonic extension procedure yields that

$$||\varphi^x_m||_{\sup} = 1,$$

thus we obtain that

$$||\varphi_m^x||_{L_\mu^2}^2 = \int_V |\varphi_m^x|^2 \, d\mu = \int_{V_x^m} |\varphi_m^x|^2 \, d\mu \leq \mu(V_x^m). \tag{3.6}$$

Since $\mu(V_0) = 0$, we get that

$$\lim_{m\to\infty} \mu(V_x^m) = \mu\left(\bigcap_{m\in\mathbb{N}} V_x^m\right) = \mu(\{x\}) = 0.$$

Hence, relations (3.5) and (3.6) imply the conclusion. □

One of the main results of this section is the next theorem, revealing the relationship between the m-Laplacian and the Laplacian.

Theorem 3.2. *Let $u \in D_\mu(V)$ and $v \in H^1(V)$. Then the following equality holds*

$$\lim_{m\to\infty} \left(\frac{N+2}{N}\right)^m \sum_{x\in V_m\setminus V_0} v(x) \cdot \Delta_m u(x) = \int_V v \cdot \Delta_\mu u \, d\mu.$$

Proof. Fix $m \in \mathbb{N}^*$, $x \in V_m \setminus V_0$, and let φ_m^x be the map defined in (2.10). Since $\varphi_m^x \in H_0^1(V)$, we get from (3.2) that

$$\langle u, \varphi_m^x \rangle = -\int_V \varphi_m^x \cdot \Delta_\mu u \, d\mu. \tag{3.7}$$

On the other hand, since $\varphi_m^x \in \mathcal{H}_m(V)$, assertion 1° of Theorem 2.2 yields that

$$\langle u, \varphi_m^x \rangle = \langle u, \varphi_m^x \rangle_m.$$

Using the definition of φ_m^x and (3.4), we obtain

$$\langle u, \varphi_m^x \rangle = -\left(\frac{N+2}{N}\right)^m \Delta_m u(x). \tag{3.8}$$

Thus, by (3.7) and (3.8),

$$\left(\frac{N+2}{N}\right)^m \sum_{x\in V_m\setminus V_0} v(x) \cdot \Delta_m u(x) = \int_V \left(\sum_{x\in V_m\setminus V_0} v(x) \cdot \varphi_m^x\right) \Delta_\mu u \, d\mu. \tag{3.9}$$

Denoting by $v_m \in \mathcal{H}_m(V)$ the function satisfying the property that $v_m|V_m = v|V_m$, we have by (2.11) that

$$v_m = \sum_{x\in V_m} v(x) \cdot \varphi_m^x,$$

so

$$\int_V v_m \cdot \Delta_\mu u \, d\mu = \int_V \left(\sum_{x\in V_m\setminus V_0} v(x) \cdot \varphi_m^x\right) \Delta_\mu u \, d\mu + \sum_{x\in V_0} v(x) \int_V \varphi_m^x \cdot \Delta_\mu u \, d\mu. \tag{3.10}$$

According to assertion 2° of Theorem 2.2,

$$\lim_{m\to\infty} \int_V v_m \cdot \Delta_\mu u \, d\mu = \int_V v \cdot \Delta_\mu u \, d\mu.$$

Thus the conclusion follows from (3.9), (3.10) and Lemma 3.1. □

Definition 3.3. Let $u \in C(V)$ and $x \in V_0$. One says that u has a normal derivative at x if the limit $\lim\limits_{m\to\infty} \left(\frac{N+2}{N}\right)^m \sum\limits_{\substack{y\in V_m \\ y \underset{m}{\sim} x}} (u(x) - u(y))$ exists in $\mathbb{R}$. In this case, this limit is denoted by $\partial_n u(x)$ and it is called the *normal derivative of u at x*. If u has a normal derivative at every point of V_0, then the map $x \in V_0 \mapsto \partial_n(x) \in \mathbb{R}$ is called the *normal derivative of u*.

Theorem 3.4. *Let $u \in D_\mu(V)$. Then the following assertions hold:*

1° *The function u has a normal derivative at every point $x \in V_0$.*

2° *The Gauss-Green formula*

$$\langle u, v\rangle = -\int_V v \cdot \Delta_\mu u \,\mathrm{d}\mu + \sum_{x\in V_0} v(x) \cdot \partial_n u(x)$$

holds for every $v \in H^1(V)$.

Proof. 1° Pick $z \in V_0$ and choose $v \in H^1(V)$ such that $v(z) = 1$ and $v(x) = 0$, for all $x \in V_0 \setminus \{z\}$. Then (3.4) yields for every $m \in \mathbb{N}^*$ that

$$\langle u, v\rangle_m = -\left(\frac{N+2}{N}\right)^m \sum_{x\in V_m\setminus V_0} v(x) \cdot \Delta_m u(x) + \left(\frac{N+2}{N}\right)^m \sum_{\substack{y\in V_m \\ y \underset{m}{\sim} z}} (u(z) - u(y)).$$

Since $\lim\limits_{m\to\infty} \langle u, v\rangle_m = \langle u, v\rangle$, Theorem 3.2 implies that u has a normal derivative at z.

2° Let $v \in H^1(V)$. The Gauss-Green formula follows from (3.4), Theorem 3.2 and the previous assertion 1°. □

4. Neumann problems on the SG

We retain the notations from the previous sections. In particular, $H^1(V)$ is equipped with the inner product $\langle\cdot,\cdot\rangle_{H^1}$ defined in assertion 5° of Theorem 2.2, where $p \in V_0$ is fixed. Furthermore, let $a \in C(V)$ be so that $a(x) > 0$ for every $x \in V$, and let $f : \mathbb{R} \to \mathbb{R}$ be continuous. We formulate the Neumann problem on the SG as the problem of finding $u \in D_\mu(V)$ such that

$$(NP) \begin{cases} -\Delta_\mu u(x) + a(x) \cdot u(x) = f(u(x)) \text{ a.e. } x \in V \\ \\ \partial_n u = 0. \end{cases}$$

A *solution of* (NP) is a function $u \in H^1(V)$ such that

$$\langle u, v\rangle + \int_V a \cdot u \cdot v \,\mathrm{d}\mu = \int_V (f\circ u) \cdot v \,\mathrm{d}\mu,\ \forall v \in H^1(V). \tag{4.1}$$

Remark 4.1. Assume that $u \in D_\mu(V)$ satisfies (4.1). Then, using (3.2), we get that

$$\int_V (-\Delta_\mu u + a \cdot v - f\circ u) \cdot v \,\mathrm{d}\mu = 0,\ \forall v \in H_0^1(V).$$

Since $H_0^1(V)$ is dense in $L_\mu^2(V)$, it follows that

$$-\Delta_\mu u + a \cdot u = f \circ u \text{ a.e. } x \in V. \tag{4.2}$$

Moreover, involving the Gauss-Green formula, (4.1) yields

$$-\int_V v\cdot\Delta_\mu u\,\mathrm{d}\mu+\sum_{x\in V_0} v(x)\cdot\partial_n u(x)+\int_V a\cdot u\cdot v\,\mathrm{d}\mu=\int_V (f\circ u)\cdot v\,\mathrm{d}\mu,\ \forall v\in H^1(V).$$

Thus, by (4.2), it follows that

$$\sum_{x\in V_0} v(x)\cdot\partial_n u(x)=0,\ \forall v\in H^1(V).$$

Hence $\partial_n u(x)=0$, for all $x\in V_0$.

We observe next that the map

$$(u,v)\in H^1(V)\times H^1(V)\mapsto\langle u,v\rangle+\int_V a\cdot u\cdot v\,\mathrm{d}\mu \tag{4.3}$$

defines an inner product which is equivalent to the inner product $\langle u,v\rangle_{H^1(V)}$. Thus the linear operator $A\colon H^1(V)\to(H^1(V))^*$ given by

$$Au(v)=\langle u,v\rangle+\int_V a\cdot u\cdot v\,\mathrm{d}\mu,\ \forall u,\,v\in H^1(V),$$

is a homeomorphism.

Furthermore, define the operator $B\colon H^1(V)\to(H^1(V))^*$ by

$$Bu(v)=\int_V (f\circ u)\cdot v\,\mathrm{d}\mu,\ \forall u,\,v\in H^1(V).$$

Note that (4.1) can be rewritten as $Au=Bu$. Thus $u\in H^1(V)$ is a solution of (NP) if and only if u is a fixed point of the operator $T\colon H^1(V)\to H^1(V)$ given by

$$T=A^{-1}\circ B. \tag{4.4}$$

Lemma 4.2. *The operator T is compact, i.e., T is continuous and maps bounded sets into relatively compact sets.*

Proof. Let i be the embedding of $(H^1(V),\|\cdot\|_{H^1})$ into $(C(V)\|\cdot\|_{\sup})$. Note that i is compact, by assertion 7° of Theorem 2.2. Denote by $D\colon(C(V),\|\cdot\|_{\sup})\to(C(V),\|\cdot\|_{\sup})$ the operator defined by

$$Du=f\circ u,\ \forall u\in C(V),$$

and by $E\colon(C(V),\|\cdot\|_{\sup})\to(H^1(V))^*$ the operator given as

$$Eu(v)=\int_v u\cdot v\,\mathrm{d}\mu,\ \forall v\in H^1(V).$$

It is clear that D is continuous. Using the continuity of i, it follows that the linear operator E is continuous. Thus, $B=E\circ D\circ i$ is continuous, and, by the the compactness of i, it is also compact. We conclude that T is compact. □

We recall now the Leray-Schauder continuation principle (see, e.g., [11, Theorem 6.A]) which will be used in the sequel.

Theorem 4.3. *Let X be a real Banach space and let $T\colon X \to X$ be a compact operator with the property that the set $M := \{x \in X \mid \exists\, t \in [0,1)$ such that $x = tT(x)\}$ is bounded. Then T has at least one fixed point.*

As an application of Theorem 4.3 we get the following result concerning problem (NP).

Theorem 4.4. *Let $a \in C(V)$ be so that $a(x) > 0$ for every $x \in V$, and let $f\colon \mathbb{R} \to \mathbb{R}$ be a continuous function with the property that there exist reals α, $\beta > 0$ and $q \in [1,2)$ such that*

$$|f(t)| \leq \alpha \cdot |t|^{q-1} + \beta, \ \forall t \in \mathbb{R}. \tag{4.5}$$

Then problem (NP) has at least one solution.

Proof. Let T be the operator defined in (4.4). We already know from Lemma 4.2 that T is compact. Put

$$M := \{u \in H^1(V) \mid \exists\, t \in [0,1) \text{ such that } u = tT(u)\}.$$

We show that M is bounded. For this, pick $u \in M$. Thus there exists $t \in [0,1)$ with $u = tT(u)$. Denoting by $||\cdot||_a$ the norm induced by the inner product given in (4.3), and using (4.5), we have

$$\begin{aligned}
||Tu||_a^2 &= ATu(Tu) = Bu(Tu) = \int_V (f \circ u) \cdot Tu \,\mathrm{d}\mu = \left| \int_V (f \circ u) \cdot Tu \,\mathrm{d}\mu \right| \\
&\leq \int_V |f \circ u| \cdot |Tu| \,\mathrm{d}\mu \leq \int_V (\alpha |u|^{q-1} + \beta) \cdot |Tu| \,\mathrm{d}\mu \\
&= \alpha t^{q-1} \int_V |Tu|^q \,\mathrm{d}\mu + \beta \int_V |Tu| \,\mathrm{d}\mu \leq \alpha \int_V |Tu|^q \,\mathrm{d}\mu + \beta \int_V |Tu| \,\mathrm{d}\mu.
\end{aligned}$$

By the Hölder inequality, the equivalence of the norms $||\cdot||_a$ and $||\cdot||_{H^1}$, and the continuity of the embedding of $(H^1(V), ||\cdot||_{H^1})$ into $(L^2_\mu(V), ||\cdot||_{L^2_\mu})$, we get that there exist positive real constants c_1, c_2 such that

$$||Tu||_{H^1}^2 \leq c_1 ||Tu||_{H^1}^q + c_2 ||Tu||_{H^1}.$$

Let $g\colon [0,\infty) \to \mathbb{R}$ be defined by $g(s) = s^2 - c_1 s^q - c_2 s$. Since $q < 2$, we have that $\lim\limits_{s\to\infty} g(s) = \infty$. It follows that the set $\{s \in [0,\infty) \mid g(s) \leq 0\}$ is bounded. This implies the boundedness of the set M. An application of Theorem 4.3 finishes the proof. □

Acknowledgements. Csaba Varga has been partially supported both by a grant of the Romanian National Authority for Scientific Research (CNCS-UEFISCDI), grant no. PN-II-ID-PCE-2011-3-0241, and by a grant of the Hungarian Research Fund (OTKA), grant no. K 115926.

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Received: March 24, 2016; Accepted: May 25, 2016.