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THE SHRINKING PROJECTION METHOD FOR A FINITE FAMILY OF DEMIMETRIC MAPPINGS WITH VARIATIONAL INEQUALITY PROBLEMS IN A HILBERT SPACE

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Abstract. In this paper, using a new nonlinear mapping called demimetric and the shrinking projection method, we prove a strong convergence theorem for finding a common element of the set of common fixed points for a finite family of these new demimetric mappings and the set of common solutions of variational inequality problems for a finite family of inverse strongly monotone mappings in a Hilbert space. Using the result, we obtain well-known and new strong convergence theorems in a Hilbert space.
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1. Introduction

Let H be a real Hilbert space and let C be a nonempty, closed and convex subset of H. For a mapping $U : C \to H$, we denote by $F(U)$ the set of fixed points of U. Let k be a real number with $0 \leq k < 1$. A mapping $U : C \to H$ is called a k-strict pseudo-contraction [3] if

$$\|Ux - Uy\|^2 \leq \|x - y\|^2 + k\|x - Ux - (y - Uy)\|^2$$

for all $x, y \in C$. If U is a k-strict pseudo-contraction and $F(U) \neq \emptyset$, then we have that, for $x \in C$ and $q \in F(U)$,

$$\|Ux - q\|^2 \leq \|x - q\|^2 + k\|x - Ux\|^2.$$

From $\|Ux - q\|^2 = \|Ux - x\|^2 + \|x - q\|^2 + 2\langle Ux - x, x - q\rangle$, we have that

$$\|Ux - x\|^2 + \|x - q\|^2 + 2\langle Ux - x, x - q\rangle \leq \|x - q\|^2 + k\|x - Ux\|^2.$$

Therefore, we have that

$$\langle x - Ux, x - q\rangle \geq \frac{1-k}{2}\|x - Ux\|^2 \tag{1.1}$$

for all $x \in C$ and $q \in F(U)$. A mapping $U : C \to H$ is called generalized hybrid [6] if there exist $\alpha, \beta \in \mathbb{R}$ such that

$$\alpha\|Ux - Uy\|^2 + (1-\alpha)\|x - Uy\|^2 \leq \beta\|Ux - y\|^2 + (1-\beta)\|x - y\|^2$$

for all $x, y \in C$. Such a mapping U is called $(\alpha,\ \beta)$-generalized hybrid. Notice that the class of generalized hybrid mappings covers several well-known mappings. For example, a (1,0)-generalized hybrid mapping is nonexpansive, i.e.,

$$\|Ux - Uy\| \leq \|x - y\|, \quad \forall x, y \in C.$$

It is nonspreading [7, 8] for $\alpha = 2$ and $\beta = 1$, i.e.,

$$2\|Ux - Uy\|^2 \leq \|Ux - y\|^2 + \|Uy - x\|^2, \quad \forall x, y \in C.$$

It is also hybrid [15] for $\alpha = \frac{3}{2}$ and $\beta = \frac{1}{2}$, i.e.,

$$3\|Ux - Uy\|^2 \leq \|x - y\|^2 + \|Ux - y\|^2 + \|Uy - x\|^2, \quad \forall x, y \in C.$$

In general, nonspreading and hybrid mappings are not continuous; see [4]. If U is generalized hybrid and $F(U) \neq \emptyset$, then we have that, for $x \in C$ and $q \in F(U)$,

$$\alpha\|q - Ux\|^2 + (1-\alpha)\|q - Ux\|^2 \leq \beta\|q - x\|^2 + (1-\beta)\|q - x\|^2$$

and hence $\|Ux - q\|^2 \leq \|x - q\|^2$. From this, we have that

$$2\langle x - q, x - Ux\rangle \geq \|x - Ux\|^2$$

and hence

$$\langle x - q, x - Ux\rangle \geq \frac{1-0}{2}\|x - Ux\|^2. \tag{1.2}$$

On the other hand, there exists such a mapping in a Banach space. Let E be a smooth Banach space and let B be a maximal monotone operator with $B^{-1}0 \neq \emptyset$. Then, for the metric resolvent J_λ of B for $\lambda > 0$, we have from [13] that, for any $x \in E$ and $q \in B^{-1}0$,

$$\langle J_\lambda x - q, J(x - J_\lambda x)\rangle \geq 0.$$

Then we get

$$\langle J_\lambda x - x + x - q, J(x - J_\lambda x)\rangle \geq 0$$

and hence

$$\langle x - q, J(x - J_\lambda x)\rangle \geq \|x - J_\lambda x\|^2 = \frac{1-(-1)}{2}\|x - J_\lambda x\|^2, \tag{1.3}$$

where J is the duality mapping on E. Motivated by (1.1), (1.2) and (1.3), Takahashi [16] introduced a new nonlinear mapping as follows: Let E be a smooth Banach space, let C be a nonempty, closed and convex subset of E and let k be a real number with $k \in (-\infty, 1)$. A mapping $U : C \to E$ with $F(U) \neq \emptyset$ is called k-demimetric if, for any $x \in C$ and $q \in F(U)$,

$$\langle x - q, J(x - Ux)\rangle \geq \frac{1-k}{2}\|x - Ux\|^2,$$

where J is the duality mapping on E. According to the definition, we get that a k-strict pseudo-contraction U with $F(U) \neq \emptyset$ is k-demimetric, an (α, β)-generalized hybrid mapping U with $F(U) \neq \emptyset$ is 0-demimetric and the metric resolvent J_λ with $B^{-1}0 \neq \emptyset$ is (-1)-demimetric.

On the other hand, we know the shrinking projection method which was introduced by Takahashi, Takeuchi and Kubota [17] for finding a fixed point of a nonexpansive mapping in a Hilbert space.

In this paper, using this new nonlinear mapping called demimetric and the shrinking projection method, we prove a strong convergence theorem for finding a common element of the set of common fixed points for a finite family of these new demimetric mappings and the set of common solutions of variational inequality problems for a finite family of inverse strongly monotone mappings in a Hilbert space. Using the result, we obtain well-known and new strong convergence theorems in a Hilbert space.

2. Preliminaries

Throughout this paper, let $\mathbb{N}$ be the set of positive integers and let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$, respectively. When $\{x_n\}$ is a sequence in H, we denote the strong convergence of $\{x_n\}$ to $x \in H$ by $x_n \to x$ and the weak convergence by $x_n \rightharpoonup x$. We have from [14] that for any $x, y \in H$ and $\lambda \in \mathbb{R}$,

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle, \tag{2.1}$$

$$\|\lambda x + (1-\lambda)y\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x - y\|^2. \tag{2.2}$$

Furthermore we have that for $x, y, u, v \in H$,

$$2\langle x - y, u - v\rangle = \|x - v\|^2 + \|y - u\|^2 - \|x - u\|^2 - \|y - v\|^2. \tag{2.3}$$

Let C be a nonempty, closed and convex subset of a Hilbert space H. A mapping $T : C \to H$ is called nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. If $T : C \to H$ is nonexpansive, then $F(T)$ is closed and convex; see [5, 14]. For a nonempty, closed and convex subset D of H, the nearest point projection of H onto D is denoted by P_D, that is, $\|x - P_D x\| \leq \|x - y\|$ for all $x \in H$ and $y \in D$. Such a mapping P_D is called the metric projection of H onto D. We know that the metric projection P_D is firmly nonexpansive, i.e., $\|P_D x - P_D y\|^2 \leq \langle P_D x - P_D y, x - y\rangle$ for all $x, y \in H$. Furthermore, $\langle x - P_D x, y - P_D x\rangle \leq 0$ holds for all $x \in H$ and $y \in D$; see [12, 14]. Using this inequality and (2.3), we have that

$$\|P_D x - y\|^2 + \|P_D x - x\|^2 \leq \|x - y\|^2, \quad \forall x \in H,\ y \in D. \tag{2.4}$$

Let H be a real Hilbert space and let C be a nonempty, closed and convex subset of H. For $\alpha > 0$, a mapping $A : C \to H$ is called α-inverse strongly monotone if

$$\langle x - y, Ax - Ay\rangle \geq \alpha\|Ax - Ay\|^2, \quad \forall x, y \in C.$$

If A is α-inverse strongly monotone and $0 < \lambda \leq 2\alpha$, then $I - \lambda A : C \to H$ is nonexpansive. In fact, we have that for all $x, y \in C$,

$$\begin{aligned}
\|(I - \lambda A)x - (I - \lambda A)y\|^2 &= \|x - y - \lambda(Ax - Ay)\|^2 \\
&= \|x - y\|^2 - 2\lambda\langle x - y, Ax - Ay\rangle + \lambda^2\|Ax - Ay\|^2 \\
&\leq \|x - y\|^2 - 2\lambda\alpha\|Ax - Ay\|^2 + \lambda^2\|Ax - Ay\|^2 \\
&= \|x - y\|^2 + \lambda(\lambda - 2\alpha)\|Ax - Ay\|^2 \\
&\leq \|x - y\|^2.
\end{aligned}$$

Thus, $I - \lambda A : C \to H$ is nonexpansive; see [1, 11, 14] for more results of inverse-strongly monotone mappings. The variational inequalty problem for $A : C \to H$ is to find a point $u \in C$ such that

$$\langle Au, x - u\rangle \geq 0, \quad \forall x \in C. \tag{2.5}$$

The set of solutions of (2.5) is denoted by $VI(C, A)$. We also have that, for $\lambda > 0$, $u = P_C(I - \lambda A)u$ if and only if $u \in VI(C, A)$. In fact, let $\lambda > 0$. Then, for $u \in C$,

$$\begin{aligned}
u = P_C(I - \lambda A)u &\iff \langle (I - \lambda A)u - u, u - y\rangle \geq 0, \quad \forall y \in C \\
&\iff \langle -\lambda Au, u - y\rangle \geq 0, \quad \forall y \in C \\
&\iff \langle Au, u - y\rangle \leq 0, \quad \forall y \in C \\
&\iff \langle Au, y - u\rangle \geq 0, \quad \forall y \in C \\
&\iff u \in VI(C, A).
\end{aligned}$$

In the case when a Banach space E is a Hilbert space, the definition of a demimetric mapping is as follows: Let H be a Hilbert space and let C be a nonempty, closed and convex subset of H. Let $k \in (-\infty, 1)$. A mapping $U : C \to H$ with $F(U) \neq \emptyset$ is called k-demimetric if, for any $x \in C$ and $q \in F(U)$,

$$\langle x - q, x - Ux\rangle \geq \frac{1 - k}{2}\|x - Ux\|^2.$$

Note again that the class of k-demimetric mappings with $k \in (-\infty, 1)$ in a Hilbert space covers k-strict pseudo-contractions with $k \in [0, 1)$, generalized hybrid mappings, the metric projections, the resolvents of a maximal monotone operator in a Hilbert space.

The following lemma which was essentially proved in [16] is important and crucial in the proof of our main result. For the sake of completeness, we give the proof.

Lemma 2.1 ([16])**.** *Let H be a Hilbert space and let C be a nonempty, closed and convex subset of H. Let k be a real number with $k \in (-\infty, 1)$ and let U be a k-demimetric mapping of C into H. Then $F(U)$ is closed and convex.*

Proof. Let us show that $F(U)$ is closed. For a sequence $\{q_n\}$ such that $q_n \to q$ and $q_n \in F(U)$, we have from the definition of U that

$$\langle q - q_n, q - Uq \rangle \geq \frac{1-k}{2} \|q - Uq\|^2.$$

From $q_n \to q$, we have $0 \geq \frac{1-k}{2} \|q - Uq\|^2$. From $1 - k > 0$, we have $\|q - Uq\|^2 = 0$ and hence $q = Uq$. This implies that $F(U)$ is closed.

Let us prove that $F(U)$ is convex. Let $p, q \in F(U)$ and set $x = \alpha p + (1 - \alpha)q$, where $\alpha \in [0, 1]$. Then we have

$$\begin{aligned}
\|x - Ux\|^2 &= \langle x - Ux, x - Ux \rangle \\
&= \langle \alpha p + (1 - \alpha)q - Ux, x - Ux \rangle \\
&= \langle \alpha p + (1 - \alpha)q - (\alpha Ux + (1 - \alpha)Ux), x - Ux \rangle \\
&= \alpha \langle p - Ux, x - Ux \rangle + (1 - \alpha)\langle q - Ux, x - Ux \rangle \\
&= \alpha \langle p - x + x - Ux, x - Ux \rangle + (1 - \alpha)\langle q - x + x - Ux, x - Ux \rangle \\
&\leq \frac{\alpha(k-1)}{2} \|x - Ux\|^2 + \alpha \|x - Ux\|^2 \\
&\quad + \frac{(1-\alpha)(k-1)}{2} \|x - Ux\|^2 + (1 - \alpha)\|x - Ux\|^2 \\
&= \frac{(k-1)}{2} \|x - Ux\|^2 + \|x - Ux\|^2
\end{aligned}$$

and hence

$$0 \leq \frac{(k-1)}{2} \|x - Ux\|^2.$$

We have from $0 > k - 1$ that $\|x - Ux\| \leq 0$ and hence $x = Ux$. This means that $F(U)$ is convex. □

The following lemma is used in the proof of our main result.

Lemma 2.2. *Let H be a Hilbert space and let C be a nonempty, closed and convex subset of H. Let $k \in (-\infty, 1)$ and let T be a k-demimetric mapping of C into H such that $F(T)$ is nonempty. Let λ be a real number with $0 < \lambda \leq 1 - k$ and define $S = (1 - \lambda)I + \lambda T$. Then S is a quasi-nonexpansive mapping of C into H.*

Proof. It is obvious that $F(T) = F(S)$. Since T be a k-demimetric mapping of C into H, we have that for any $x \in C$ and $z \in F(S)$,

$$\begin{aligned}
\langle x - z, x - Sx \rangle &= \langle x - z, x - (1 - \lambda)x - \lambda Tx \rangle = \lambda \langle x - z, x - Tx \rangle \\
&\geq \lambda \frac{1-k}{2} \|x - Tx\|^2 = \lambda^2 \frac{1-k}{2\lambda} \|x - Tx\|^2 \\
&= \frac{1-k}{2\lambda} \|\lambda x - \lambda Tx\|^2 = \frac{1-k}{2\lambda} \|x - Sx\|^2 \\
&\geq \frac{\lambda}{2\lambda} \|x - Sx\|^2 = \frac{1}{2} \|x - Sx\|^2.
\end{aligned}$$

Then S is a 0-demimetric mapping. Furthermore, we have from (2.3) that for any $x \in C$ and $z \in F(S)$,

$$\begin{aligned}
&\frac{1}{2}\|x - Sx\|^2 \leq \langle x - z, x - Sx\rangle \\
&\quad\Longleftrightarrow \|x - Sx\|^2 \leq 2\langle x - z, x - Sx\rangle \\
&\quad\Longleftrightarrow \|x - Sx\|^2 \leq \|x - Sx\|^2 + \|x - z\|^2 - \|Sx - z\|^2 \\
&\quad\Longleftrightarrow \|Sx - z\|^2 \leq \|x - z\|^2 \\
&\quad\Longleftrightarrow \|Sx - z\| \leq \|x - z\|.
\end{aligned}$$

Therefore, S is quasi-nonexpansive. □

3. Main result

Let H be a Hilbert space and let C be a nonempty, closed and convex subset of H. A mapping $U : C \to H$ is called demiclosed if, for a sequence $\{x_n\}$ in C such that $x_n \rightharpoonup w$ and $x_n - Ux_n \to 0$, then $w = Uw$ holds. For example, if C is a nonempty, closed and convex subset of H and T is a nonexpansive mapping of C of H, then T is demiclosed; see [2]. In fact, let $\{x_n\}$ be a sequence in C such that $x_n \rightharpoonup u$ and $x_n - Ux_n \to 0$. Since C is weakly closed, we have that $u \in C$. Furthermore, we have from $x_n \rightharpoonup u$ that $\{x_n\}$ is bounded and then $\{Tx_n\}$ is bounded. Thus, we have that

$$\begin{aligned}
\|u - Tu\|^2 &= \|u - x_n + x_n - Tu\|^2 \\
&= \|u - x_n\|^2 + \|x_n - Tu\|^2 + 2\langle u - x_n, x_n - Tu\rangle \\
&= \|u - x_n\|^2 + \|x_n - Tx_n + Tx_n - Tu\|^2 + 2\langle u - x_n, x_n - u + u - Tu\rangle \\
&= \|u - x_n\|^2 + \|x_n - Tx_n\|^2 + \|Tx_n - Tu\|^2 + 2\langle x_n - Tx_n, Tx_n - Tu\rangle \\
&\quad - 2\|u - x_n\|^2 + 2\langle u - x_n, u - Tu\rangle \\
&\leq \|u - x_n\|^2 + \|x_n - Tx_n\|^2 + \|x_n - u\|^2 + 2\langle x_n - Tx_n, Tx_n - Tu\rangle \\
&\quad - 2\|u - x_n\|^2 + 2\langle u - x_n, u - Tu\rangle \\
&= \|x_n - Tx_n\|^2 + 2\langle x_n - Tx_n, Tx_n - Tu\rangle + 2\langle u - x_n, u - Tu\rangle \\
&\to 0.
\end{aligned}$$

Therefore, we have that $u = Tu$.

In this section, using the shrinking projection method, we prove a strong convergence theorem for finding a common element of the set of common fixed points for a finite family of demimetric mappings and the set of common solutions of variational inequalty problems for a finite family of inverse strongly monotone mappings in a Hilbert space.

Theorem 3.1. *Let H be a Hilbert space and let C be a nonempty, closed and convex subset of H. Let $\{k_1, \dots, k_M\} \subset (-\infty, 1)$ and $\{\mu_1, \dots, \mu_N\} \subset (0, \infty)$. Let $\{T_j\}_{j=1}^M$ be a finite family of k_j-demimetric and demiclosed mappings of C into H and let $\{B_i\}_{i=1}^N$ be a finite family of μ_i-inverse strongly monotone mappings of C into H.*

Assume that $\cap_{j=1}^{M} F(T_j) \cap (\cap_{i=1}^{N} VI(C, B_i)) \neq \emptyset$. *Let* $x_1 \in C$ *and* $C_1 = C$. *Let* $\{x_n\}$ *be a sequence generated by*

$$\begin{cases} z_n = \sum_{j=1}^{M} \xi_j((1-\lambda_n)I + \lambda_n T_j)x_n, \\ w_n = \sum_{i=1}^{N} \sigma_i P_C(I - \eta_n B_i)x_n, \\ y_n = \alpha_n x_n + \beta_n z_n + \gamma_n w_n, \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \leq \|x_n - z\|\}, \\ x_{n+1} = P_{C_{n+1}} x_1, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $a, b, c \in \mathbb{R}$, $\{\lambda_n\}, \{\eta_n\} \subset (0, \infty)$, $\{\xi_1, \ldots, \xi_M\}, \{\sigma_1, \ldots, \sigma_N\} \subset (0,1)$ *and* $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \subset (0,1)$ *satisfy the following conditions:*

(1) $0 < a \leq \lambda_n \leq \min\{1-k_1, \ldots, 1-k_M\}$, $0 < b \leq \eta_n \leq 2\min\{\mu_1, \ldots, \mu_N\}$;
(2) $\sum_{j=1}^{M} \xi_j = 1$ *and* $\sum_{i=1}^{N} \sigma_i = 1$;
(3) $0 < c \leq \alpha_n, \beta_n, \gamma_n < 1$ *and* $\alpha_n + \beta_n + \gamma_n = 1$.

Then $\{x_n\}$ *converges strongly to a point* $z_0 \in \cap_{j=1}^{M} F(T_j) \cap (\cap_{i=1}^{N} VI(C, B_i))$, *where* $z_0 = P_{\cap_{j=1}^{M} F(T_j) \cap (\cap_{i=1}^{N} VI(C,B_i))} x_1$.

Proof. Since B_i is μ_i-inverse strongly monotone for all $i \in \{1, \ldots, N\}$ and $0 < b \leq \eta_n \leq 2\mu_i$, $P_C(I - \eta_n B_i)$ is nonexpansive and $F(P_C(I - \eta_n B_i)) = VI(C, B_i)$ is closed and convex. Furthermore, we know from Lemma 2.1 that $F(T_j)$ is closed and convex. Therefore, we have that $\cap_{j=1}^{M} F(T_j) \cap (\cap_{i=1}^{N} VI(C, B_i))$ is nonempty, closed and convex. Thus we have that $P_{\cap_{j=1}^{M} F(T_j) \cap (\cap_{i=1}^{N} VI(C,B_i))}$ is well defined. Since

$$\begin{aligned} \|y_n - z\| \leq \|x_n - z\| &\iff \|y_n - z\|^2 \leq \|x_n - z\|^2 \\ &\iff \|y_n\|^2 - \|x_n\|^2 - 2\langle y_n - x_n, z\rangle \leq 0, \end{aligned}$$

it is obvious that C_n are closed and convex for all $n \in \mathbb{N}$. Let us show that $\cap_{j=1}^{M} F(T_j) \cap (\cap_{i=1}^{N} VI(C, B_i)) \subset C_n$ for all $n \in \mathbb{N}$. It is obvious that

$$\cap_{j=1}^{M} F(T_j) \cap (\cap_{i=1}^{N} VI(C, B_i)) \subset C_1 = C.$$

Suppose that $\cap_{j=1}^{M} F(T_j) \cap (\cap_{i=1}^{N} VI(C, B_i)) \subset C_k$ for some $k \in \mathbb{N}$. Then we have from Lemma 2.2 that for $z \in \cap_{j=1}^{M} F(T_j) \cap (\cap_{i=1}^{N} VI(C, B_i))$,

$$\begin{aligned} \|z_k - z\| &= \|\sum_{j=1}^{M} \xi_j((1-\lambda_k)I + \lambda_k T_j)x_k - z\| \\ &\leq \sum_{j=1}^{M} \xi_j \|((1-\lambda_k)I + \lambda_k T_j)x_k - z\| \qquad (3.1) \\ &\leq \sum_{j=1}^{M} \xi_j \|x_k - z\| = \|x_k - z\|. \end{aligned}$$

Furthermore, we have that

$$\begin{aligned}\|w_k - z\| &= \|\sum_{i=1}^{N} \sigma_i P_C(I-\eta_k B_i)x_k - z\| \\ &\leq \sum_{i=1}^{N} \sigma_i \|P_C(I-\eta_k B_i)x_k - z\| \\ &\leq \sum_{i=1}^{N} \sigma_i \|x_k - z\| = \|x_k - z\|.\end{aligned} \tag{3.2}$$

Thus we have that

$$\begin{aligned}\|y_k - z\| &= \|\alpha_k x_k + \beta_k z_k + \gamma_k w_k - z\| \\ &\leq \alpha_k\|x_k - z\| + \beta_k\|z_k - z\| + \gamma_k\|w_k - z\| \\ &\leq \alpha_k\|x_k - z\| + \beta_k\|x_k - z\| + \gamma_k\|x_k - z\| \\ &= \|x_k - z\|.\end{aligned} \tag{3.3}$$

This implies $z \in C_{k+1}$. Therefore, we have by mathematical induction that

$$\cap_{j=1}^{M} F(T_j) \cap (\cap_{i=1}^{N} VI(C,B_i)) \subset C_n$$

for all $n \in \mathbb{N}$. Thus $x_{n+1} = P_{C_{n+1}}x_1$ is well defined.

Since $\cap_{j=1}^{M} F(T_j) \cap (\cap_{i=1}^{N} VI(C,B_i))$ is nonempty, closed and convex, there exists $z_0 \in \cap_{j=1}^{M} F(T_j) \cap (\cap_{i=1}^{N} VI(C,B_i))$ such that $z_0 = P_{\cap_{j=1}^{M} F(T_j) \cap (\cap_{i=1}^{N} VI(C,B_i))}x_1$. By $x_{n+1} = P_{C_{n+1}}x_1$, we have that

$$\|x_1 - x_{n+1}\| \leq \|x_1 - y\|$$

for all $y \in C_{n+1}$. Since $z_0 \in \cap_{j=1}^{M} F(T_j) \cap (\cap_{i=1}^{N} VI(C,B_i)) \subset C_{n+1}$, we have that

$$\|x_1 - x_{n+1}\| \leq \|x_1 - z_0\|. \tag{3.4}$$

This means that $\{x_n\}$ is bounded. From $x_n = P_{C_n}x_1$ and $x_{n+1} \in C_{n+1} \subset C_n$, we have that

$$\|x_1 - x_n\| \leq \|x_1 - x_{n+1}\|.$$

Thus $\{\|x_1 - x_n\|\}$ is bounded and nondecreasing. Then there exists the limit of $\{\|x_1 - x_n\|\}$. Put $\lim_{n\to\infty}\|x_n - x_1\| = c$. For any $m, n \in \mathbb{N}$ with $m \geq n$, we have $C_m \subset C_n$. From $x_m = P_{C_m}x_1 \in C_m \subset C_n$ and (2.4), we have that

$$\|x_m - P_{C_n}x_1\|^2 + \|P_{C_n}x_1 - x_1\|^2 \leq \|x_1 - x_m\|^2.$$

This implies that

$$\|x_m - x_n\|^2 \leq \|x_1 - x_m\|^2 - \|x_n - x_1\|^2 \leq c^2 - \|x_n - x_1\|^2. \tag{3.5}$$

Since $c^2 - \|x_n - x_1\|^2 \to 0$ as $n \to \infty$, we have that $\{x_n\}$ is a Caushy sequence. By the completeness of H and the closedness of C, there exists a point $u \in C$ such that $\lim_{n\to\infty} x_n = u$.

Let us show that $u \in \cap_{j=1}^{M} F(T_j)$. From (3.5), we have $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$. By $x_{n+1} \in C_{n+1}$, we have that

$$\begin{aligned} \|y_n - x_n\| &\le \|y_n - x_{n+1}\| + \|x_{n+1} - x_n\| \\ &\le \|x_n - x_{n+1}\| + \|x_{n+1} - x_n\| \\ &\le 2\|x_n - x_{n+1}\|. \end{aligned} \tag{3.6}$$

This implies that

$$\lim_{n\to\infty} \|y_n - x_n\| = 0. \tag{3.7}$$

Let $z \in \cap_{j=1}^{M} F(T_j) \cap (\cap_{i=1}^{N} VI(C, B_i))$. Using [10], we have from (3.1) and (3.2) that

$$\begin{aligned} \|y_n - z\|^2 &= \alpha_n \|x_n - z\|^2 + \beta_n \|z_n - z\|^2 + \gamma_n \|w_n - z\|^2 \\ &\quad - \alpha_n\beta_n \|z_n - x_n\|^2 - \alpha_n\gamma_n \|w_n - x_n\|^2 - \gamma_n\beta_n \|z_n - w_n\|^2 \\ &\le \alpha_n \|x_n - z\|^2 + \beta_n \|x_n - z\|^2 + \gamma_n \|x_n - z\|^2 \\ &\quad - \alpha_n\beta_n \|z_n - x_n\|^2 - \alpha_n\gamma_n \|w_n - x_n\|^2 - \gamma_n\beta_n \|z_n - w_n\|^2 \\ &= \|x_n - z\|^2 - \alpha_n\beta_n \|z_n - x_n\|^2 - \alpha_n\gamma_n \|w_n - x_n\|^2 - \gamma_n\beta_n \|z_n - w_n\|^2 \end{aligned}$$

and hence

$$\begin{aligned} &c^2\|x_n - z_n\|^2 + c^2\|w_n - x_n\|^2 + c^2\|z_n - w_n\|^2 \\ &\qquad \le \alpha_n\beta_n \|z_n - x_n\|^2 + \alpha_n\gamma_n \|w_n - x_n\|^2 + \gamma_n\beta_n \|z_n - w_n\|^2 \\ &\qquad \le \|x_n - z\|^2 - \|y_n - z\|^2 \\ &\qquad \le \|x_n - y_n\|(\|x_n - z\| + \|y_n - z\|). \end{aligned}$$

From $c > 0$ and (3.7) we have that

$$\lim_{n\to\infty} \|z_n - x_n\| = 0, \ \lim_{n\to\infty} \|w_n - x_n\| = 0, \ \lim_{n\to\infty} \|z_n - w_n\| = 0. \tag{3.8}$$

Since T_j is k_j-demimetric for all $j \in \{1, \dots, M\}$, we have that for $z \in \cap_{j=1}^{M} F(T_j)$,

$$\begin{aligned} \langle x_n - z, x_n - z_n \rangle &= \langle x_n - z, x_n - \sum_{j=1}^{M} \xi_j ((1-\lambda_n)I + \lambda_n T_j) x_n \rangle \\ &= \sum_{j=1}^{M} \xi_j \langle x_n - z, x_n - ((1-\lambda_n)I + \lambda_n T_j) x_n \rangle \\ &= \sum_{j=1}^{M} \xi_j \lambda_n \langle x_n - z, x_n - T_j x_n \rangle \\ &\ge \sum_{j=1}^{M} \xi_j \lambda_n \frac{1-k_j}{2} \|x_n - T_j x_n\|^2 \\ &\ge \sum_{j=1}^{M} \xi_j a \frac{1-k_j}{2} \|x_n - T_j x_n\|^2. \end{aligned}$$

We have from $\lim_{n\to\infty} \|z_n - x_n\| = 0$ that

$$\lim_{n\to\infty} \|x_n - T_j x_n\| = 0, \quad \forall j \in \{1, \dots, M\}.$$

Since T_j are demiclosed for all $j \in \{1, \dots, M\}$ and $\lim_{n\to\infty} x_n = u$, we have $u \in \cap_{j=1}^M F(T_j)$.

Let us show that $u \in \cap_{i=1}^N VI(C, B_i)$. Since $P_C(I - \eta_n B_i)$ is nonexpansive for all $i \in \{1, \dots, N\}$, we have that for $z \in \cap_{i=1}^N VI(C, B_i)$,

$$\begin{aligned}
\langle x_n - z, x_n - w_n\rangle &= \langle x_n - z, x_n - \sum_{i=1}^N \sigma_i P_C(I - \eta_n B_i)x_n\rangle \\
&= \sum_{i=1}^N \sigma_i \langle x_n - z, x_n - P_C(I - \eta_n B_i)x_n\rangle \\
&\geq \sum_{i=1}^N \sigma_i \frac{1}{2} \|x_n - P_C(I - \eta_n B_i)x_n\|^2.
\end{aligned}$$

We have from $\lim_{n\to\infty} \|w_n - x_n\| = 0$ that

$$\lim_{n\to\infty} \|x_n - P_C(I - \eta_n B_i)x_n\| = 0, \quad \forall i \in \{1, \dots, N\}.$$

Since $\{\eta_n\}$ is bounded, we have that there exists a subsequence $\{\eta_{n_l}\}$ of $\{\eta_n\}$ such that $\lim_{l\to\infty} \eta_{n_l} = \eta$ and $0 < b \leq \eta \leq 2\min\{\mu_1, \dots, \mu_N\}$. For such η, we have that for any $i \in \{1, \dots, N\}$,

$$\begin{aligned}
\|x_{n_l} - P_C(I - \eta B_i)x_{n_l}\| &\leq \|x_{n_l} - P_C(I - \eta_{n_l} B_i)x_{n_l}\| \\
&\quad + \|P_C(I - \eta_{n_l} B_i)x_{n_l} - P_C(I - \eta B_i)x_{n_l}\| \\
&\leq \|x_{n_l} - P_C(I - \eta_{n_l} B_i)x_{n_l}\| \\
&\quad + \|(I - \eta_{n_l} B_i)x_{n_l} - (I - \eta B_i)x_{n_l}\| \\
&= \|x_{n_l} - P_C(I - \eta_{n_l} B_i)x_{n_l}\| + |\eta_{n_l} - \eta| \|B_i x_{n_l}\|.
\end{aligned}$$

On the other hand, we have that for $y \in C$ and $i \in \{1, \dots, N\}$,

$$\begin{aligned}
b\|B_i x_n\| &\leq \eta_n \|B_i x_n\| = \|\eta_n B_i x_n\| \\
&= \|x_n - (y - \eta_n B_i y) + y - \eta_n B_i y - (x_n - \eta_n B_i x_n)\| \\
&\leq \|x_n - y\| + \eta_n \|B_i y\| + \|(I - \eta_n B_i)y - (I - \eta_n B_i)x_n\| \\
&\leq \|x_n - y\| + \max\{\mu_1, \dots, \mu_N\} \|B_i y\| + \|y - x_n\|.
\end{aligned}$$

Since $\{x_n\}$ is bounded, we have that $\{B_i x_n\}$ is bounded for all $i \in \{1, \dots, N\}$. Thus we have that

$$\lim_{l\to\infty} \|x_{n_l} - P_C(I - \eta B_i)x_{n_l}\| = 0, \quad \forall i \in \{1, \dots, N\}.$$

Since $\lim_{n\to\infty} x_n = u$ and $P_C(I - \eta B_i)$ are nonexpansive for all $i \in \{1, \dots, N\}$, we have $u \in \cap_{i=1}^N VI(C, B_i)$.

From $z_0 = P_{\cap_{j=1}^M F(T_j)\cap(\cap_{i=1}^N VI(C,B_i))}x_1$, $u \in \cap_{j=1}^M F(T_j) \cap (\cap_{i=1}^N VI(C,B_i))$, (3.4) and $x_n \to u$, we have that

$$\|x_1 - z_0\| \leq \|x_1 - u\| = \lim_{n\to\infty} \|x_1 - x_n\| \leq \|x_1 - z_0\|.$$

Then $u = z_0$. Therefore, we have $x_n \to u = z_0$. This completes the proof. □

4. Applications

In this section, we apply Theorem 3.1 to obtain well-known and new strong convergence theorems in Hilbert spaces. We know the following lemmas obtained by Marino and Xu [9] and Kocourek, Takahashi and Yao [6]; see also [18, 19].

Lemma 4.1 ([9, 18]). *Let H be a Hilbert space and let C be a nonempty, closed and convex subset of H. Let k be a real number with $0 \leq k < 1$ and $U : C \to H$ be a k-strict pseudo-contraction. If $x_n \rightharpoonup z$ and $x_n - Ux_n \to 0$, then $z \in F(U)$.*

Lemma 4.2 ([6, 19]). *Let H be a Hilbert space, let C be a nonempty, closed and convex subset of H and let $U : C \to H$ be generalized hybrid. If $x_n \rightharpoonup z$ and $x_n - Ux_n \to 0$, then $z \in F(U)$.*

The following is a strong convergence theorem for a finite family of strict pseudo-contractions in a Hilbert space.

Corollary 4.3. *Let H be a Hilbert space and let C be a nonempty, closed and convex subset of H. Let $\{k_1, \dots, k_M\} \subset [0,1)$ and let $\{T_j\}_{j=1}^M$ be a finite family of k_j-strict pseudo-contractions of C into H. Assume that $\cap_{j=1}^M F(T_j) \neq \emptyset$. Let $x_1 \in C$ and $C_1 = C$. Let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} z_n = \sum_{j=1}^M \xi_j((1-\lambda_n)I + \lambda_n T_j)x_n, \\ y_n = \alpha_n x_n + \beta_n z_n, \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \leq \|x_n - z\|\}, \\ x_{n+1} = P_{C_{n+1}}x_1, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $a, c \in \mathbb{R}$, $\{\lambda_n\} \subset (0,\infty)$, $\{\xi_1, \dots, \xi_M\} \subset (0,1)$ and $\{\alpha_n\}, \{\beta_n\} \subset (0,1)$ satisfy the following conditions:

(1) $0 < a \leq \lambda_n \leq \min\{1-k_1, \dots, 1-k_M\}$;
(2) $\sum_{j=1}^M \xi_j = 1$;
(3) $0 < c \leq \alpha_n, \beta_n < 1$ *and* $\alpha_n + \beta_n = 1$.

Then $\{x_n\}$ converges strongly to a point $z_0 \in \cap_{j=1}^M F(T_j)$, where $z_0 = P_{\cap_{j=1}^M F(T_j)}x_1$.

Proof. Since T_j is a k_j-strict pseudo-contraction of C into H such that $F(T_j) \neq \emptyset$, from (1.1), T_j is k_j-demimetric. Furthermore, from Lemma 4.1, T_j is demiclosed. Furthermore, if $B_i = 0$ for all $i \in \{1, \dots, N\}$ in Theorem 3.1, then B_i is a 1-inverse strongly monotone mapping. Putting $\eta_n = 1$ for all $n \in \mathbb{N}$ in Theorem 3.1, we have that $w_n = x_n$ for all $n \in \mathbb{N}$. Furthermore, replacing $\beta_n + \gamma_n$ by β_n, we have the desired result from Theorem 3.1. □

The following is a strong convergence theorem for a finite family of generalized hybrid mappings and a finite family of nonexpansive mappings in a Hilbert space.

Corollary 4.4. *Let H be a Hilbert space and let C be a nonempty, closed and convex subset of H. Let $\{T_j\}_{j=1}^M$ be a finite family of generalized hybrid mappings of C into H and let $\{U_i\}_{i=1}^N$ be a finite family of nonexpansive mappings of C into H. Assume that $\cap_{j=1}^M F(T_j) \cap (\cap_{i=1}^N F(U_i)) \neq \emptyset$. Let $x_1 \in C$ and $C_1 = C$. Let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} z_n = \sum_{j=1}^M \xi_j((1-\lambda_n)I + \lambda_n T_j)x_n, \\ w_n = \sum_{i=1}^N \sigma_i P_C((1-\eta_n)I + \eta_n U_i)x_n, \\ y_n = \alpha_n x_n + \beta_n z_n + \gamma_n w_n, \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \leq \|x_n - z\|\}, \\ x_{n+1} = P_{C_{n+1}}x_1, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $a, b, c \in \mathbb{R}$, $\{\lambda_n\}, \{\eta_n\} \subset (0, \infty)$, $\{\xi_1, \ldots, \xi_M\}, \{\sigma_1, \ldots, \sigma_N\} \subset (0,1)$ and $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \subset (0,1)$ satisfy the following conditions:

(1) $0 < a \leq \lambda_n \leq 1$, $0 < b \leq \eta_n \leq 1$;
(2) $\sum_{j=1}^M \xi_j = 1$ *and* $\sum_{i=1}^N \sigma_i = 1$;
(3) $0 < c \leq \alpha_n, \beta_n, \gamma_n < 1$ *and* $\alpha_n + \beta_n + \gamma_n = 1$.

Then $\{x_n\}$ converges strongly to a point $z_0 \in \cap_{j=1}^M F(T_j) \cap (\cap_{i=1}^N F(U_i))$, where $z_0 = P_{\cap_{j=1}^M F(T_j) \cap (\cap_{i=1}^N F(U_i))} x_1$.

Proof. Since T_j is a generalized hybrid mapping of C into H such that $F(T_j) \neq \emptyset$, from (1.2), T_j is 0-demimetric. Furthermore, from Lemma 4.2, T_j is demiclosed. Since U_i is nonexpansive, $B_i = I - U_i$ is a $\frac{1}{2}$-inverse strongly monotone mapping. We also have from $\cap_{i=1}^N F(U_i) \neq \emptyset$ that

$$\cap_{i=1}^N VI(C, I - U_i) = \cap_{i=1}^N F(P_C U_i) = \cap_{i=1}^N F(U_i).$$

Therefore, we have the desired result from Theorem 3.1. $\square$

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