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FIXED POINT THEOREMS FOR VARIOUS TYPES OF F-CONTRACTIONS IN COMPLETE b-METRIC SPACES

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Abstract. We generalize various types of F-contractions defined by Wardowski, Vetro and others to b-metric spaces and prove fixed point theorems for them. The examples given show that these generalizations extend the existing results in a significant way.
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1. Introduction

Banach's fixed point theorem is an important result in the theory of metric spaces. Over the years, various generalizations of it appeared in the literature, following a number of different lines of thought. One such idea was to relax the conditions imposed on the space itself, while another possibility was to generalize the contractive condition.

A well-known generalization of metric spaces are b-metric spaces introduced by Czerwik in [9], while the contractive condition in Banach's theorem has been weakened in several ways: see for example the results of Ćirić, Hardy-Rogers, Reich, Suzuki in [6, 10, 12, 14] and the recent works of Wardowski in [15, 16].

In the last two decades a number of generalizations appeared, combining the two ideas mentioned above. For example, Ćirić-, Hardy-Rogers-, Suzuki-type contractions and fixed point theorems for them in b-metric spaces were discussed in [2, 3, 4, 5].

So far in the literature there are only a few examples considering Wardowski's F-contractions in b-metric spaces (see [1] and [7]). Our goal in this paper is to develop fixed point theory in this direction: we study F-contractions and their generalizations in the context of b-metric spaces. In order to achieve this, we use Wardowski's paper [15] as a starting point. In that paper the author imposed three general conditions on functions F and one extra contractive condition concerning the operator. In our work we prove fixed point theorems for different notions of F-contractions in b-metric spaces, without using Wardowski's condition (F2).

The results we present improve and generalize some of the results in the literature on F-contractions. The technique used in the proofs also points out that Wardowski's original (F2) condition may be omitted from the currently used definitions of the different types of F-contractions.

2. Preliminaries

First we recall b-metric spaces and those of their properties which we are going to use later.

Definition 2.1. Let X be a non-empty set and $s \geq 1$ a real number. A function $d : X \times X \to [0, \infty)$ is called a *b-metric* if the following conditions are satisfied, for every $x, y, z \in X$:

(B1) $d(x, y) = 0$ if and only if $x = y$;
(B2) $d(x, y) = d(y, x)$;
(B3) $d(x, y) \leq s\,[d(x, z) + d(z, y)]$.

In this case (X, d) is called a *b-metric space* with constant $s \geq 1$.

Convergent sequences and Cauchy sequences in b-metric spaces, continuous operators on b-metric spaces, etc. are defined the same way as in metric spaces. The limit of a convergent sequence is unique and every convergent sequence is a Cauchy sequence. A b-metric space is called complete if every Cauchy sequence is convergent. Czerwik in [9] generalized Banach's fixed point theorem to the b-metric case.

Examples and more details on b-metric spaces can also be found in the articles [2, 3, 4] and in the book [11].

One of the main difficulties when proving fixed point theorems in b-metric spaces arises from the fact that the distance functional $d\colon X \times X \to [0, \infty)$ is usually not continuous. The following lemma will help to deal with this problem.

Lemma 2.2. (see also [11]) *If (X, d) is a b-metric space with constant $s \geq 1$, $x^*, y^* \in X$ and $(x_n)_{n\in\mathbb{N}}$ is a convergent sequence in X with $\lim_{n\to\infty} x_n = x^*$ then*

$$\frac{1}{s} d(x^*, y^*) \leq \liminf_{n\to\infty} d(x_n, y^*) \leq \limsup_{n\to\infty} d(x_n, y^*) \leq s d(x^*, y^*).$$

Proof. If we apply twice the relaxed triangle inequality (B3), we get for every $n \in \mathbb{N}$

$$\frac{1}{s} d(x^*, y^*) - d(x_n, x^*) \leq d(x_n, y^*) \leq s d(x^*, y^*) + s d(x_n, x^*).$$

If we take lim inf on the left-hand side inequality and lim sup on the right-hand side inequality, we obtain the desired property.

Our results are based on the following $\mathcal{F}_{s,\tau}$ class of functions, defined in two steps.

Definition 2.3. A function $F\colon (0, \infty) \to \mathbb{R}$ belongs to $\mathcal{F}$ if it satisfies the following conditions:

(F1) F is strictly increasing;
(F2) there exists $k \in (0,1)$ such that $\lim_{x\to 0^+} x^k F(x) = 0$.

Note that we omitted Wardowski's (F2) condition from the definition. Explicitly, we will not require that

(WF2) if $(\alpha_n)_{n\in\mathbb{N}}$ is a sequence of positive real numbers then $\lim_{n\to\infty} \alpha_n = 0$ if and only if $\lim_{n\to\infty} F(\alpha_n) = -\infty$.

The reason for this is that Lemma 2.4 stated below will suffice in the proofs.

Lemma 2.4. *If $F\colon (0,\infty) \to \mathbb{R}$ is an increasing function and $(\alpha_n)_{n\in\mathbb{N}} \subset (0,+\infty)$ is a decreasing sequence such that $\lim_{n\to\infty} F(\alpha_n) = -\infty$ then $\lim_{n\to\infty} \alpha_n = 0$.*

Proof. Since $(\alpha_n)_{n\in\mathbb{N}}$ is decreasing and bounded below, it is also convergent. Let $\lim_{n\to\infty} \alpha_n = a \geq 0$ and suppose that $a > 0$. Since $\alpha_n \geq a$ and F is increasing, it follows that $F(a) \leq F(\alpha_n)$, for all $n \geq 0$. If we let $n \to \infty$ then we obtain $F(a) \leq \lim_{n\to\infty} F(\alpha_n) = -\infty$ which is a contradiction, hence $a = 0$.

The well-known examples of functions $F \in \mathcal{F}$ are $\ln x$, $\ln x + x$, $\ln(x^2+x)$ and $-\frac{1}{\sqrt{x}}$ (see [15]). The functions given in the following two examples will belong to $\mathcal{F}$ in our sense, but not in Wardowski' sense.

Example 2.5. Let $a > 0$ and $F\colon (0,\infty) \to \mathbb{R}$, $F(x) = x^a$. It is easy to see that F satisfies both (F1) and (F2). However, F does not satisfy Wardowski's (F2). Indeed, if $\alpha_n = \frac{1}{n}$, for every $n \in \mathbb{N}^*$ then $\lim_{n\to\infty} \alpha_n = 0$ and $\lim_{n\to\infty} F(\alpha_n) = 0 \neq -\infty$.

Example 2.6. Let $F\colon (0,\infty) \to \mathbb{R}$, $F(x) = \ln(x+1)$. Clearly F is strictly increasing, and since $\lim_{x\to 0^+} \sqrt{x}\ln(x+1) = 0$, (F2) is satisfied for $k = \frac{1}{2}$. On the other hand, if $\alpha_n = \frac{1}{n}$, for every $n \in \mathbb{N}^*$ then $\lim_{n\to\infty} \alpha_n = 0$ and $\lim_{n\to\infty} F(\alpha_n) = 0$.

When we pass from metric spaces to b-metric spaces, we will need an extra compatibility condition. The weakest possible such condition is the one that appears in the following definition:

Definition 2.7. Let $s \geq 1$ and $\tau > 0$. We say that $F \in \mathcal{F}$ belongs to $\mathcal{F}_{s,\tau}$ if it also satisfies

(F$s\tau$) if $\inf F = -\infty$ and $x, y, z \in (0,\infty)$ are such that $\tau + F(sx) \leq F(y)$ and $\tau + F(sy) \leq F(z)$ then
$$\tau + F(s^2 x) \leq F(sy).$$

We make two important remarks. First, if $\inf F \neq -\infty$ then (F$s\tau$) is satisfied, for all $s \geq 1$ and $\tau > 0$. Second, when $s = 1$ and $\tau > 0$ is arbitrary, condition (F$s\tau$) is a tautology, hence in this case the family $\mathcal{F}_{s,\tau}$ is $\mathcal{F}$.

In [7] the authors introduce the following condition ((F4) in Definition 3.1):

(F'$s\tau$) if $(\alpha_n)_{n\in\mathbb{N}} \subset (0,\infty)$ is a sequence such that $\tau + F(s\alpha_n) \leq F(\alpha_{n-1})$, for all $n \in \mathbb{N}$ and for some $\tau > 0$, then $\tau + F(s^n\alpha_n) \leq F(s^{n-1}\alpha_{n-1})$, for all $n \in \mathbb{N}^*$.

They use this condition to prove a fixed point theorem for multivalued F-contractions when F is continuous from the right (Theorem 3.4 in [7]). The equivalence of these two conditions is proven in the following proposition.

Proposition 2.8. *If F is increasing then (F$s\tau$) is equivalent to (F'$s\tau$).*
Proof. We distinguish two cases. If $\inf F \neq -\infty$ then (F$s\tau$) is trivial. On the other hand, (F'$s\tau$) also holds, since in this case there exists no sequence (α_n) of positive real numbers such that $\tau + F(s\alpha_n) \leq F(\alpha_{n-1})$, for all $n \in \mathbb{N}^*$. Indeed if α_n was such a sequence then

$$\tau + F(\alpha_n) \leq \tau + F(s\alpha_n) \leq F(\alpha_{n-1}), \quad \forall n \in \mathbb{N}^* \tag{2.1}$$

would hold. By induction, inequality (2.1) implies

$$F(\alpha_n) \leq F(\alpha_0) - n\tau, \quad \forall n \in \mathbb{N},$$

hence $\inf F = -\infty$, which is a contradiction.

In the case when $\inf F = -\infty$, first we prove that (F$s\tau$) implies (F'$s\tau$). Suppose that $(\alpha_n)_{n\in\mathbb{N}}$ is a sequence of positive numbers such that $\tau + F(s\alpha_n) \leq F(\alpha_{n-1})$, for all $n \in \mathbb{N}^*$. We proceed by induction to prove that

$$\tau + F(s^n\alpha_n) \leq F(s^{n-1}\alpha_{n-1}), \quad \forall n \in \mathbb{N}^*. \tag{2.2}$$

For $n = 1$ the statement is trivial. Suppose that (2.2) holds for a fixed $n \in \mathbb{N}^*$ and let us prove it for $n+1$. First we choose $x = \alpha_{n+1}$, $y = \alpha_n$ and $z = s^{n-1}\alpha_{n-1}$. We can use now (F$s\tau$), because

$$\tau + F(sy) = \tau + F(s\alpha_n) \leq \tau + F(s^n\alpha_n) \leq F(s^{n-1}\alpha_{n-1}) = F(z),$$

hence $\tau + F(s^2\alpha_{n+1}) \leq F(s\alpha_n)$. Next we choose $x = s\alpha_{n+1}$, $y = s\alpha_n$, $z = s^{n-1}\alpha_{n-1}$ and prove similarly that $\tau + F(s^3\alpha_{n+1}) \leq F(s^2\alpha_n)$. In $n-2$ more steps we get the desired inequality. It follows that (F$s\tau$) implies (F'$s\tau$).

Finally, we prove that if $\inf F = -\infty$ then (F'$s\tau$) implies (F$s\tau$). Let $x, y, z \in (0, \infty)$ be such that $\tau + F(sx) \leq F(y)$ and $\tau + F(sy) \leq F(z)$. We are going to construct a sequence $(\alpha_n)_{n\in\mathbb{N}}$ that satisfies the condition imposed in (F'$s\tau$). Let $\alpha_0 = z$, $\alpha_1 = y$ and $\alpha_2 = x$. To give the rest of the terms of (α_n), we observe that since $\inf F = -\infty$, we can pick an $\alpha_n > 0$ such that $\tau + F(s\alpha_n) \leq F(\alpha_{n-1})$, for every $n \geq 3$. We apply now (F'$s\tau$) for this sequence $(\alpha_n)_{n\in\mathbb{N}}$: when $n = 2$ we obtain $\tau + F(s^2x) \leq F(sy)$.

Remark 2.9. In practice it is easier to check (F$s\tau$) than (F'$s\tau$). Except of the function $\frac{-1}{\sqrt{x}}$, all the other examples of functions belonging to $\mathcal{F}$ given before also belong to $\mathcal{F}_{s,\tau}$, for any $s \geq 1$ and $\tau > 0$. We prove this for $F\colon (0,\infty) \to \mathbb{R}$, $F(x) = \ln x + x$. It is enough to prove that for any $x, y \in (0, \infty)$ such that $\tau + F(sx) \leq F(y)$, the inequality $\tau + F(s^2x) \leq F(sy)$ is also satisfied.

Thus we know that $\tau + \ln(sx) + sx \leq \ln y + y$. This inequality is equivalent to

$$\ln\frac{sx}{y} + sx - y \leq -\tau.$$

Since F is increasing, $sx \le y$ follows from the inequality imposed on x and y. Hence $s^2x - sy \le sx - y \le 0$ and

$$\ln \frac{s^2x}{sy} + s^2x - sy \le \ln \frac{sx}{y} + sx - y \le -\tau,$$

thus $\tau + F(s^2x) \le F(sy)$ holds as well.

3. A fixed point theorem for F-contractions in b-metric spaces

In this section we define F-contractions in b-metric spaces and we prove a fixed point theorem for them. We also investigate the well-posedness of the fixed point problem of F-contractions in b-metric spaces (for more details and further references on the well-posedness of fixed point problems see [13]).

Definition 3.1. Let (X, d) be a b-metric space with constant $s \ge 1$ and $T\colon X \to X$ an operator. If there exist $\tau > 0$ and $F \in \mathcal{F}_{s,\tau}$ such that for all $x, y \in X$ the inequality $d(Tx, Ty) > 0$ implies

(F) $\tau + F(s \cdot d(Tx, Ty)) \le F(d(x, y))$,

then T is called an *F-contraction.*

Theorem 3.2. *If (X, d) is a complete b-metric space with constant $s \ge 1$ and $T\colon X \to X$ is an F-contraction for some $F \in \mathcal{F}_{s,\tau}$ then T has a unique fixed point x^*. Furthermore, for any $x_0 \in X$ the sequence $x_{n+1} = Tx_n$ is convergent and $\lim_{n\to\infty} x_n = x^*$.*

Proof. First we prove that T has at most one fixed point. Suppose that x^* and y^* are two different fixed points of T, thus $Tx^* = x^* \ne y^* = Ty^*$. It follows that $d(Tx^*, Ty^*) = d(x^*, y^*) > 0$, hence we can apply (F) to get

$$\tau + F(s \cdot d(Tx^*, Ty^*)) \le F(d(x^*, y^*)) \le F(s \cdot d(x^*, y^*)) = F(s \cdot d(Tx^*, Ty^*)).$$

This inequality implies $\tau \le 0$, which is a contradiction, hence T can have at most one fixed point.

Next we prove the existence of a fixed point. Let $x_0 \in X$ be arbitrary. We construct the sequence $x_{n+1} = Tx_n$ and we denote by $\gamma_n = d(x_{n+1}, x_n)$ the consecutive distances. If there exists $n_0 \in \mathbb{N}$ such that $x_{n_0+1} = x_{n_0}$ then we have $Tx_{n_0} = x_{n_0}$. Thus $x^* = x_{n_0}$ is a fixed point of T and the proof is finished. In the case when $x_{n+1} \ne x_n$, $\forall n \in \mathbb{N}$, we have $\gamma_n > 0$, for every $n \in \mathbb{N}$. Hence (F) implies $F(s\gamma_{n+1}) \le F(\gamma_n) - \tau$, for every $n \in \mathbb{N}$. By Proposition 2.8, we get

$$F(s^{n+1}\gamma_{n+1}) \le F(s^n\gamma_n) - \tau, \quad \forall n \in \mathbb{N}, \tag{3.1}$$

and hence

$$F(s^n\gamma_n) \le F(s^{n-1}\gamma_{n-1}) - \tau \le F(s^{n-2}\gamma_{n-2}) - 2\tau \le \cdots \le F(\gamma_0) - n\tau, \quad \forall n \in \mathbb{N}.$$

It follows that

$$F(s^n\gamma_n) \le F(\gamma_0) - n\tau, \quad \forall n \in \mathbb{N}. \tag{3.2}$$

Since $\lim_{n\to\infty} F(\gamma_0) - n\tau = -\infty$, inequality (3.2) implies $\lim_{n\to\infty} F(s^n\gamma_n) = -\infty$. On the other hand, by inequality (3.1), the sequence $(s^n\gamma_n)_{n\in\mathbb{N}}$ is decreasing and we can apply Lemma 2.4 to get $\lim_{n\to\infty} s^n\gamma_n = 0$. According to (F2), there exists a $k \in (0,1)$ such that $\lim_{n\to\infty}(s^n\gamma_n)^k F(s^n\gamma_n) = 0$. Multiplying (3.2) by $(s^n\gamma_n)^k$ results

$$0 \le n(s^n\gamma_n)^k\tau + (s^n\gamma_n)^k F(s^n\gamma_n) \le (s^n\gamma_n)^k F(\gamma_0), \quad \forall n \in \mathbb{N}.$$

By the above, when $n \to \infty$, we obtain $\lim_{n\to\infty} n(s^n\gamma_n)^k = 0$. This inequality implies that there exists $n_1 \in \mathbb{N}$ such that $n(s^n\gamma_n)^k \le 1$, for all $n \ge n_1$. Thus

$$s^n\gamma_n \le \frac{1}{n^{\frac{1}{k}}}, \quad \forall n \ge n_1. \tag{3.3}$$

Next we prove that $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence. For all $n \in \mathbb{N}$ and $p \in \mathbb{N}^*$ the following chain of inequalities holds:

$$\begin{aligned}
d(x_{n+p}, x_n) &\le sd(x_{n+p}, x_{n+1}) + s\gamma_n \\
&\le s^2 d(x_{n+p}, x_{n+2}) + s^2\gamma_{n+1} + s\gamma_n \\
&\le s^3 d(x_{n+p}, x_{n+3}) + s^3\gamma_{n+2} + s^2\gamma_{n+1} + s\gamma_n \\
&\vdots \\
&\le s^{p-1}\gamma_{n+p-1} + s^{p-1}\gamma_{n+p-2} + \cdots + s^2\gamma_{n+1} + s\gamma_n \\
&\le s^p\gamma_{n+p-1} + s^{p-1}\gamma_{n+p-2} + \cdots + s^2\gamma_{n+1} + s\gamma_n \\
&= \frac{1}{s^{n-1}}\left(s^{n+p-1}\gamma_{n+p-1} + s^{n+p-2}\gamma_{n+p-2} + \cdots + s^{n+1}\gamma_{n+1} + s^n\gamma_n\right) \\
&= \frac{1}{s^{n-1}}\sum_{i=n}^{n+p-1} s^i\gamma_i \le \frac{1}{s^{n-1}}\sum_{i=n}^{\infty} s^i\gamma_i.
\end{aligned}$$

Hence, for all $n \ge n_1$ and $p \in \mathbb{N}^*$ inequality (3.3) implies

$$d(x_{n+p}, x_n) \le \frac{1}{s^{n-1}}\sum_{i=n}^{\infty} s^i\gamma_i \le \frac{1}{s^{n-1}}\sum_{i=n}^{\infty}\frac{1}{i^{\frac{1}{k}}} \to 0,$$

thus $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence. Since (X, d) is complete, there exists $x^* \in X$ such that $\lim_{n\to\infty} x_n = x^*$.

On the other hand, since $\tau + F(d(Tx, Ty)) \le \tau + F(s \cdot d(Tx, Ty)) \le F(d(x, y))$ holds for all such $x, y \in X$ for which $d(Tx, Ty) > 0$, and because F is increasing,

$$d(Tx, Ty) \le d(x, y), \quad \forall x, y \in X.$$

This implies

$$d(x_{n+1}, Tx^*) \le d(x_n, x^*), \quad \forall n \ge 0.$$

It follows by Lemma 2.2 that

$$0 \le s^{-1}d(x^*, Tx^*) \le \liminf_{n\to\infty} d(x_n, Tx^*) \le \limsup_{n\to\infty} d(x_n, Tx^*) \le \limsup_{n\to\infty} d(x_n, x^*) = 0,$$

hence $x^* = Tx^*$. We proved that the operator T has a unique fixed point and for every $x_0 \in X$ the sequence $x_{n+1} = Tx_n$ converges to this fixed point, thus the proof is finished.

The following example illustrates a situation when Banach's fixed point theorem for complete b-metric spaces cannot be applied, while the conditions of Theorem 3.2 are satisfied.

Example 3.3. Let $r_n = 2^{\frac{n}{2}} n$, for every $n \in \mathbb{N}$, $X = \{r_n \mid n \in \mathbb{N}\}$ and define the functional $d\colon X \times X \to [0,\infty)$, $d(x,y) = (x-y)^2$. It is easy to check that $(X, d, s = 2)$ is a complete b-metric space, but it is not a metric space. Define $T\colon X \to X$ by setting $T(r_0) = r_0$ and $T(r_n) = r_{n-1}$, for every $n \geq 1$. We are going to prove that there exists a $\tau > 0$ such that T is an F-contraction for $F\colon (0,\infty) \to \mathbb{R}$, $F(x) = x$, while T is not a contraction in the b-metric sense. Indeed, since

$$\lim_{n\to\infty} \frac{2d(Tr_n, Tr_0)}{d(r_n, r_0)} = \lim_{n\to\infty} \frac{2(r_{n-1} - r_0)^2}{(r_n - r_0)^2} = \lim_{n\to\infty} \frac{2\left(2^{\frac{n-1}{2}}(n-1)\right)^2}{\left(2^{\frac{n}{2}} n\right)^2} = 1,$$

thus Banach's fixed point theorem for b-metric spaces cannot be applied for T.

Next we prove that the conditions imposed on T in Theorem 3.2 are satisfied. Condition (F) translates to

(F) for every $x, y \in X$ such that $Tx \neq Ty$, we have

$$2d(Tx, Ty) - d(x, y) \leq -\tau.$$

We prove this in two steps. First, for every $n \geq 2$

$$\begin{aligned} 2d(Tr_n, Tr_0) - d(r_n, r_0) &= 2\left(2^{\frac{n-1}{2}}(n-1)\right)^2 - \left(2^{\frac{n}{2}} n\right)^2 \\ &= 2^n(1-2n) \\ &\leq -1. \end{aligned}$$

Second, for every $n, k \in \mathbb{N}^*$ we have

$$\begin{aligned} &2d(Tr_{n+k}, Tr_n) - d(r_{n+k}, r_n) = \\ &\quad = \left(2^{\frac{n+k}{2}}(n+k-1) - 2^{\frac{n}{2}}(n-1)\right)^2 - \left(2^{\frac{n+k}{2}}(n+k) - 2^{\frac{n}{2}} n\right)^2 \\ &\quad = \left(2^{\frac{n}{2}} - 2^{\frac{n+k}{2}}\right)\left(2^{\frac{n+k}{2}}(2n+2k-1) - 2^{\frac{n}{2}}(2n-1)\right) \\ &\quad = \left(1 - 2^{\frac{k}{2}}\right) \cdot 2^n \left(2^{\frac{k}{2}}(2n+2k-1) - (2n-1)\right) \\ &\quad \leq -1. \end{aligned}$$

By the above, $\tau = 1$ satisfies all the required properties and thus T is an F-contraction.

Theorem 3.4. *If (X, d) is a complete b-metric space with constant $s \geq 1$ and $T\colon X \to X$ is an F-contraction for some differentiable $F \in \mathcal{F}_{s,\tau}$ with $\lim_{x\to\infty} F'(x) < \infty$, then the fixed point problem for T is well-posed: i.e. for any sequence $(x_n)_{n\in\mathbb{N}}$ that satisfies*

$\lim_{n\to\infty} d(x_n, Tx_n) = 0$, *we have* $\lim_{n\to\infty} d(x_n, x^*) = 0$, *where* x^* *denotes the unique fixed point of* T.

Proof. In the most general case, the sequence $(x_n)_{n\in\mathbb{N}}$ can be decomposed into two subsequences $(x_n) = (y_{n_k})_{k\in\mathbb{N}} \cup (z_{n_k})_{k\in\mathbb{N}}$, where for any $k \in \mathbb{N}$ we have $Ty_{n_k} = x^*$ and $Tz_{n_k} \neq x^*$. Thus it is enough to prove the following two assertions:

(i) if $(x_n)_{n\in\mathbb{N}}$ is such that for all $n \in \mathbb{N}$ $Tx_n = x^*$, then $\lim_{n\to\infty} d(x_n, x^*) = 0$;

(ii) if $(x_n)_{n\in\mathbb{N}}$ is such that for all $n \in \mathbb{N}$ $Tx_n \neq x^*$, then $\lim_{n\to\infty} d(x_n, x^*) = 0$.

The first case is trivial, since $d(x_n, x^*) = d(x_n, Tx_n)$, and $\lim_{n\to\infty} d(x_n, Tx_n) = 0$.

We prove (ii). If $T(x_n) \neq x^*$ for any $n \in \mathbb{N}$ then we can apply (F) to get

$$F(d(x_n, x^*)) - F(sd(Tx_n, x^*)) \geq \tau.$$

Since $d(x_n, x^*) > sd(Tx_n, x^*)$, we obtain

$$\frac{F(d(x_n, x^*)) - F(sd(Tx_n, x^*))}{d(x_n, x^*) - sd(Tx_n, x^*)} \geq \frac{\tau}{d(x_n, x^*) - sd(Tx_n, x^*)}.$$

It follows that there exists a c_n between $d(x_n, x^*)$ and $sd(Tx_n, x^*)$ such that

$$F'(c_n) = \frac{F(d(x_n, x^*)) - F(sd(Tx_n, x^*))}{d(x_n, x^*) - sd(Tx_n, x^*)} \geq \frac{\tau}{d(x_n, x^*) - sd(Tx_n, x^*)}.$$

The last inequality implies

$$0 < \frac{\tau}{F'(c_n)} \leq d(x_n, x^*) - sd(Tx_n, x^*) \leq sd(x_n, Tx_n).$$

Since $\lim_{n\to\infty} d(x_n, Tx_n) = 0$, it follows that $\lim_{n\to\infty} F'(c_n) = \infty$. By the condition imposed on F', this can happen only if $\lim_{n\to\infty} c_n = 0$. Hence $\lim_{n\to\infty} sd(Tx_n, x^*) = 0$, which implies $\lim_{n\to\infty} d(x_n, x^*) = 0$.

4. A fixed point theorem for F-weak contractions in b-metric spaces

In order to combine Ćirić-type fixed point theorems with the notion of F-contractions, Wardowski and Dung introduced in [16] F-weak contractions and proved a fixed point theorem for them. This notion is a natural generalization of classical Ćirić-type contractions, in the direction of F-contractions. Our goal in this section is to extend the notion of F-weak contractions to b-metric spaces and prove a fixed point theorem for them.

Definition 4.1. Let (X, d) be a b-metric space with constant $s \geq 1$ and $T\colon X \to X$ an operator. If there exists $\tau > 0$ and $F \in \mathcal{F}_{s,\tau}$ such that for all $x, y \in X$ the inequality $d(Tx, Ty) > 0$ implies

$$(\mathrm{F}_w)\quad \tau + F(s \cdot d(Tx, Ty)) \leq F\left(\max\left\{d(x,y), d(x,Tx), d(y,Ty), \tfrac{d(x,Ty)+d(y,Tx)}{2s}\right\}\right),$$

then T is called an *F-weak contraction.*

Theorem 4.2. *Let* (X, d) *be a complete* b*-metric space with constant* $s \geq 1$ *and* $T\colon X \to X$ *an* F*-weak contraction for some* $F \in \mathcal{F}_{s,\tau}$*. Then* T *has at most one fixed point and for any* $x_0 \in X$ *the sequence* $x_{n+1} = Tx_n$ *is convergent in* X*. Furthermore,* *if either T or F is continuous then T has a unique fixed point x^* and for all $x_0 \in X$ the sequence $x_{n+1} = Tx_n$ converges to x^*.*

Proof. We prove first that T has at most one fixed point. Suppose that x^* and y^* are two different fixed points of T, thus $Tx^* = x^* \neq y^* = Ty^*$. It follows that $d(Tx^*, Ty^*) = d(x^*, y^*) > 0$, hence we can apply ($\mathrm{F}_w$) to get

$$\begin{aligned}
&\tau + F(s \cdot d(Tx^*, Ty^*)) \leq \\
&\leq F\left(\max\left\{d(x^*,y^*), d(x^*,Tx^*), d(y^*,Ty^*), \frac{d(x^*,Ty^*) + d(y^*,Tx^*)}{2s}\right\}\right) \\
&\leq F\left(\max\left\{d(x^*,y^*), d(x^*,x^*), d(y^*,y^*), \frac{d(x^*,y^*) + d(y^*,x^*)}{2s}\right\}\right) \\
&= F(d(x^*,y^*)) \leq F(s \cdot d(x^*,y^*)) \\
&= F(s \cdot d(Tx^*, Ty^*)).
\end{aligned}$$

The last inequality implies $\tau \leq 0$, which is a contradiction. Hence T has at most one fixed point.

Next we prove the existence of a fixed point. Let $x_0 \in X$. Define the sequence $x_{n+1} = Tx_n$ and denote by $\gamma_n = d(x_{n+1}, x_n)$ the consecutive distances. If there exists $n_0 \in \mathbb{N}$ such that $x_{n_0+1} = x_{n_0}$ then $Tx_{n_0} = x_{n_0}$, and thus $x^* = x_{n_0}$ is a fixed point of T, finishing the proof. We can suppose now that $x_{n+1} \neq x_n$, for all $n \in \mathbb{N}$, hence we can apply (F_w) for any $n \in \mathbb{N}$ to get

$$\begin{aligned}
F(s\gamma_n) &\leq F\left(\max\left\{\gamma_n, \gamma_n, \gamma_{n-1}, \frac{d(x_{n-1}, x_{n+1})}{2s}\right\}\right) - \tau \\
&\leq F\left(\max\left\{\gamma_n, \gamma_{n-1}, \frac{s\gamma_n + s\gamma_{n-1}}{2s}\right\}\right) - \tau \\
&= F(\max\{\gamma_n, \gamma_{n-1}\}) - \tau.
\end{aligned}$$

If there exists $n \in \mathbb{N}$ such that $\max\{\gamma_n, \gamma_{n-1}\} = \gamma_n$ then

$$F(s\gamma_n) \leq F(\gamma_n) - \tau < F(\gamma_n) \leq F(s\gamma_n),$$

which is a contradiction. We conclude that $\max\{\gamma_n, \gamma_{n-1}\} = \gamma_{n-1}$, for all $n \in \mathbb{N}$, and thus we have

$$F(s\gamma_n) \leq F(\gamma_{n-1}) - \tau, \quad \forall n \in \mathbb{N}.$$

It follows that we can use the argument presented in the proof of Theorem 3.2 here as well to obtain first

$$\lim_{n\to\infty} F(s^n\gamma_n) \leq F(\gamma_0) - n\tau \leq \lim_{n\to\infty} F(\gamma_0) - n\tau = -\infty,$$

then $\lim_{n\to\infty} s^n\gamma_n = 0$ and finally, by condition (F2), that $(x_n)_{n\in\mathbb{N}}$ is a Cauchy-sequence. Since (X, d) is complete, there exists $x^* \in X$ such that $\lim_{n\to\infty} x_n = x^*$.

It remains to prove that x^* is a fixed point of T if either T or F is continuous. First, if T is continuous then $T(x^*) = \lim_{n\to\infty} T(x_n) = \lim_{n\to\infty} x_{n+1} = x^*$, hence x^* is indeed a fixed point of T. Second, if F is continuous we distinguish two cases. If there exists a subsequence $(x_{n_i})_{i\in\mathbb{N}} \subset (x_n)_{n\in\mathbb{N}}$ such that $x_{n_i} = Tx^*$, for all $i \in \mathbb{N}$ then $x^* = \lim_{i\to\infty} x_{n_i} = \lim_{i\to\infty} Tx^* = Tx^*$. If there is no such subsequence of $(x_n)_{n\in\mathbb{N}}$

then there exists an $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$ we have $d(Tx_n, Tx^*) > 0$. We can apply (F_w) to get the following inequality, for every $n \geq n_0$:

$$\tau + F\left(s \cdot d\left(x_{n+1}, Tx^*\right)\right) \leq \leq F\left(\max\left\{d(x_n, x^*), \gamma_n, d(x^*, Tx^*), \frac{d(x_n, Tx^*) + d(x^*, x_{n+1})}{2s}\right\}\right). \tag{4.1}$$

Aiming for a contradiction, we suppose that $a = d(x^*, Tx^*) > 0$ and we denote $\beta_n = d(x_n, x^*)$. Since $\lim_{n\to\infty} x_n = x^*$, there exists an $n_1 \in \mathbb{N}$ such that for every $n \geq n_1$ both $\beta_n < \frac{a}{2}$ and $\gamma_n < \frac{a}{2}$ hold. As a consequence, for every $n \geq n_1$

$$\max\left\{\beta_n, \gamma_n, a, \frac{d(x_n, Tx^*) + \beta_{n+1}}{2s}\right\} \leq \max\left\{\beta_n, \gamma_n, a, \frac{s\beta_n + sa + \beta_{n+1}}{2s}\right\} \leq \max\left\{\beta_n, \gamma_n, a, \frac{s\frac{a}{2} + sa + \frac{a}{2}}{2s}\right\} = a. \tag{4.2}$$

Since F is increasing, inequalities (4.1) and (4.2) for every $n \geq \max\{n_0, n_1\}$ imply

$$\tau + F\left(s \cdot d\left(x_{n+1}, Tx^*\right)\right) \leq F(d(x^*, Tx^*)). \tag{4.3}$$

On the other hand, Lemma 2.2 and the continuity and monotonicity of F give

$$\tau + F(d(x^*, Tx^*)) \leq \tau + F(s \cdot \liminf_{n\to\infty} d(x_n, Tx^*)) \leq \tau + \liminf_{n\to\infty} F(s \cdot d(x_n, Tx^*)). \tag{4.4}$$

If we pass to lim inf in (4.3) and then apply (4.4), we get

$$\tau + F(d(x^*, Tx^*)) \leq F(d(x^*, Tx^*)),$$

which contradicts $\tau > 0$. Hence we proved that $0 = a = d(x^*, Tx^*)$, and thus x^* is a fixed point of T.

5. F-contractions of Hardy-Rogers type

In this section we prove a Hardy-Rogers type fixed point theorem for F-weak contractions in b-metric spaces, which generalizes Theorem 3.1 in [8] to the b-metric case.

Definition 5.1. Let (X, d) be a b-metric space with constant $s \geq 1$, $a, b, c, e, f \geq 0$ real numbers and $T\colon X \to X$ an operator. If there exist $\tau > 0$ and $F \in \mathcal{F}_s$ such that for all $x, y \in X$ the inequality $d\,(Tx, Ty) > 0$ implies

(F_{HR}) $\tau + F\left(s \cdot d\left(Tx, Ty\right)\right) \leq$
$\leq F\left(ad(x, y) + bd(x, Tx) + cd(y, Ty) + ed(x, Ty) + fd(y, Tx)\right)$

then T is called an *F-weak contraction of Hardy-Rogers type.*

Theorem 5.2. *Suppose that (X, d) is a complete b-metric space with constant $s \geq 1$ and $T\colon X \to X$ is an F-weak contraction of Hardy-Rogers type. If either $a + b + c + (s + 1)e < 1$ or $a + b + c + (s + 1)f < 1$ holds then for every $x_0 \in X$ the sequence $x_{n+1} = Tx_n$ converges to a fixed point of T. Moreover, if $a + e + f < s$ holds as well then T has exactly one fixed point.*

Proof. Let $x_0 \in X$ and define $x_{n+1} = Tx_n$, for every $n \in \mathbb{N}$. As before, denote $\gamma_n = d(x_{n+1}, x_n)$. If there exists $n_0 \in \mathbb{N}$ such that $x_{n_0+1} = x_{n_0}$ then x_{n_0} is a fixed point of T and $x_n = x_{n_0}$, for every $n \geq n_0$. On the other hand, when $x_{n+1} \neq x_n$, for all $n \in \mathbb{N}$, we can apply (F$_{HR}$) for $x = x_n$ and $y = x_{n+1}$, the properties of F and the relaxed triangle inequality to obtain the following chain of inequalities:

$$\begin{aligned} \tau + F(s\gamma_{n+1}) &\leq F\left(a\gamma_n + b\gamma_n + c\gamma_{n+1} + e \cdot d(x_n, x_{n+2})\right) \\ &\leq F\left(a\gamma_n + b\gamma_n + c\gamma_{n+1} + se\gamma_n + se\gamma_{n+1}\right) \\ &= F\left((a+b+se)\gamma_n + (c+se)\gamma_{n+1}\right). \end{aligned} \tag{5.1}$$

Since F is strictly increasing, it follows that

$$s\gamma_{n+1} < (a+b+se)\gamma_n + (c+se)\gamma_{n+1},$$

and thus for every $n \geq n_0$ we have

$$\left(1 - \frac{c}{s} - e\right) s\gamma_{n+1} < (a+b+se)\gamma_n. \tag{5.2}$$

In the case when $a+b+c+(s+1)e < 1$ holds, we obtain

$$1 - \frac{c}{s} - e \geq 1 - c - e > a + b + se \geq 0$$

and hence inequality (5.2) implies $s\gamma_{n+1} < \gamma_n$, for every $n \in \mathbb{N}$. We can use now inequality (5.1) and that F is strictly increasing to obtain for every $n \in \mathbb{N}$

$$\begin{aligned} \tau + F(s\gamma_{n+1}) &\leq F((a+b+se)\gamma_n + (c+se)\gamma_{n+1}) \\ &< F(a+b+se)\gamma_n + (c+e)\gamma_n) \\ &< F(\gamma_n). \end{aligned}$$

In the other case, when $a+b+c+(s+1)f < 1$, we obtain the same inequality analogously, if we start with $x = x_{n+1}$ and $y = x_n$ in condition (F$_{HR}$).

It follows that $\tau + F(s\gamma_{n+1}) < F(\gamma_n)$ for every $n \in \mathbb{N}$ in either case. We can now use the technique presented in the proof of Theorem 3.2 to prove that $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence. Thus, by completeness of X, there exists $x^* \in X$ such that $\lim_{n\to\infty} x_n = x^*$.

Let us prove that x^* is a fixed point of T. Assume that $Tx^* \neq x^*$.

If $Tx_n = Tx^*$ for infinitely many values of n then there exists a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$ that takes the constant values x^* for all $k \in \mathbb{N}$, hence it converges to Tx^*. In this case, the uniqueness of the limit of the sequence $(x_n)_{n\in\mathbb{N}}$ implies $Tx^* = x^*$.

In the other case, when there are only finitely many values of $n \in \mathbb{N}$ for which $Tx_n = x^*$, there exists an $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ we have $Tx_n \neq Tx^*$. In this case we can write the following chain of inequalities, for every $n \geq n_0$:

$$\begin{aligned} d(x^*, Tx^*) \leq& s[d(x^*, x_{n+1}) + d(Tx_n, Tx^*)] \\ \leq& s \cdot d(x^*, x_{n+1}) + a \cdot d(x_n, x^*) + b \cdot d(x_n, Tx_n) \\ &+ c \cdot d(x^*, Tx^*) + e \cdot d(x_n, Tx^*) + f \cdot d(x^*, x_{n+1}), \end{aligned}$$

where we used the relaxed triangle inequality, condition (F_{HR}) with $x = x_n$, $y = x^*$ and the monotonicity of F.

If we pass to lim sup in the inequality above and use Lemma 2.2, we obtain

$$(1-c)d(x^*, Tx^*) \le se \cdot d(x^*, Tx^*). \tag{5.3}$$

Similarly, if we choose $x = x^*$ and $y = x_n$ in (F_{HR}), we obtain

$$(1-b)d(x^*, Tx^*) \le sf \cdot d(x^*, Tx^*). \tag{5.4}$$

Depending on whether $a+b+c+(1+s)e < 1$ or $a+b+c+(1+s)f < 1$ holds, either inequality (5.3) or inequality (5.4) is a contradiction, hence $Tx^* = x^*$.

In the last step we prove that T cannot have more than one fixed point when inequality $a+e+f < s$ holds as well. Let us assume that x^* and y^* are two different fixed points of T. Since $Tx^* = x^* \ne y^* = Ty^*$, we have

$$\begin{aligned} s \cdot d(x^*, y^*) &= s \cdot d(Tx^*, Ty^*) \\ &< ad(x^*, y^*) + bd(x^*, Tx^*) + cd(y^*, Ty^*) + ed(x^*, Ty^*) + fd(y^*, Tx^*) \\ &= (a+e+f)d(x^*, y^*) < s \cdot d(x^*, y^*), \end{aligned}$$

which is a contradiction, and thus T cannot have more than one fixed points.

Remark 5.3. Theorem 5.2 generalizes also a Hardy-Rogers type fixed point theorem that appeared in [16] as Corollary 2.5 to the b-metric case. We also can omit the continuity assumptions on T or F.

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