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DISLOCATED QUASI-METRIC AND GENERALIZED CONTRACTIONS

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Abstract. The paper contains some fixed point theorems for generalized contractions in dislocated quasi-metric spaces. The simplest requirement is condition: $p(f(y), f(x)) \leq g(p(y, x))$, for all $x, y \in X$, where p is a dislocated quasi-metric on X (if $p(x, y) = p(y, x) = 0$, then $x = y$; $0 \leq p(x, z) \leq p(x, y) + p(y, z)$) and g is a comparison function of a general type. Our results are far extensions of some known fixed point theorems for dislocated quasi-metric spaces.

Key Words and Phrases: Dislocated quasi-metric, fixed point, generalized contraction, fixed point, cyclic mapping.

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Introduction

In [8] some fixed point theorems were proved for dislocated metric spaces defined by Hitzler and Seda in [2]. The aim of the present paper is to extend the results of [8, Section 3], to the case of dislocated quasi-metric spaces defined by Zeyada, Hassan and Ahmed in [10]. Consequently, our theorems strongly generalize the results of Zoto and Hoxha proved in [11].

In Section 1 the definitions of a dislocated metric and of a dislocated quasi-metric are presented. This section is devoted to the study of some properties of the respective spaces, and completeness is of our particular interest.

Section 2 is devoted to fixed point theorems for general contractions. The simplest requirement is condition (2.1): $p(f(y), f(x)) \leq \varphi(p(y, x))$, for all $x, y \in X$, where p is a dislocated quasi-metric on X, $f: X \to X$ is a mapping, and the comparison function $\varphi: [0, \infty) \to [0, \infty)$ belongs to a wide class of mappings defined in [7]. The main classical results are Theorem 2.5 (a far extension of the celebrated theorems of Matkowski [5, Theorem 1.2], and of Boyd-Wong [1, Theorem 1]), and a more general Theorem 2.7. The most sophisticated ones are the theorems for cyclic mappings (see Definition 2.8): Theorem 2.11, and Theorem 2.12, which is proved with the use of cross mappings defined in [6]. Our theorems extend also the main results of Zoto and Hoxha in [11], i.e. Theorems 3.6, 3.8.

1. Dislocated metric and dislocated quasi-metric

The notion of dislocated metric was introduced by Hitzler and Seda in [2], and the notion of dislocated quasi-metric was introduced by Zeyada, Hassan and Ahmed in [10].

Definition 1.1. Let X be a nonempty set, and $p : X \times X \to [0, \infty)$ a mapping satisfying

$$\text{if } p(x,y) = p(y,x) = 0 \text{ then } x = y, \qquad x, y \in X, \tag{1.1}$$

$$p(x,z) \leq p(x,y) + p(y,z), \qquad x, y, z \in X. \tag{1.2}$$

Then p is called a **dislocated quasi-metric** (briefly a dq-metric), and (X, p) is called a **dislocated quasi-metric space** (briefly a dq-metric space). If, in addition

$$p(x,y) = p(y,x), \qquad x, y \in X \tag{1.3}$$

holds, then p is called a **dislocated metric** (briefly a d-metric), and (X, p) is called a **dislocated metric space** (briefly a d-metric space).

The topology of a d-metric (or a dq-metric) space (X, p) is generated by balls $B(x,r) = \{y \in X : p(x,y) < r\}$. Clearly, $x \in B(x,r)$ does not necessarily hold, but the family of all balls generates the respective smallest topology for $X = \bigcup\{B(x,r) : x \in X, r > 0\}$ [3, Theorem 12, p. 47].

Let us recall the subsequent two definitions.

Definition 1.2 (cf. [10, Definition 2.1])**.** A sequence $(x_n)_{n\in\mathbb{N}}$ in dq-metric space (X, p) is called **Cauchy** if the following condition is satisfied

$$\lim_{m,n\to\infty} \min\{p(x_m, x_n), p(x_n, x_m)\} = 0 \tag{1.4}$$

Definition 1.3 ([10, Definitions 2.2, 2.3])**.** A dq-metric space is called **complete** if each Cauchy sequence $(x_n)_{n\in\mathbb{N}}$ in X **converges** to an $x \in X$, i.e. the following condition is satisfied

$$\lim_{n\to\infty} p(x, x_n) = \lim_{n\to\infty} p(x_n, x) = 0. \tag{1.5}$$

Unfortunately, these notions are not well suited to the topology of dq-metric spaces. We prefer to replace “complete” by a more precise term “**0-complete**”.

Proposition 1.4. *Let (X, p) be a dq-metric space. Then from condition (1.5) it follows that*

$$\lim_{m,n\to\infty} p(x_m, x_n) = 0, \tag{1.6}$$

and $p(x,x) = 0$.

Proof. Condition (1.2) yields

$$\lim_{m,n\to\infty} p(x_m, x_n) \leq \lim_{m\to\infty} p(x_m, x) + \lim_{n\to\infty} p(x, x_n) = 0,$$

$$\text{and } p(x,x) \leq \lim_{n\to\infty} p(x, x_n) + \lim_{n\to\infty} p(x_n, x) = 0.$$

□

Hence we obtain

Corollary 1.5. *If for a sequence $(x_n)_{n\in\mathbb{N}}$ in a dq-metric space (X,p) condition (1.5) holds, then (1.4) is equivalent to (1.6). In particular, if (X,p) is a 0-complete dq-metric space, then (1.4) and (1.6) are equivalent.*

Lemma 1.6. *Let (X,p) be a dq-metric space. Then γ defined by*

$$\gamma(x,y) = [p(x,y) + p(y,x)]/2 \qquad x,y \in X \tag{1.7}$$

is a d-metric on X. If (X,p) is 0-complete, then (X,γ) is 0-complete.

Proof. It is clear that γ is a d-metric on X. If (1.6) holds for p replaced by γ, then (1.4) is satisfied. Consequently, if $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in (X,γ) and (X,p) is 0-complete, then there exists an $x \in X$ such that

$$\lim_{n\to\infty} p(x,x_n) = \lim_{n\to\infty} p(x_n,x) = 0.$$

This equality yields $\lim_{n\to\infty}\gamma(x,x_n) = 0$, and thus (X,γ) is 0-complete. □

Lemma 1.7. *Let (X,γ) be a d-metric space. Then δ defined by*

$$\delta(x,y) = \gamma(x,y) \text{ if } x \neq y, \text{ and } \delta(x,x) = 0, \qquad x,y \in X \tag{1.8}$$

is a metric on X, and (X,γ) is 0-complete iff (X,δ) is complete.

Proof. Clearly, δ is a metric on X. Assume that (X,δ) is a complete metric space, and $\lim_{m,n\to\infty}\gamma(x_m,x_n) = 0$. Then

$$0 \le \lim_{m,n\to\infty}\delta(x_m,x_n) \le \lim_{m,n\to\infty}\gamma(x_m,x_n) = 0$$

means that there exists an $x \in X$ such that $\lim_{n\to\infty}\delta(x,x_n) = 0$. If there are infinitely many $x_m = x$, then

$$0 \le \gamma(x,x_n) \le \gamma(x,x_m) + \gamma(x_m,x_n) = \gamma(x_m,x_m) + \gamma(x_m,x_n),$$

and $\lim_{m,n\to\infty}\gamma(x_m,x_n) = 0$ yield $\lim_{n\to\infty}\gamma(x,x_n) = 0$. If only finite $x_m = x$, then

$$\lim_{n\to\infty}\gamma(x,x_n) = \lim_{n\to\infty}\delta(x,x_n) = 0.$$

Consequently, if (X,δ) is complete, then (X,γ) is 0-complete.

Assume that (X,γ) is 0-complete, and $\lim_{m,n\to\infty}\delta(x_m,x_n) = 0$. If there exist infinitely many x_m equal, say to an x, then

$$0 \le \delta(x,x_n) \le \delta(x,x_m) + \delta(x_m,x_n) = \delta(x_m,x_n)$$

means that $\lim_{n\to\infty}\delta(x,x_n) = 0$. If $(x_n)_{n\in\mathbb{N}}$ does not contain any constant subsequence, then there exists a subsequence $(x_{k_n})_{n\in\mathbb{N}}$ such that $\delta(x_{k_m},x_{k_n}) = \gamma(x_{k_m},x_{k_n})$, and now $\lim_{m,n\to\infty}\delta(x_m,x_n) = 0$ yields $\lim_{m,n\to\infty}\gamma(x_{k_m},x_{k_n}) = 0$. Therefore, there exists an $x \in X$ such that $\lim_{n\to\infty}\gamma(x,x_{k_n}) = 0$ (as (X,γ) is 0-complete). Now,

$$0 \le \delta(x,x_n) \le \delta(x,x_{k_n}) + \delta(x_{k_n},x_n) \le \gamma(x,x_{k_n}) + \delta(x_{k_n},x_n),$$

and $\lim_{m,n\to\infty}\delta(x_m,x_n) = 0$ mean that $\lim_{n\to\infty}\delta(x,x_n) = 0$, i.e. (X,δ) is complete. □

Now, Corollary 1.5, and Lemmas 1.6, 1.7 yield

Corollary 1.8. *If a dq-metric space (X,p) is 0-complete, then for γ given by (1.7) d-metric space (X,γ) is 0-complete. Any d-metric space (X,γ) is 0-complete iff for δ defined by (1.8) metric space (X,δ) is complete.*

In [8, Definition 2.3], the following notion was presented for dislocated strongly quasi-metric spaces (for which (1.1) is replaced by $p(x,y) = 0$ yields $x = y$, $x, y \in X$): (X,p) is 0-complete if for every sequence $(x_n)_{n\in\mathbb{N}}$ in X such that $\lim_{n>m\to\infty} p(x_n, x_m) = 0$ there exists an $x \in X$ such that $\lim_{n\to\infty} p(x, x_n) = 0$.

Both ideas of 0-completeness coincide for d-metric spaces. Indeed, if (X,p) is a d-metric space then in view of (1.3) conditions (1.4), (1.6) are equivalent. In addition, $\lim_{n>m\to\infty} p(x_n, x_m) = \lim_{n\neq m\to\infty} p(x_n, x_m)$, and $p(x_n, x_n) \le 2p(x_n, x_m)$ mean that (1.6) is equivalent to $\lim_{n>m\to\infty} p(x_n, x_m) = 0$.

2. Generalized contractions

In the present section we extend the results of [8, Section 3], obtained for d-metric, to suite the case of dq-metric.

We are interested in mappings $f\colon X \to X$ satisfying

$$p(f(y), f(x)) \le \varphi(p(y,x)) \tag{2.1}$$

or

$$p(f(y), f(x)) \le \min\{\varphi(ml_f(y,x)), \varphi(mr_f(y,x))\} \tag{2.2}$$

for

$$\begin{aligned} ml_f(y,x) &= \max\{p(y,x), p(f(y),y), p(f(x),x)\}, \text{ and} \\ mr_f(y,x) &= \max\{p(y,x), p(y,f(y)), p(x,f(x))\}, \end{aligned} \tag{2.3}$$

where (X,p) is a dq-metric space, and φ is a comparison function.

According to the notations from [7] $\mathbf{\Phi}$ is a class of mappings $\varphi\colon [0,\infty) \to [0,\infty)$ such that $\varphi(\alpha) < \alpha$, $\alpha > 0$; and $\varphi \in \mathbf{\Phi_0}$ iff $\varphi \in \Phi$ and $\varphi(0) = 0$. In turn, $\mathbf{\Phi_P}$ consists of mappings $\varphi\colon [0,\infty) \to [0,\infty)$ for which every sequence $(a_n)_{n\in\mathbb{N}}$ such that $a_{n+1} \le \varphi(a_n)$, $n \in \mathbb{N}$ converges to zero. It appears [7, Proposition 16], that $\Phi_P \subset \Phi_0$, and if $\varphi \in \Phi_0$ satisfies

$$\limsup_{\beta\to\alpha^+} \varphi(\beta) < \alpha, \qquad \alpha > 0, \tag{2.4}$$

then $\varphi \in \Phi_P$. Consequently, (see [7]), if $\varphi \in \Phi_0$ is upper semicontinuous from the right (see [1]), then $\varphi \in \Phi_P$; also, if $\varphi \in \Phi_0$ is nondecreasing and $\lim_{n\to\infty} \varphi^n(\alpha) = 0$, $\alpha > 0$ (see [5]), then $\varphi \in \Phi_P$.

There exist non-monotone mappings $\varphi \in \Phi_P$ for which (2.4) does not hold (see [8, Example]).

The subsequent lemma is a modification of [8, Lemma 3.1].

Lemma 2.1. *Let X be a nonempty set, and let $p : X \times X \to [0,\infty)$, $f : X \to X$ be mappings satisfying condition (2.1) or (2.2), for all $x, y \in X$ and a $\varphi \in \Phi$. Then condition*

$$\begin{aligned} p(f^2(x), f(x)) &\le \varphi(p(f(x), x)) \text{ and} \\ p(f(x), f^2(x)) &\le \varphi(p(x, f(x)), \qquad x \in X \end{aligned} \tag{2.5}$$

holds. In addition, if $\varphi \in \Phi_P$, then we have $\lim_{n\to\infty} p(f^{n+1}(x), f^n(x)) = \lim_{n\to\infty} p(f^n(x), f^{n+1}(x)) = \lim_{n\to\infty} p(f^n(x), f^n(x)) = 0$, $x \in X$.

Proof. For notational simplicity let us adopt $x_n = f^n(x)$, $n \in \mathbb{N}$. We have

$$ml_f(x_1, x) = \max\{p(x_1, x), p(x_2, x_1)\}.$$

Suppose $p(x_1, x) < p(x_2, x_1)$. Then (2.2) yields

$$0 < \alpha = p(x_2, x_1) \le \varphi(ml_f(x_1, x)) = \varphi(p(x_2, x_1)) = \varphi(\alpha),$$

a contradiction ($\varphi \in \Phi$). Now, $ml_f(x_1, x) = p(x_1, x)$ holds, and we obtain the first part of (2.5) (which for (2.1) is trivial). Now, for $a_n = p(x_{n+1}, x_n)$, $n \in \mathbb{N}$, and $\varphi \in \Phi_P$ we get $\lim_{n\to\infty} a_n = 0$ (note that $\varphi(0) = 0$). A similar reasoning for mr_f proves the second part of (2.5), and $\lim_{n\to\infty} p(x_n, x_{n+1}) = 0$. We also have

$$0 \le \lim_{n\to\infty} p(x_n, x_n) \le \lim_{n\to\infty} p(x_n, x_{n+1}) + \lim_{n\to\infty} p(x_{n+1}, x_n) = 0$$

□

The notion of f-orbitally completeness presented in [11] should be better suited to the topology of dq-metric spaces. Therefore, we suggest the subsequent idea.

Definition 2.2 (cf. [11, Definition 3.3])**.** Let (X, p) be a dq-metric space, and let $f\colon X \to X$ be a mapping. Then (X, p) is called **f-orbitally 0-complete** if for every sequence $(x_n)_{n\in\mathbb{N}}$ satisfying (1.4) and contained in any orbit $\{x_0, f(x_0), f^2(x_0), \ldots\}$ $(x_0 \in X)$, there exists an $x \in X$ such that (1.5) holds.

Remark 2.3. The results of [8, Section 3] (excepting [8, Theorem 3.10]), stay valid for "0-complete" replaced by "f-orbitally 0-complete" ("f^t-orbitally complete" for Theorem 3.5).

The next lemma is a modification of [8, Lemma 3.2], proved for d-metric spaces.

Lemma 2.4. *Let (X, p) be an f-orbitally 0-complete dq-metric space for a mapping f satisfying condition (2.1) or (2.2), for all $x, y \in X$ and a $\varphi \in \Phi_0$. If for $x_n = f^n(x_0)$, $\lim_{m,n\to\infty} p(x_m, x_n) = 0$ holds, then f has a unique fixed point; if $x = f(x)$, then $p(x, x) = 0$, and $\lim_{n\to\infty} p(x, x_n) = \lim_{n\to\infty} p(x_n, x) = 0$.*

Proof. Let $x \in X$ be such that $\lim_{n\to\infty} p(x, x_n) = \lim_{n\to\infty} p(x_n, x) = 0$. For condition (2.1) we have

$$\begin{aligned} p(f(x), x) &\le p(f(x), x_{n+1}) + p(x_{n+1}, x) \\ &\le \varphi(p(x, x_n)) + p(x_{n+1}, x) \le p(x, x_n) + p(x_{n+1}, x), \end{aligned}$$

($\varphi \in \Phi_0$) and

$$\begin{aligned} p(x, f(x)) &\le p(x, x_{n+1}) + p(x_{n+1}, f(x)) \\ &\le p(x, x_{n+1}) + \varphi(p(x_n, x)) \le p(x, x_{n+1}) + p(x_n, x). \end{aligned}$$

Therefore, we have $p(x, f(x)) = p(f(x), x) = 0$, which yields $x = f(x)$ (see (1.1)) and $p(x, x) = 0$.

For condition (2.2), $p(f(x), x) > 0$ and large n we have

$$\begin{aligned} 0 < p(f(x), x) &\le p(f(x), x_{n+1}) + p(x_{n+1}, x) \le \varphi(ml_f(x, x_n)) + p(x_{n+1}, x) \\ &= \varphi(\max\{p(x, x_n), p(f(x), x), p(x_{n+1}, x_n)\}) + p(x_{n+1}, x) \\ &= \varphi(p(f(x), x)) + p(x_{n+1}, x). \end{aligned}$$

Consequently, $0 < p(f(x), x) \leq \varphi(p(f(x), x))$ holds, a contradiction ($\varphi \in \Phi$), and $p(f(x), x) = 0$. In a similar way, by applying mr_f we prove that $p(x, f(x)) = 0$, and we get $x = f(x)$.

If y is a fixed point of f, then

$$0 \leq p(f(y), y) = p(y, y) = p(f(y), f(y)) \leq \varphi(p(y, y)) = \varphi(ml_f(y, y))$$

means that $p(y, y) = 0$.

Suppose x, y are two fixed points of f. Then

$$p(y, x) = p(f(y), f(x)) \leq \varphi(p(y, x)) = \varphi(\max\{p(y, x), 0, 0\}) = \varphi(ml_f(y, x))$$

means that $p(y, x) = 0$, and similarly $p(x, y) = 0$, i.e. $x = y$. □

Now, we are ready to prove the following analog of [8, Theorem 3.3] (we refine the respective proof from [8]).

Theorem 2.5. *Let (X, p) be an f-orbitally 0-complete dq-metric space for a mapping f satisfying condition (2.1) or (2.2), for all $x, y \in X$ and a $\varphi \in \Phi_0$ having property (2.4) or a $\varphi \in \Phi_P$ such that*

$$\limsup_{\beta \to \alpha^-} \varphi(\beta) < \alpha, \qquad \alpha > 0 \tag{2.6}$$

(e.g. if φ is nondecreasing) holds. Then f has a unique fixed point; if $x = f(x)$, then $p(x, x) = 0$, and $\lim_{n\to\infty} p(x, f^n(x_0)) = \lim_{n\to\infty} p(f^n(x_0), x) = 0$, $x_0 \in X$.

Proof. It is sufficient to prove that $\lim_{m,n\to\infty} p(x_m, x_n) = 0$ holds for $x_n = f^n(x_0)$, $n \in \mathbb{N}$ (see Corollary 1.5 and Lemma 2.4). Suppose that there are infinitely many $k, n \in \mathbb{N}$ such that $p(f^{n+1+k}(x_0), f^k(x_0)) \geq \epsilon > 0$. Let $n = n(k) > 0$ be the smallest numbers satisfying this inequality for infinitely many large k. For simplicity let us adopt $x = f^k(x_0)$, and $x_n = f^n(x)$, $n \in \mathbb{N}$. We have

$$\epsilon \leq p(x_{n+1}, x) \leq p(x_{n+1}, x_n) + p(x_n, x) < p(x_{n+1}, x_n) + \epsilon,$$

which for $n = n(k)$ means that

$$\lim_{k\to\infty} p(x_{n+1}, x) = \lim_{k\to\infty} p(x_n, x) = \epsilon,$$

as $\lim_{k\to\infty} p(x_{n+1}, x_n) = \lim_{k\to\infty} p(x_1, x) = 0$ (see Lemma 2.1). Now for $y = x_n$ condition (2.2) yields

$$\begin{aligned} \epsilon \leq p(x_{n+1}, x) &\leq p(x_{n+1}, x_1) + p(x_1, x) \leq \varphi(ml_f(x_n, x)) + p(x_1, x) \\ &= \varphi(\max\{p(x_n, x), p(x_{n+1}, x_n), p(x_1, x)\}) + p(x_1, x), \end{aligned}$$

and we obtain (from (2.1) as well)

$$\epsilon \leq \varphi(p(x_n, x)) + p(x_1, x)$$

for large k. Now, $p(x_n, x) < \epsilon$, $\lim_{k\to\infty} p(x_n, x) = \epsilon$, and condition (2.6) yield

$$\epsilon \leq \limsup_{k\to\infty} \varphi(p(x_n, x)) < \epsilon,$$

a contradiction. Similarly, $p(x_{n+1}, x) \geq \epsilon$,

$$p(x_{n+1}, x) - p(x_{n+1}, x_{n+2}) - p(x_1, x) \leq p(x_{n+2}, x_1) \leq \varphi(p(x_{n+1}, x)),$$

and condition (2.4) yield

$$\epsilon \leq \limsup_{k\to\infty} \varphi(p(x_{n+1}, x)) < \epsilon,$$

a contradiction. Therefore, $\lim_{n>m\to\infty} p(x_n, x_m) = 0$ holds. In a similar way, by applying mr_f we prove that $\lim_{n>m\to\infty} p(x_m, x_n) = 0$, which with

$$p(x_n, x_n) \leq p(x_n, x_{n+1}) + p(x_{n+1}, x_n)$$

yield $\lim_{m,n\to\infty} p(x_m, x_n) = 0$. □

The previous theorem is a far extension of [11, Theorems 3.6, 3.8].

Let us recall the following.

Lemma 2.6 ([7, Lemma 29])**.** *Let $f\colon X \to X$ be a mapping such that f^t for a $t \in \mathbb{N}$ has a unique fixed point, say x. Then x is the unique fixed point of f. If, in addition, $x \in \lim_{n\to\infty}(f^t)^n(x_0)$, $x_0 \in X$, then $x \in \lim_{n\to\infty} f^n(x_0)$, $x_0 \in X$ holds.*

Now, Theorem 2.5 and Lemma 2.6 yield

Theorem 2.7. *Let (X, p) be an f^t-orbitally 0-complete dq-metric space for a mapping f satisfying condition (2.1) or (2.2), for all $x, y \in X$, with f replaced by f^t for a $t \in \mathbb{N}$, and a $\varphi \in \Phi_0$ having property (2.4) or a $\varphi \in \Phi_P$ such that (2.6) holds. Then f has a unique fixed point; if $x = f(x)$, then $p(x, x) = 0$, and $\lim_{n\to\infty} p(x, f^n(x_0)) = \lim_{n\to\infty} p(f^n(x_0), x) = 0$, $x_0 \in X$.*

Kirk, Srinivasan and Veeramani [4] suggested the idea of cyclic mappings which was later formalized by Rus in [9] as cyclic representation of $X = X_1 \cup \cdots \cup X_t$ with respect to f. The next definition means the same, but is more compact.

Definition 2.8 ([8, Definition 3.6])**.** A mapping $f\colon X \to X$ is called **cyclic** on $X_1, \ldots, X_t$ (for a $t > 1$) if $\emptyset \neq X = X_1 \cup \cdots \cup X_t$, and $f(X_j) \subset X_{j++}$, $j = 1, \ldots, t$, where $j{+}{+} = j + 1$ for $j = 1, \ldots, t-1$, and $t{+}{+} = 1$.

Clearly, $X_j \neq \emptyset$ for a j in Definition 2.8, and hence $X_j \neq \emptyset$, $j = 1, \ldots, t$.

The proof of Lemma 2.1 works also for the following.

Lemma 2.9. *Let $p\colon X \times X \to [0, \infty)$ be a mapping, and let $f\colon X \to X$ be cyclic on $X_1, \ldots, X_t$. Assume that (2.1) or (2.2) is satisfied for all $x \in X_j$, $y \in X_{j++}$, $j = 1, \ldots, t$, and a $\varphi \in \Phi$. Then condition (2.5) holds, and if $\varphi \in \Phi_P$, then $\lim_{n\to\infty} p(f^{n+1}(x), f^n(x)) = \lim_{n\to\infty} p(f^n(x), f^{n+1}(x)) = \lim_{n\to\infty} p(f^n(x), f^n(x)) = 0$, $x \in X$.*

If we consider n such that $x \in X_j$ and $x_n \in X_{j++}$ for a $j \in \{1, \ldots, t\}$, then the proof of Lemma 2.4 yields the following analog.

Lemma 2.10. *Let (X, p) be an f-orbitally 0-complete dq-metric space for a mapping f cyclic on $X_1, \ldots, X_t$. Assume that (2.1) or (2.2) is satisfied for all $x \in X_j$, $y \in X_{j++}$, $j = 1, \ldots, t$, and a $\varphi \in \Phi_0$. If for $x_n = f^n(x_0)$, $\lim_{m,n\to\infty} p(x_m, x_n) = 0$ holds, then f has a unique fixed point; if $x = f(x)$, then $p(x, x) = 0$, and $\lim_{n\to\infty} p(x, x_n) = \lim_{n\to\infty} p(x_n, x) = 0$.*

Lemmas 2.9, 2.10 yield the following extension of Theorem 2.5.

Theorem 2.11 (cf. [8, Theorem 3.9])**.** *Let a dq-metric space (X,p) be f-orbitally 0-complete for an f cyclic on $X_1,\dots,X_t$. Assume that (2.1) or (2.2) is satisfied for all $x\in X_j$, $y\in X_{j++}$, $j=1,\dots,t$, and a $\varphi\in\Phi_0$. If in addition, φ has property (2.4) or $\varphi\in\Phi_P$ and satisfies (2.6), then f has a unique fixed point; if $x=f(x)$, then $p(x,x)=0$, and $\lim_{n\to\infty}p(x,f^n(x_0))=\lim_{n\to\infty}p(f^n(x_0),x)=0$, $x_0\in X$.*

Proof. We apply a reasoning similar to the one presented in the proof of [8, Theorem 3.9]. It is sufficient to prove that $\lim_{m,n\to\infty}p(x_m,x_n)=0$ holds for $x_n=f^n(x_0)$, $n\in\mathbb{N}$ (see Corollary 1.5 and Lemma 2.10). Suppose that there are infinitely many $k,n\in\mathbb{N}$ such that $p(x_{(n+1)t+k+1},x_k)\geq\epsilon>0$. Let $n=n(k)>0$ be the smallest numbers satisfying this inequality for infinitely many large k. For simplicity let us adopt $x=f^k(x_0)$, and $x_n=f^n(x)$, $n\in\mathbb{N}$. Clearly, $x\in X_j$ yields $x_{nt+1},x_{(n+1)t+1}\in X_{j++}$. We have

$$\begin{aligned}\epsilon\leq p(x_{(n+1)t+1},x)&\leq p(x_{(n+1)t+1},x_{nt+1})+p(x_{nt+1},x)\\&<p(x_{(n+1)t+1},x_{nt+1})+\epsilon\leq p(x_{(n+1)t+1},x_{(n+1)t})+\cdots+p(x_{nt+2},x_{nt+1})+\epsilon,\end{aligned}$$

which for $n=n(k)$ means that

$$\lim_{k\to\infty}p(x_{(n+1)t+1},x)=\lim_{k\to\infty}p(x_{nt+1},x)=\epsilon,$$

as $\lim_{m\to\infty}p(x_{m+1},x_m)=0$ (see Lemma 2.9). Now for $y=x_{nt+1}$ condition (2.2) yields

$$\begin{aligned}\epsilon\leq p(x_{(n+1)t+1},x)&\leq p(x_{(n+1)t+1},x_{(n+1)t})+\cdots+p(x_{nt+2},x_1)+p(x_1,x)\\&\leq p(x_{(n+1)t+1},x_{(n+1)t})+\cdots+\varphi(ml_f(x_{nt+1},x))+p(x_1,x)\\&=p(x_{(n+1)t+1},x_{(n+1)t})+\cdots\\&+\varphi(\max\{p(x_{nt+1},x),p(x_{nt+2},x_{nt+1}),p(x_1,x)\})+p(x_1,x),\end{aligned}$$

and we obtain (from (2.1) as well)

$$\epsilon\leq p(x_{(n+1)t+1},x_{(n+1)t})+\cdots+\varphi(p(x_{nt+1},x))+p(x_1,x)$$

for large k. Now, $p(x_{nt+1},x)<\epsilon$, $\lim_{k\to\infty}p(x_{nt+1},x)=\epsilon$, and condition (2.6) yield

$$\epsilon\leq\limsup_{k\to\infty}\varphi(p(x_{nt+1},x))<\epsilon,$$

a contradiction. Similarly, $p(x_{(n+1)t+1},x)\geq\epsilon$,

$$\begin{aligned}p(x_{(n+1)t+1},x)-p(x_{(n+1)t+1},x_{(n+1)t+2})-p(x_1,x)&\\\leq p(x_{(n+1)t+2},x_1)\leq\varphi(p(x_{(n+1)t+1},x)),&\end{aligned}$$

and condition (2.4) yield

$$\epsilon\leq\limsup_{k\to\infty}\varphi(p(x_{(n+1)t+1},x))<\epsilon,$$

a contradiction. Now, it is clear that $\lim_{m,n\to\infty}p(x_{m+nt+1},x_m)=0$. Consequently,

$$\begin{aligned}&\lim\nolimits_{m,n\to\infty}p(x_{m+nt+s},x_m)\\&\leq\lim\nolimits_{m,n\to\infty}(p(x_{m+nt+s},x_{m+nt+s-1})+\cdots\\&+p(x_{m+nt+2},x_{m+nt+1})+p(x_{m+nt+1},x_m))=0\end{aligned}$$

for any $s \in \{2,\ldots,t\}$, i.e. $\lim_{n>m\to\infty} p(x_n,x_m)=0$. In a similar way, by applying mr_f we prove that $\lim_{n>m\to\infty} p(x_m,x_n)=0$, and the final inequality of the proof of Theorem 2.5 shows that $\lim_{m,n\to\infty} p(x_m,x_n)=0$. □

Let us present cyclic mappings of the second type, i.e. those for (2.1) or (2.2) with $x,y\in X_j$, $j=1,\ldots,t$. It is convenient to apply the idea of cross mappings introduced in [6].

Let $F_j : X_j \to 2^{Ej++}$, $j=1,\ldots,t$ (for a $t>1$) be multivalued mappings. Then for $Y = X_1\times\cdots\times X_t$, $E=E_1\times\cdots\times E_t$ we define a **cross mapping** $F\colon Y\to 2^E$ as follows [6, (3.1)]:

$$F(x) = F_t(x_t)\times F_1(x_1)\times\cdots\times F_{t-1}(x_{t-1}),\qquad x=(x_1,\ldots,x_t)\in Y. \tag{2.7}$$

We can see that for $E_j\subset X_j$, $j=1,\ldots,t$ the composition $F_t\circ F_{t-1}\circ\cdots\circ F_1$ has a fixed point in X_1 iff F has a fixed point. This concept is very efficient for multivalued mappings (see [6, Section 3]). Let us apply cross mappings to prove the following extension of [8, Theorem 3.10].

Theorem 2.12. *Let (X,p) be a dq-metric space, and let $f : X\to X$ be cyclic on 0-complete sets $X_1,\ldots,X_t$. Assume that (2.1) or (2.2) is satisfied for all $x,y\in X_j$, $j=1,\ldots,t$, and a nondecreasing $\varphi\in\Phi_P$. Then f has a unique fixed point; if $x=f(x)$, then we have $p(x,x)=0$, and $\lim_{n\to\infty} p(x,f^n(x_0)) = \lim_{n\to\infty} p(f^n(x_0),x)=0$, $x_0\in X$.*

Proof. Let us consider $Y = X_1\times\cdots\times X_t$ and

$$q(y,x) = \max\{p(y_1,x_1),\ldots,p(y_t,x_t)\},\qquad x,y\in Y.$$

Then (Y,q) is a dq-metric space. If φ is nondecreasing and (2.1) or (2.2) is satisfied for p, then it is also satisfied for q, as e.g. $\max\{\varphi(a),\varphi(b)\}=\varphi(\max\{a,b\})$. In addition, Y is h-orbitally 0-complete for the cross mapping $h\colon Y\to Y$ defined by

$$h(x) = (f(x_t), f(x_1),\ldots,f(x_{t-1})),\qquad x\in Y.$$

In view of Theorem 2.5 the mapping h has a unique fixed point. This means that f^t has a unique fixed point. Now we apply Lemma 2.6. □

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