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GENERAL COMPOSITE ITERATIVE METHODS FOR GENERAL SYSTEMS OF VARIATIONAL INEQUALITIES

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Abstract. In this paper, we introduce a general composite implicit scheme and a general composite explicit scheme for finding a solution of general system of variational inequalities in a real Hilbert space. We establish the strong convergence of these two general composite schemes to a solution of the general system of variational inequalities which is the unique solution of some variational inequality. Applications to variational inequalities are given.
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1. Introduction

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and induced norm $\| \cdot \|$, C be a nonempty closed convex subset of H.

Let $A : C \to H$ be a nonlinear mapping on C. The classical variational inequality problem (VIP) [15] is to find $x^* \in C$ such that

$$\langle Ax^*, x - x^* \rangle \geq 0, \quad \forall x \in C. \tag{1.1}$$

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The solution set of VIP (1.1) is denoted by VI(C, F). The VIP (1.1) has been extensively studied both in theory and algorithms. See, e.g., [21], [16], [19], [3], [4], [13] and the references therein.

Let $F_1, F_2 : C \to H$ be two mappings. We consider the following general system of variational inequalities (GSVI) of finding $(x^*, y^*) \in C \times C$ such that

$$\begin{cases} \langle \nu_1 F_1 y^* + x^* - y^*, x - x^* \rangle \geq 0, \quad \forall x \in C, \\ \langle \nu_2 F_2 x^* + y^* - x^*, x - y^* \rangle \geq 0, \quad \forall x \in C, \end{cases} \tag{1.2}$$

where $\nu_1 > 0$ and $\nu_2 > 0$ are two constants. The solution set of GSVI (1.2) is denoted by GSVI(C, F_1, F_2). Recently, many authors have been devoting the study of the GSVI (1.2); see e.g., [11], [24], [9], [7], [8], [2], [5], [10], [20], [14], [1], [6] and the references therein.

In this paper, we introduce a general composite implicit scheme and a general composite explicit scheme for finding a solution of GSVI (1.2) in a real Hilbert space H. Further, we establish the strong convergence of these two general composite schemes to a solution of GSVI (1.2), which is also the unique solution of some variational inequality.

2. Preliminaries

We need the following notions and facts.

A mapping $F : C \to H$ is said to be

(i) L-Lipschitz if there exists a constant $L \geq 0$ such that

$$\|Fx - Fy\| \leq L\|x - y\|, \quad \forall x, y \in C.$$

(ii) monotone if

$$\langle Fx - Fy, x - y \rangle \geq 0, \quad \forall x, y \in C;$$

(iii) η-strongly monotone if there exists a constant $\eta > 0$ such that

$$\langle Fx - Fy, x - y \rangle \geq \eta\|x - y\|^2, \quad \forall x, y \in C;$$

(iv) α-inverse-strongly monotone if there exists a constant $\alpha > 0$ such that

$$\langle Fx - Fy, x - y \rangle \geq \alpha\|Fx - Fy\|^2, \quad \forall x, y \in C.$$

It can be easily seen that if T is nonexpansive, then $I - T$ is monotone. It is also easy to see that the projection P_C is 1-ism.

On the other hand, it is obvious that if $F : C \to H$ is α-inverse-strongly monotone, then F is monotone and $\frac{1}{\alpha}$-Lipschitz continuous. Moreover, we also have that, for all $u, v \in C$ and $\lambda > 0$,

$$\|(I - \lambda F)u - (I - \lambda F)v\|^2 \leq \|u - v\|^2 + \lambda(\lambda - 2\alpha)\|Fu - Fv\|^2. \tag{2.1}$$

So, if $\lambda \leq 2\alpha$, then $I - \lambda F$ is a nonexpansive mapping from C to H.

A mapping $T : C \to C$ is called k-strictly pseudocontractive (or a k-strict pseudocontraction) if there exists a constant $k \in [0, 1)$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|(I - T)x - (I - T)y\|^2, \quad \forall x, y \in C.$$

The mapping T is pseudocontractive if and only if

$$\langle Tx - Ty, x - y \rangle \leq \|x - y\|^2, \quad \forall x, y \in C.$$

T is strongly pseudocontractive if and only if there exists a constant $\lambda \in (0,1)$ such that

$$\langle Tx - Ty, x - y\rangle \le \lambda \|x - y\|^2, \quad \forall x, y \in C.$$

The mapping T is also said to be pseudocontractive if $k = 1$ and T is said to be strongly pseudocontractive if there exists a positive constant $\lambda \in (0,1)$ such that $T + (1-\lambda)I$ is pseudocontractive.

For any sequence $\{x_n\}$, we use $x_n \rightharpoonup x$ for weak convergence and $x_n \to x$ for strong converges. Moreover, we use $\omega_w(x_n)$ to denote the weak ω-limit set of the sequence $\{x_n\}$, i.e.,

$$\omega_w(x_n) := \{x \in H : x_{n_i} \rightharpoonup x \text{ for some subsequence } \{x_{n_i}\} \text{ of } \{x_n\}\}.$$

We denote by $\mathrm{Fix}(T)$ the set of fixed points of T and by $\mathbf{R}$ the set of all real numbers.The metric (or nearest point) projection from H onto C is the mapping $P_C : H \to C$ which assigns to each point $x \in H$ the unique point $P_C x \in C$ satisfying the property

$$\|x - P_C x\| = \inf_{y \in C} \|x - y\| =: d(x, C).$$

The following properties of projections are useful for our results.

Proposition 2.1. *Given any $x \in H$ and $z \in C$. One has*

(i) $z = P_C x \iff \langle x - z, y - z\rangle \le 0,\ \forall y \in C$;

(ii) $z = P_C x \iff \|x - z\|^2 \le \|x - y\|^2 - \|y - z\|^2,\ \forall y \in C$;

(iii) $\langle P_C x - P_C y, x - y\rangle \ge \|P_C x - P_C y\|^2,\ \forall y \in H$, which hence implies that P_C is nonexpansive and monotone.

A mapping $T : H \to H$ is said to be firmly nonexpansive if $2T - I$ is nonexpansive, or equivalently, if T is 1-inverse strongly monotone (1-ism),

$$\langle x - y, Tx - Ty\rangle \ge \|Tx - Ty\|^2, \quad \forall x, y \in H.$$

Proposition 2.2. (see [9]) *For given $\bar{x}, \bar{y} \in C$, $(\bar{x}, \bar{y})$ is a solution of the GSVI (1.2) if and only if $\bar{x}$ is a fixed point of the mapping $G : C \to C$ defined by*

$$Gx = P_C(I - \nu_1 F_1) P_C(I - \nu_2 F_2)x, \quad \forall x \in C,$$

where $\bar{y} = P_C(I - \nu_2 F_2)\bar{x}$.

In particular, if the mapping $F_j : C \to H$ is ζ_j-inverse-strongly monotone for $j = 1, 2$, then the mapping G is nonexpansive provided $\nu_j \in (0, 2\zeta_j]$ for $j = 1, 2$. We denote by Ξ the fixed point set of the mapping G.

We need some facts and tools in a real Hilbert space H which are listed as lemmas below.

Lemma 2.1. *Let X be a real inner product space. Then there holds the following inequality*

$$\|x + y\|^2 \le \|x\|^2 + 2\langle y, x + y\rangle, \quad \forall x, y \in X.$$

Lemma 2.2. *Let H be a real Hilbert space. Then the following hold:*

(a) $\|x - y\|^2 = \|x\|^2 - \|y\|^2 - 2\langle x - y, y\rangle$ for all $x, y \in H$;

(b) $\|\lambda x + \mu y\|^2 = \lambda\|x\|^2 + \mu\|y\|^2 - \lambda\mu\|x - y\|^2$ for all $x, y \in H$ and $\lambda, \mu \in [0,1]$ with $\lambda + \mu = 1$;

(c) If $\{x_n\}$ is a sequence in H such that $x_n \rightharpoonup x$, it follows that

$$\limsup_{n\to\infty} \|x_n - y\|^2 = \limsup_{n\to\infty} \|x_n - x\|^2 + \|x - y\|^2, \quad \forall y \in H.$$

It is clear that, in a real Hilbert space H, $T : C \to C$ is k-strictly pseudocontractive if and only if the following inequality holds:

$$\langle Tx - Ty, x - y\rangle \le \|x - y\|^2 - \frac{1-k}{2}\|(I - T)x - (I - T)y\|^2, \quad \forall x, y \in C.$$

This immediately implies that if T is a k-strictly pseudocontractive mapping, then $I - T$ is $\frac{1-k}{2}$-inverse strongly monotone.

Lemma 2.3. (see [17, Proposition 2.1]) *Let C be a nonempty closed convex subset of a real Hilbert space H and $T : C \to C$ be a mapping.*

(i) If T is a k-strictly pseudocontractive mapping, then T satisfies the Lipschitzian condition

$$\|Tx - Ty\| \le \frac{1+k}{1-k}\|x - y\|, \quad \forall x, y \in C.$$

(ii) If T is a k-strictly pseudocontractive mapping, then the mapping $I - T$ is semiclosed at 0, that is, if $\{x_n\}$ is a sequence in C such that $x_n \rightharpoonup \tilde{x}$ and $(I-T)x_n \to 0$, then $(I - T)\tilde{x} = 0$.

(iii) If T is k-(quasi-)strict pseudocontraction, then the fixed-point set $\mathrm{Fix}(T)$ of T is closed and convex so that the projection $P_{\mathrm{Fix}(T)}$ is well defined.

Lemma 2.4. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $T : C \to C$ be a k-strictly pseudocontractive mapping. Let γ and δ be two nonnegative real numbers such that $(\gamma + \delta)k \le \gamma$. Then*

$$\|\gamma(x - y) + \delta(Tx - Ty)\| \le (\gamma + \delta)\|x - y\|, \quad \forall x, y \in C.$$

Let C be a nonempty closed convex subset of a real Hilbert space H. We introduce some notations. Let λ be a number in $(0, 1]$ and let $\mu > 0$. Associating with a nonexpansive mapping $T : C \to C$, we define the mapping $T^\lambda : C \to H$ by

$$T^\lambda x := Tx - \lambda\mu F(Tx), \quad \forall x \in C,$$

where $F : C \to H$ is an operator such that, for some positive constants $\kappa, \eta > 0$, F is κ-Lipschitzian and η-strongly monotone on C; that is, F satisfies the conditions:

$$\|Fx - Fy\| \le \kappa\|x - y\| \quad \text{and} \quad \langle Fx - Fy, x - y\rangle \ge \eta\|x - y\|^2$$

for all $x, y \in C$.

3. Main results

Let C be a nonempty closed convex subset of a real Hilbert space H. Throughout this section, we always assume the following:

$F : C \to H$ is a κ-Lipschitzian and η-strongly monotone operator with positive constants $\kappa, \eta > 0$, and $F_j : C \to H$ is ζ_j-inverse strongly monotone for $j = 1, 2$;

A is a $\bar{\gamma}$-strongly positive bounded linear operator on H with $\bar{\gamma} \in (1, 2)$, i.e., there exists a constant $\bar{\gamma} > 0$ such that

$$\langle Ax, x\rangle \ge \bar{\gamma}\|x\|^2, \quad \forall x \in H;$$

$V : C \to H$ is an l-Lipschitzian mapping with constant $l \ge 0$;

$0 < \mu < \frac{2\eta}{\kappa^2}$ and $0 \leq \gamma l < \tau$ with $\tau = 1 - \sqrt{1 - \mu(2\eta - \mu\kappa^2)}$;

$G_t := P_C(I - \nu_1(t)F_1)P_C(I - \nu_2(t)F_2)$, $t \in (0, \min\{1, \frac{2-\bar{\gamma}}{\tau-\gamma l}\})$ and

$G := P_C(I - \nu_1 F_1)P_C(I - \nu_2 F_2)$ with $0 < \nu_j \leq \nu_j(t) \leq 2\zeta_j$ and $\lim_{t\to 0} \nu_j(t) = \nu_j$ for $j = 1, 2$;

$G_n := P_C(I - \nu_{1,n}F_1)P_C(I - \nu_{2,n}F_2)$ with $0 < \nu_j \leq \nu_{j,n} \leq 2\zeta_j$ and $\lim_{n\to\infty} \nu_{j,n} = \nu_j$ for $j = 1, 2$;

$\Xi \neq \emptyset$ and P_Ξ is the metric projection of H onto Ξ;

$\{\alpha_n\} \subset [0, 1]$ and $\{\beta_n\} \subset (0, 1]$.

By Proposition 2.2, we know that G_t and G_n are nonexpansive and $\mathrm{Fix}(G) = \mathrm{Fix}(G_t) = \mathrm{Fix}(G_n)$.

In this section, we introduce the first general composite scheme that generates a net $\{x_t\}_{t\in(0,\min\{1,\frac{2-\bar{\gamma}}{\tau-\gamma l}\})}$ implicitly as follows:

$$x_t = P_C[(I - \theta_t A)G_t x_t + \theta_t(t\gamma V x_t + (I - t\mu F)G_t x_t)]. \tag{3.1}$$

We prove the strong convergence of $\{x_t\}$ as $t \to 0$ to a fixed point $\tilde{x}$ of G (i.e., $\tilde{x} \in \Xi$), which is a unique solution to the VIP

$$\langle (A - I)\tilde{x}, p - \tilde{x}\rangle \geq 0, \quad \forall p \in \Xi. \tag{3.2}$$

For arbitrarily given $x_0 \in C$, we also propose the second general composite explicit scheme, which generates a sequence $\{x_n\}$ in an explicit way:

$$\begin{cases} y_n = \alpha_n \gamma V x_n + (I - \alpha_n \mu F)G_n x_n, \\ x_{n+1} = P_C[(I - \beta_n A)G_n x_n + \beta_n y_n], \quad \forall n \geq 0, \end{cases} \tag{3.3}$$

and establish the strong convergence of $\{x_n\}$ as $n \to \infty$ to a fixed point $\tilde{x}$ of G (i.e., $\tilde{x} \in \Xi$), which is also the unique solution to VIP (3.2).

Now, for $t \in (0, \min\{1, \frac{2-\bar{\gamma}}{\tau-\gamma l}\})$, and $\theta_t \in (0, \|A\|^{-1}]$, consider a mapping $Q_t : C \to C$ defined by

$$Q_t x = P_C[(I - \theta_t A)G_t x + \theta_t(t\gamma V x + (I - t\mu F)G_t x)], \quad \forall x \in C.$$

It is easy to see that Q_t is a contractive mapping with constant $1 - \theta_t(\bar{\gamma} - 1 + t(\tau - \gamma l))$. By the Banach contraction principle, Q_t has a unique fixed point, denoted by x_t, which uniquely solves the fixed point equation (3.1).

We summary the basic properties of $\{x_t\}$ as follows.

Proposition 3.1. *Let $\{x_t\}$ be defined via (3.1). Then*

(i) $\{x_t\}$ is bounded for $t \in (0, \min\{1, \frac{2-\bar{\gamma}}{\tau-\gamma l}\})$;

(ii) $\lim_{t\to 0} \|x_t - G_t x_t\| = 0$ provided $\lim_{t\to 0} \theta_t = 0$;

(iii) $x_t : (0, \min\{1, \frac{2-\bar{\gamma}}{\tau-\gamma l}\}) \to H$ is locally Lipschitzian provided

$$\theta_t : (0, \min\{1, \frac{2-\bar{\gamma}}{\tau-\gamma l}\}) \to (0, \|A\|^{-1}]$$

is locally Lipschitzian, and $\nu_j(t) : (0, \min\{1, \frac{2-\bar{\gamma}}{\tau-\gamma l}\}) \to [\nu_j, 2\zeta_j]$ is locally Lipschitzian for $j = 1, 2$;

(iv) x_t defines a continuous path from $(0, \min\{1, \frac{2-\bar{\gamma}}{\tau-\gamma l}\})$ into H provided

$$\theta_t : (0, \min\{1, \frac{2-\bar{\gamma}}{\tau-\gamma l}\}) \to (0, \|A\|^{-1}]$$

is continuous, and $\nu_j(t) : (0, \min\{1, \frac{2-\bar{\gamma}}{\tau-\gamma l}\}) \to [\nu_j, 2\zeta_j]$ *is continuous for* $j = 1, 2$.

We are now in a position to prove the following theorem for strong convergence of the net $\{x_t\}$ as $t \to 0$, which guarantees the existence of solutions of the variational inequality (3.2).

Theorem 3.1. *Let the net* $\{x_t\}$ *be defined via (3.1). If* $\lim_{t\to 0} \theta_t = 0$*, then* x_t *converges strongly to a fixed point* $\tilde{x}$ *of* G *as* $t \to 0$*, which solves the VIP (3.2). Equivalently, we have* $P_{\Xi}(2I - A)\tilde{x} = \tilde{x}$.

Proof. We first note that we have the uniqueness of solutions of the VIP (3.2) which is a consequence of the strong monotonicity of $A - I$ Next, we prove that $x_t \to \tilde{x}$ as $t \to 0$. Observing $\mathrm{Fix}(G) = \mathrm{Fix}(G_t)$ by Proposition 2.2, from (3.1), we write, for given $p \in \Xi$,

$$x_t - p = x_t - w_t + w_t - p = x_t - w_t + (I - \theta_t A)(G_t x_t - G_t p)$$
$$+\theta_t[t(\gamma V x_t - \mu F p) + (I - t\mu F)G_t x_t - (I - t\mu F)p] + \theta_t(I - A)p,$$

where $w_t = (I - \theta_t A)G_t x_t + \theta_t(t\gamma V x_t + (I - t\mu F)G_t x_t)$. Then, by Proposition 2.1 (i), we have

$$\begin{aligned}\|x_t - p\|^2 &= \langle x_t - w_t, x_t - p\rangle + \langle (I - \theta_t A)(G_t x_t - G_t p, x_t - p\rangle + \theta_t[t\langle \gamma V x_t - \mu F p, x_t - p\rangle \\ &\quad + \langle (I - t\mu F)G_t x_t - (I - t\mu F)p, x_t - p\rangle] + \theta_t\langle (I - A)p, x_t - p\rangle \\ &\le (1 - \theta_t\bar{\gamma})\|x_t - p\|^2 + \theta_t[(1 - t\tau)\|x_t - p\|^2 + t\gamma l\|x_t - p\|^2 \\ &\quad + t\langle (\gamma V - \mu F)p, x_t - p\rangle] + \theta_t\langle (I - A)p, x_t - p\rangle \\ &= [1 - \theta_t(\bar{\gamma} - 1 + t(\tau - \gamma l))]\|x_t - p\|^2 + \theta_t(t\langle (\gamma V - \mu F)p, x_t - p\rangle + \langle (I - A)p, x_t - p\rangle).\end{aligned}$$

Therefore,

$$\|x_t - p\|^2 \le \frac{1}{\bar{\gamma} - 1 + t(\tau - \gamma l)}(t\langle (\gamma V - \mu F)p, x_t - p\rangle + \langle (I - A)p, x_t - p\rangle). \tag{3.4}$$

Since the net $\{x_t\}_{t\in(0,\min\{1,\frac{2-\bar{\gamma}}{\tau-\gamma l}\})}$ is bounded (due to Proposition 3.1 (i)), we know that if $\{t_n\}$ is a subsequence in $(0, \min\{1, \frac{2-\bar{\gamma}}{\tau-\gamma l}\})$ such that $t_n \to 0$ and $x_{t_n} \rightharpoonup x^*$, then from (3.4), we obtain $x_{t_n} \to x^*$. Let us show that $x^* \in \Xi$. To this end, note that $G : C \to C$ defined by $G := P_C(I - \nu_1 F_1)P_C(I - \nu_2 F_2)$ for $0 < \nu_j \le 2\zeta_j$ for $j = 1, 2$. Then G is nonexpansive with $\mathrm{Fix}(G) = \Xi$ (due to Proposition 2.2). By the definition of x_t and the nonexpansivity of $P_C(I - \nu_j F_j), j = 1, 2$ we get

$$\begin{aligned}&\|Gx_{t_n} - x_{t_n}\| \\ &\le \|Gx_{t_n} - G_{t_n}x_{t_n}\| + \|G_{t_n}x_{t_n} - x_{t_n}\| \\ &\le \|P_C(I - \nu_1 F_1)P_C(I - \nu_2 F_2)x_{t_n} - P_C(I - \nu_1(t_n)F_1)P_C(I - \nu_2(t_n)F_2)x_{t_n}\| \\ &\quad + \|G_{t_n}x_{t_n} - (I - \theta_{t_n}A)G_{t_n}x_{t_n} - \theta_{t_n}(t_n\gamma V x_{t_n} + (I - t_n\mu F)G_{t_n}x_{t_n})\| \\ &\le \|P_C(I - \nu_1 F_1)P_C(I - \nu_2 F_2)x_{t_n} - P_C(I - \nu_1 F_1)P_C(I - \nu_2(t_n)F_2)x_{t_n}\| \\ &\quad + \|P_C(I - \nu_1 F_1)P_C(I - \nu_2(t_n)F_2)x_{t_n} - P_C(I - \nu_1(t_n)F_1)P_C(I - \nu_2(t_n)F_2)x_{t_n}\| \\ &\quad + \theta_{t_n}\|(I - A)G_{t_n}x_{t_n} + t_n(\gamma V x_{t_n} - \mu F G_{t_n}x_{t_n})\| \\ &\le \|(I - \nu_2 F_2)x_{t_n} - (I - \nu_2(t_n)F_2)x_{t_n}\| \\ &\quad + \|(I - \nu_1 F_1)P_C(I - \nu_2(t_n)F_2)x_{t_n} - (I - \nu_1(t_n)F_1)P_C(I - \nu_2(t_n)F_2)x_{t_n}\| \\ &\quad + \theta_{t_n}\|(I - A)G_{t_n}x_{t_n} + t_n(\gamma V x_{t_n} - \mu F G_{t_n}x_{t_n})\| \\ &= |\nu_2(t_n) - \nu_2|\|F_2 x_{t_n}\| + |\nu_1(t_n) - \nu_1|\|F_1 P_C(I - \nu_2(t_n)F_2)x_{t_n}\| \\ &\quad + \theta_{t_n}\|(I - A)G_{t_n}x_{t_n} + t_n(\gamma V x_{t_n} - \mu F G_{t_n}x_{t_n})\|.\end{aligned}$$

Since $\theta_{t_n} \to 0$ and $\nu_j(t_n) \to \nu_j$ as $t_n \to 0$ for $j = 1, 2$, we have $(I - G)x_{t_n} \to 0$ as $t_n \to 0$. Thus it follows from [12] that $x^* \in \mathrm{Fix}(G)$. By Proposition 2.2 we get $x^* \in \Xi$.

Finally, let us show that x^* is a solution of the VIP (3.2). Since

$$x_t = x_t - w_t + (I - \theta_t A)G_t x_t + \theta_t(t\gamma V x_t + (I - t\mu F)G_t x_t),$$

we have

$$x_t - G_t x_t = x_t - w_t + \theta_t(I - A)G_t x_t + \theta_t t(\gamma V x_t - \mu F G_t x_t).$$

Since G_t is nonexpansive (due to Proposition 2.2), $I - G_t$ is monotone. So, from the monotonicity of $I - G_t$, it follows that, for $p \in \Xi = \mathrm{Fix}(G_t)$,

$$\begin{aligned} 0 &\le \langle (I - G_t)x_t - (I - G_t)p, x_t - p\rangle = \langle (I - G_t)x_t, x_t - p\rangle \\ &= \langle\langle x_t - w_t, x_t - p\rangle + \theta_t\langle (I - A)G_t x_t, x_t - p\rangle + \theta_t t\langle \gamma V x_t - \mu F G_t x_t, x_t - p\rangle \\ &\le \theta_t\langle (I - A)G_t x_t, x_t - p\rangle + \theta_t t\langle \gamma V x_t - \mu F G_t x_t, x_t - p\rangle \\ &= \theta_t\langle (I - A)x_t, x_t - p\rangle + \theta_t\langle (I - A)(G_t - I)x_t, x_t - p\rangle + \theta_t t\langle \gamma V x_t - \mu F G_t x_t, x_t - p\rangle. \end{aligned}$$

This implies that

$$\langle (A - I)x_t, x_t - p\rangle \le \langle (I - A)(G_t - I)x_t, x_t - p\rangle + t\langle \gamma V x_t - \mu F G_t x_t, x_t - p\rangle. \quad (3.5)$$

Now, replacing t in (3.5) with t_n and letting $n \to \infty$, noticing the boundedness of $\{\gamma V x_{t_n} - \mu F G_{t_n} x_{t_n}\}$ and the fact that $(I - A)(G_{t_n} - I)x_{t_n} \to 0$ as $n \to \infty$ (due to Proposition 3.1 (ii)), we obtain

$$\langle (A - I)x^*, x^* - p\rangle \le 0.$$

That is, $x^* \in \Xi$ is a solution of the VIP (3.2); hance $x^* = \tilde{x}$ by uniqueness. In summary, we have proven that each cluster point of $\{x_t\}$ (as $t \to 0$) equals $\tilde{x}$. Consequently, $x_t \to \tilde{x}$ as $t \to 0$.

The VIP (3.2) can be rewritten as

$$\langle (2I - A)\tilde{x} - \tilde{x}, \tilde{x} - p\rangle \ge 0, \quad \forall p \in \Xi.$$

Using Proposition 2.1 (i), the last inequality is equivalent to the fixed point equation

$$P_\Xi(2I - A)\tilde{x} = \tilde{x}.$$

□

Taking $F = \frac{1}{2}I$, $\mu = 2$ and $\gamma = 1$ in Theorem 3.1, we get

Corollary 3.1. *Let $\{x_t\}$ be defined by*

$$x_t = P_C[(I - \theta_t A)G_t x_t + \theta_t(tV x_t + (1 - t)G_t x_t)].$$

If $\lim_{t\to 0}\theta_t = 0$, then $\{x_t\}$ converges strongly as $t \to 0$ to a fixed point $\tilde{x}$ of G (i.e., $\tilde{x} \in \Xi$), which is the unique solution of the VIP (3.2).

Next, we prove the following result in order to establish the strong convergence of the sequence $\{x_n\}$ generated by the general composite explicit scheme (3.3).

Theorem 3.2. *Let $\{x_n\}$ be the sequence generated by the explicit scheme (3.3), where $\{\alpha_n\}$ and $\{\beta_n\}$ satisfy the following condition:*

(C1) $\{\alpha_n\} \subset [0, 1]$, $\{\beta_n\} \subset (0, 1]$ and $\alpha_n \to 0$, $\beta_n \to 0$ as $n \to \infty$.

Let LIM *be a Banach limit. Then*

$$\mathrm{LIM}_n\langle (A - I)\tilde{x}, \tilde{x} - x_n\rangle \le 0,$$

where $\tilde{x} = \lim_{t\to 0^+} x_t$ *with* x_t *being defined by*

$$x_t = P_C[(I-\theta_t A)Gx_t + \theta_t(t\gamma V x_t + (I - t\mu F)Gx_t)], \tag{3.6}$$

where $G : C \to C$ *is defined by* $Gx = P_C(I-\nu_1 F_1)P_C(I-\nu_2 F_2)x$ *for* $0 < \nu_j \leq 2\zeta_j, j = 1, 2.$

Proof. First, note that from the condition (C1), without loss of generality, we may assume that $0 < \beta_n \leq \|A\|^{-1}$ for all $n \geq 0$.

Let $\{x_t\}$ be the net generated by (3.6). Since G is a nonexpansive self-mapping on C, by Theorem 3.1 with $G_t = G$ and Proposition 2.2, there exists $\lim_{t\to 0} x_t \in \mathrm{Fix}(G) = \Xi$. Denote it by $\tilde{x}$. Moreover, $\tilde{x}$ is the unique solution of the VIP (3.2). From Proposition 3.1 (i) with $G_t = G$, we know that $\{x_t\}$ is bounded and so are the nets $\{Vx_t\}$ and $\{FGx_t\}$.

First of all, let us show that $\{x_n\}$ is bounded. To this end, take $p \in \mathrm{Fix}(G) = \mathrm{Fix}(G_n)$, then we get

$$\begin{aligned} \|y_n - p\| &= \|\alpha_n\gamma V x_n + (I-\alpha_n\mu F)G_n x_n - p\| \\ &= \|\alpha_n(\gamma V x_n - \mu F p) + (I-\alpha_n\mu F)G_n x_n - (I-\alpha_n\mu F)G_n p\| \\ &\leq (1-\alpha_n(\tau-\gamma l))\|x_n - p\| + \alpha_n\|(\gamma V - \mu F)p\|, \end{aligned}$$

and hence we obtain

$$\|x_n - p\| \leq \max\{\|x_0 - p\|, \frac{\|(\gamma V-\mu F)p\| + \|I-A\|\|p\|}{\bar{\gamma}-1}\}, \quad \forall n \geq 0.$$

This implies that $\{x_n\}$ is bounded and so are $\{Gx_n\}, \{G_n x_n\}, \{FG_n x_n\}, \{Vx_n\}$ and $\{y_n\}$. Thus, utilizing the control condition (C1), we get

$$\begin{aligned} \|x_{n+1} - G_n x_n\| &= \|P_C[(I-\beta_n A)G_n x_n + \beta_n y_n] - G_n x_n\| \\ &\leq \|(I-\beta_n A)G_n x_n + \beta_n y_n - G_n x_n\| \\ &= \beta_n\|y_n - AG_n x_n\| \to 0 \quad \text{as } n\to\infty, \end{aligned}$$

and

$$\begin{aligned} \|Gx_t - x_{n+1}\| &\leq \|Gx_t - Gx_n\| + \|Gx_n - G_n x_n\| + \|G_n x_n - x_{n+1}\| \\ &\leq \|x_t - x_n\| + \|G_n x_n - x_{n+1}\| \\ &\quad + \|P_C(I-\nu_1 F_1)P_C(I-\nu_2 F_2)x_n - P_C(I-\nu_{1,n}F_1)P_C(I-\nu_{2,n}F_2)x_n\| \\ &\leq \|x_t - x_n\| + \|G_n x_n - x_{n+1}\| \\ &\quad + \|P_C(I-\nu_1 F_1)P_C(I-\nu_2 F_2)x_n - P_C(I-\nu_1 F_1)P_C(I-\nu_{2,n}F_2)x_n\| \\ &\quad + \|P_C(I-\nu_1 F_1)P_C(I-\nu_{2,n} F_2)x_n - P_C(I-\nu_{1,n}F_1)P_C(I-\nu_{2,n}F_2)x_n\| \\ &\leq \|x_t - x_n\| + \|G_n x_n - x_{n+1}\| + \|(I-\nu_2 F_2)x_n - (I-\nu_{2,n}F_2)x_n\| \\ &\quad + \|(I-\nu_1 F_1)P_C(I-\nu_{2,n}F_2)x_n - (I-\nu_{1,n}F_1)P_C(I-\nu_{2,n}F_2)x_n\| \\ &\leq \|x_t - x_n\| + \|x_{n+1} - G_n x_n\| \\ &\quad + \|\nu_{2,n} - \nu_2|\|F_2 x_n\| + |\nu_{1,n} - \nu_1|\|F_1 P_C(I-\nu_{2,n}F_2)x_n\| \\ &= \|x_t - x_n\| + \epsilon_n, \end{aligned} \tag{3.7}$$

where $\epsilon_n = \|x_{n+1} - G_n x_n\| + \|\nu_{2,n} - \nu_2\| \|F_2 x_n\| + |\nu_{1,n} - \nu_1| \|F_1 P_C (I - \nu_{2,n} F_2) x_n\| \to 0$ as $n \to \infty$. Also observing that A is strongly positive, we have

$$\langle Ax_t - Ax_n, x_t - x_n \rangle = \langle A(x_t - x_n), x_t - x_n \rangle \geq \bar{\gamma} \|x_t - x_n\|^2. \tag{3.8}$$

For simplicity, we write $w_t = (I - \theta_t A)Gx_t + \theta_t(t\gamma V x_t + (I - t\mu F)Gx_t)$. Then we obtain that $x_t = P_C w_t$ and

$$\begin{aligned} x_t - x_{n+1} &= x_t - w_t + (I - \theta_t A)Gx_t + \theta_t(t\gamma V x_t + (I - t\mu F)Gx_t) - x_{n+1} \\ &= (I - \theta_t A)Gx_t - (I - \theta_t A)x_{n+1} + \theta_t(t\gamma V x_t \\ &\quad + (I - t\mu F)Gx_t - Ax_{n+1}) + x_t - w_t. \end{aligned}$$

Applying Lemma 2.1, we have

$$\begin{aligned} &\|x_t - x_{n+1}\|^2 \leq \|(I - \theta_t A)Gx_t - (I - \theta_t A)x_{n+1}\|^2 \\ &+2\theta_t \langle Gx_t - t(\mu F Gx_t - \gamma V x_t) - Ax_{n+1}, x_t - x_{n+1} \rangle + 2\langle x_t - w_t, x_t - x_{n+1} \rangle \\ &\leq \|(I - \theta_t A)Gx_t - (I - \theta_t A)x_{n+1}\|^2 + 2\theta_t \langle Gx_t - t(\mu F Gx_t - \gamma V x_t) - Ax_{n+1}, x_t - x_{n+1} \rangle \\ &\leq (1 - \theta_t \bar{\gamma})^2 \|Gx_t - x_{n+1}\|^2 + 2\theta_t \langle Gx_t - x_t, x_t - x_{n+1} \rangle \\ &-2\theta_t t \langle \mu F Gx_t - \gamma V x_t, x_t - x_{n+1} \rangle + 2\theta_t \langle x_t - Ax_{n+1}, x_t - x_{n+1} \rangle. \end{aligned} \tag{3.9}$$

Using (3.7) and (3.8) in (3.9), we obtain

$$\begin{aligned} &\|x_t - x_{n+1}\|^2 \\ &\leq (1 - \theta_t \bar{\gamma})^2 \|Gx_t - x_{n+1}\|^2 + 2\theta_t \langle Gx_t - x_t, x_t - x_{n+1} \rangle \\ &\quad + 2\theta_t t \langle \gamma V x_t - \mu F Gx_t, x_t - x_{n+1} \rangle + 2\theta_t \langle x_t - Ax_{n+1}, x_t - x_{n+1} \rangle \\ &\leq (1 - \theta_t \bar{\gamma})^2 (\|x_t - x_n\| + \epsilon_n)^2 + 2\theta_t \|Gx_t - x_t\| \|x_t - x_{n+1}\| \\ &\quad + 2\theta_t t \|\gamma V x_t - \mu F Gx_t\| \|x_t - x_{n+1}\| + 2\theta_t \langle x_t - Ax_{n+1}, x_t - x_{n+1} \rangle \\ &= (\theta_t^2 \bar{\gamma} - 2\theta_t) \bar{\gamma} \|x_t - x_n\|^2 + \|x_t - x_n\|^2 + (1 - \theta_t \bar{\gamma})^2 (2\|x_t - x_n\| \epsilon_n + \epsilon_n^2) \\ &\quad + 2\theta_t \|Gx_t - x_t\| \|x_t - x_{n+1}\| + 2\theta_t t \|\gamma V x_t - \mu F Gx_t\| \|x_t - x_{n+1}\| \\ &\quad + 2\theta_t \langle x_t - Ax_{n+1}, x_t - x_{n+1} \rangle \\ &\leq (\theta_t^2 \bar{\gamma} - 2\theta_t) \langle Ax_t - Ax_n, x_t - x_n \rangle + \|x_t - x_n\|^2 + (1 - \theta_t \bar{\gamma})^2 (2\|x_t - x_n\| \epsilon_n + \epsilon_n^2) \\ &\quad + 2\theta_t \|Gx_t - x_t\| \|x_t - x_{n+1}\| + 2\theta_t t \|\gamma V x_t - \mu F Gx_t\| \|x_t - x_{n+1}\| \\ &\quad + 2\theta_t \langle x_t - Ax_{n+1}, x_t - x_{n+1} \rangle \\ &= \theta_t^2 \bar{\gamma} \langle Ax_t - Ax_n, x_t - x_n \rangle + \|x_t - x_n\|^2 + (1 - \theta_t \bar{\gamma})^2 (2\|x_t - x_n\| \epsilon_n + \epsilon_n^2) \\ &\quad + 2\theta_t \|Gx_t - x_t\| \|x_t - x_{n+1}\| + 2\theta_t t \|\gamma V x_t - \mu F Gx_t\| \|x_t - x_{n+1}\| \\ &\quad + 2\theta_t [\langle x_t - Ax_{n+1}, x_t - x_{n+1} \rangle - \langle Ax_t - Ax_n, x_t - x_n \rangle] \\ &= \theta_t^2 \bar{\gamma} \langle A(x_t - x_n), x_t - x_n \rangle + \|x_t - x_n\|^2 + (1 - \theta_t \bar{\gamma})^2 (2\|x_t - x_n\| \epsilon_n + \epsilon_n^2) \\ &\quad + 2\theta_t \|Gx_t - x_t\| \|x_t - x_{n+1}\| + 2\theta_t t \|\gamma V x_t - \mu F Gx_t\| \|x_t - x_{n+1}\| \\ &\quad + 2\theta_t [\langle (I - A)x_t, x_t - x_{n+1} \rangle + \langle A(x_t - x_{n+1}), x_t - x_{n+1} \rangle - \langle A(x_t - x_n), x_t - x_n \rangle]. \end{aligned} \tag{3.10}$$

Applying the Banach limit LIM to (3.10), from $\epsilon_n \to 0$ we have

$$\begin{aligned}
&\mathrm{LIM}_n\|x_t - x_{n+1}\|^2 \qquad (3.11)\\
&\le \theta_t^2\bar{\gamma}\mathrm{LIM}_n\langle A(x_t - x_n), x_t - x_n\rangle + \mathrm{LIM}_n\|x_t - x_n\|^2\\
&+ 2\theta_t\|Gx_t - x_t\|\mathrm{LIM}_n\|x_t - x_{n+1}\| + 2\theta_t t\|\gamma V x_t - \mu FGx_t\|\mathrm{LIM}_n\|x_t - x_{n+1}\|\\
&+ 2\theta_t[\mathrm{LIM}_n\langle (I - A)x_t, x_t - x_{n+1}\rangle + \mathrm{LIM}_n\langle A(x_t - x_{n+1}), x_t - x_{n+1}\rangle\\
&- \mathrm{LIM}_n\langle A(x_t - x_n), x_t - x_n\rangle].
\end{aligned}$$

Using the property $\mathrm{LIM}_n a_n = \mathrm{LIM}_n a_{n+1}$ of the Banach limit in (3.11), we obtain

$$\begin{aligned}
&\mathrm{LIM}_n\langle (A - I)x_t, x_t - x_n\rangle \qquad (3.12)\\
&= \mathrm{LIM}_n\langle (A - I)x_t, x_t - x_{n+1}\rangle\\
&\le \frac{\theta_t\bar{\gamma}}{2}\mathrm{LIM}_n\langle A(x_t - x_n), x_t - x_n\rangle + \frac{1}{2\theta_t}[\mathrm{LIM}_n\|x_t - x_n\|^2 - \mathrm{LIM}_n\|x_t - x_{n+1}\|^2]\\
&+ \|Gx_t - x_t\|\mathrm{LIM}_n\|x_t - x_n\| + t\|\gamma V x_t - \mu FGx_t\|\mathrm{LIM}_n\|x_t - x_n\|\\
&+ \mathrm{LIM}_n\langle A(x_t - x_{n+1}), x_t - x_{n+1}\rangle - \mathrm{LIM}_n\langle A(x_t - x_n), x_t - x_n\rangle\\
&\le \frac{\theta_t\bar{\gamma}}{2}\mathrm{LIM}_n\langle A(x_t - x_n), x_t - x_n\rangle + \|Gx_t - x_t\|\mathrm{LIM}_n\|x_t - x_n\|\\
&+ t\|\gamma V x_t - \mu FGx_t\|\mathrm{LIM}_n\|x_t - x_n\|.
\end{aligned}$$

Since as $t \to 0$,

$$\theta_t\langle A(x_t - x_n), x_t - x_n\rangle \le \theta_t\|A\|\|x_t - x_n\|^2 \le \theta_t K \to 0, \qquad (3.13)$$

where $\|A\|\|x_t - x_n\|^2 \le K$,

$$\|Gx_t - x_t\| \to 0 \quad \text{ans} \quad t\|\gamma V x_t - \mu FGx_t\| \to 0 \quad \text{as } t \to 0, \qquad (3.14)$$

we conclude from (3.12)-(3.14) that

$$\begin{aligned}
&\mathrm{LIM}_n\langle (A - I)\tilde{x}, \tilde{x} - x_n\rangle\\
&\le \limsup_{t\to 0}\mathrm{LIM}_n\langle (A - I)x_t, x_t - x_n\rangle\\
&\le \limsup_{t\to 0}\tfrac{\theta_t\bar{\gamma}}{2}\mathrm{LIM}_n\langle A(x_t - x_n), x_t - x_n\rangle + \limsup_{t\to 0}\|Gx_t - x_t\|\mathrm{LIM}_n\|x_t - x_n\|\\
&+\limsup_{t\to 0} t\|\gamma V x_t - \mu FGx_t\|\mathrm{LIM}_n\|x_t - x_n\|\\
&= 0.
\end{aligned}$$

This completes the proof. □

Now, using Theorem 3.2, we establish the strong convergence of the sequence $\{x_n\}$ generated by the general composite explicit scheme (3.3) to a fixed point $\tilde{x}$ of G (i.e., $\tilde{x} \in \Xi$), which is also the unique solution of the VIP (3.2).

Theorem 3.3. *Let $\{x_n\}$ be the sequence generated by the explicit scheme (3.3), where $\{\alpha_n\}$ and $\{\beta_n\}$ satisfy the following conditions:*

(C1) $\{\alpha_n\} \subset [0, 1]$, $\{\beta_n\} \subset (0, 1]$ and $\alpha_n \to 0$, $\beta_n \to 0$ as $n \to \infty$;

(C2) $\sum_{n=0}^{\infty}\beta_n = \infty$.

If $\{x_n\}$ is weakly asymptotically regular (i.e., $x_{n+1} - x_n \rightharpoonup 0$), then x_n converges strongly to a fixed point $\tilde{x}$ of G (i.e., $\tilde{x} \in \Xi$), which is the unique solution of the VIP (3.2).

Proof. First, note that from the condition (C1), without loss of generality, we may assume that $\alpha_n \tau < 1$ and $\frac{2\beta_n(\bar{\gamma}-1)}{1-\beta_n} < 1$ for all $n \geq 0$.

Let x_t be defined by (3.6), that is,

$$x_t = P_C[(I - \theta_t A)Gx_t + \theta_t(Gx_t - t(\mu FGx_t - \gamma Vx_t)],$$

for $t \in (0, \min\{1, \frac{2-\bar{\gamma}}{\tau-\gamma l}\})$, where $G = P_C(I - \nu_1 F_1)P_C(I - \nu_2 F_2)$ for $0 < \nu_j \leq 2\zeta_j$, and $\lim_{t\to 0} x_t := \tilde{x} \in \mathrm{Fix}(G) = \Xi$). Then $\tilde{x}$ is the unique solution of the VIP (3.2).

We divide the rest of the proof into several steps.

Step 1. We see that

$$\|x_n - p\| \leq \max\{\|x_0 - p\|, \frac{\|(\gamma V - \mu F)p\| + \|I - A\|\|p\|}{\bar{\gamma} - 1}\}, \quad \forall n \geq 0,$$

for all $p \in \Xi$ as in the proof of Theorem 3.2. Hence $\{x_n\}$ is bounded and so are $\{Gx_n\}, \{G_n x_n\}$, $\{FG_n x_n\}, \{Vx_n\}$ and $\{y_n\}$.

Step 2. We show that $\limsup_{n\to\infty}\langle (I - A)\tilde{x}, x_n - \tilde{x}\rangle \leq 0$. To this end, put

$$a_n := \langle (A - I)\tilde{x}, \tilde{x} - x_n\rangle, \quad \forall n \geq 0.$$

Then, by Theorem 3.2 we get $\mathrm{LIM}_n a_n \leq 0$ for any Banach limit LIM. Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}(a_{n+1} - a_n) = \limsup_{j\to\infty}(a_{n_j+1} - a_{n_j})$$

and $x_{n_j} \rightharpoonup u \in H$. This implies that $x_{n_j+1} \rightharpoonup u$ since $\{x_n\}$ is weakly asymptotically regular. Therefore, we have

$$w - \lim_{j\to\infty}(\tilde{x} - x_{n_j+1}) = w - \lim_{j\to\infty}(\tilde{x} - x_{n_j}) = \tilde{x} - u,$$

and so

$$\limsup_{n\to\infty}(a_{n+1} - a_n) = \lim_{j\to\infty}\langle (A - I)\tilde{x}, (\tilde{x} - x_{n_j+1}) - (\tilde{x} - x_{n_j})\rangle = 0.$$

Then, by [18, Proposition 2] we obtain $\limsup_{n\to\infty} a_n \leq 0$, that is,

$$\limsup_{n\to\infty}\langle (I - A)\tilde{x}, x_n - \tilde{x}\rangle = \limsup_{n\to\infty}\langle (A - I)\tilde{x}, \tilde{x} - x_n\rangle \leq 0.$$

Step 3. We show that $\lim_{n\to\infty}\|x_n - \tilde{x}\| = 0$. Indeed, for simplicity, we write $w_n = (I - \beta_n A)G_n x_n + \beta_n y_n$ for all $n \geq 0$. Then $x_{n+1} = P_C w_n$. Utilizing (3.3) and $G_n\tilde{x} = \tilde{x}$, we have

$$y_n - \tilde{x} = (I - \alpha_n\mu F)G_n x_n - (I - \alpha_n\mu F)G_n\tilde{x} + \alpha_n(\gamma Vx_n - \mu F\tilde{x}),$$

and

$$x_{n+1} - \tilde{x} = x_{n+1} - w_n + (I - \beta_n A)(G_n x_n - G_n\tilde{x}) + \beta_n(y_n - \tilde{x}) + \beta_n(I - A)\tilde{x}.$$

Applying Lemma 2.1 and [23, Lemma 31], we obtain

$$\begin{aligned}\|y_n - \tilde{x}\|^2 &= \|(I - \alpha_n\mu F)G_n x_n - (I - \alpha_n\mu F)G_n\tilde{x} + \alpha_n(\gamma Vx_n - \mu F\tilde{x})\|^2 \\ &\leq \|(I - \alpha_n\mu F)G_n x_n - (I - \alpha_n\mu F)G_n\tilde{x}\|^2 + 2\alpha_n\langle \gamma Vx_n - \mu F\tilde{x}, y_n - \tilde{x}\rangle \\ &\leq (1 - \alpha_n\tau)^2\|x_n - \tilde{x}\|^2 + 2\alpha_n\|\gamma Vx_n - \mu F\tilde{x}\|\|y_n - \tilde{x}\| \\ &\leq \|x_n - \tilde{x}\|^2 + 2\alpha_n\|\gamma Vx_n - \mu F\tilde{x}\|\|y_n - \tilde{x}\|,\end{aligned}$$

and hence

$$
\begin{aligned}
&\|x_{n+1} - \tilde{x}\|^2 \qquad (3.15)\\
&= \|(I - \beta_n A)(G_n x_n - G_n \tilde{x}) + \beta_n (y_n - \tilde{x}) + \beta_n (I - A)\tilde{x} + x_{n+1} - w_n\|^2\\
&\le \|(I - \beta_n A)(G_n x_n - G_n \tilde{x})\|^2 + 2\beta_n \langle y_n - \tilde{x}, x_{n+1} - \tilde{x}\rangle + 2\beta_n \langle (I - A)\tilde{x}, x_{n+1} - \tilde{x}\rangle\\
&\quad + 2\langle x_{n+1} - w_n, x_{n+1} - \tilde{x}\rangle\\
&\le \|(I - \beta_n A)(G_n x_n - G_n \tilde{x})\|^2 + 2\beta_n \langle y_n - \tilde{x}, x_{n+1} - \tilde{x}\rangle + 2\beta_n \langle (I - A)\tilde{x}, x_{n+1} - \tilde{x}\rangle\\
&\le (1 - \beta_n \bar{\gamma})^2 \|x_n - \tilde{x}\|^2 + 2\beta_n \|y_n - \tilde{x}\| \|x_{n+1} - \tilde{x}\| + 2\beta_n \langle (I - A)\tilde{x}, x_{n+1} - \tilde{x}\rangle\\
&\le (1 - \beta_n \bar{\gamma})^2 \|x_n - \tilde{x}\|^2 + \beta_n (\|y_n - \tilde{x}\|^2 + \|x_{n+1} - \tilde{x}\|^2) + 2\beta_n \langle (I - A)\tilde{x}, x_{n+1} - \tilde{x}\rangle\\
&\le (1 - \beta_n \bar{\gamma})^2 \|x_n - \tilde{x}\|^2 + \beta_n [\|x_n - \tilde{x}\|^2 + 2\alpha_n \|\gamma V x_n - \mu F \tilde{x}\| \|y_n - \tilde{x}\|]\\
&\quad + \beta_n \|x_{n+1} - \tilde{x}\|^2 + 2\beta_n \langle (I - A)\tilde{x}, x_{n+1} - \tilde{x}\rangle\\
&= [(1 - \beta_n \bar{\gamma})^2 + \beta_n] \|x_n - \tilde{x}\|^2 + 2\alpha_n \beta_n \|\gamma V x_n - \mu F \tilde{x}\| \|y_n - \tilde{x}\|\\
&\quad + \beta_n \|x_{n+1} - \tilde{x}\|^2 + 2\beta_n \langle (I - A)\tilde{x}, x_{n+1} - \tilde{x}\rangle.
\end{aligned}
$$

It then follows from (3.15) that

$$
\begin{aligned}
\|x_{n+1} - \tilde{x}\|^2 &\le \frac{(1 - \beta_n \bar{\gamma})^2 + \beta_n}{1 - \beta_n} \|x_n - \tilde{x}\|^2 + \frac{\beta_n}{1 - \beta_n} [2\alpha_n \|\gamma V x_n - \mu F \tilde{x}\| \|y_n - \tilde{x}\|\\
&\quad + 2\langle (I - A)\tilde{x}, x_{n+1} - \tilde{x}\rangle]\\
&= (1 - \frac{2\beta_n(\bar{\gamma} - 1)}{1 - \beta_n}) \|x_n - \tilde{x}\|^2\\
&\quad + \frac{2\beta_n(\bar{\gamma} - 1)}{1 - \beta_n} \cdot \frac{1}{2(\bar{\gamma} - 1)} [2\alpha_n \|\gamma V x_n - \mu F \tilde{x}\| \|y_n - \tilde{x}\|\\
&\quad + \beta_n \bar{\gamma}^2 \|x_n - \tilde{x}\|^2 + 2\langle (I - A)\tilde{x}, x_{n+1} - \tilde{x}\rangle]\\
&= (1 - \omega_n) \|x_n - \tilde{x}\|^2 + \omega_n \delta_n,
\end{aligned}
$$

where $\omega_n = \frac{2\beta_n(\bar{\gamma}-1)}{1-\beta_n}$ and

$$
\delta_n = \frac{1}{2(\bar{\gamma} - 1)} [2\alpha_n \|\gamma V x_n - \mu F \tilde{x}\| \|y_n - \tilde{x}\| + \beta_n \bar{\gamma}^2 \|x_n - \tilde{x}\|^2 + 2\langle (I - A)\tilde{x}, x_{n+1} - \tilde{x}\rangle].
$$

It can be readily seen from Step 2 and conditions (C1) and (C2) that $\omega_n \to 0$, $\sum_{n=0}^{\infty} \omega_n = \infty$ and $\limsup_{n\to\infty} \delta_n \le 0$. By [22, Lemma 2.1] with $r_n = 0$, we conclude that $\lim_{n\to\infty} \|x_n - \tilde{x}\| = 0$. This completes the proof. □

Corollary 3.2. *Let $\{x_n\}$ be the sequence generated by the explicit scheme (3.3). Assume that the sequences $\{\alpha_n\}$ and $\{\beta_n\}$ satisfy the conditions (C1) and (C2) in Theorem 3.3. If $\{x_n\}$ is asymptotically regular, then $\{x_n\}$ converges strongly to a fixed point $\tilde{x}$ of G which is the unique solution of the VIP (3.2).*

Putting $\mu = 2$, $F = \frac{1}{2}I$ and $\gamma = 1$ in Theorem 3.3, we obtain the following.

Corollary 3.3. *Let $\{x_n\}$ be generated by the following iterative scheme:*

$$
\begin{cases}
y_n = \alpha_n V x_n + (1 - \alpha_n) G_n x_n,\\
x_{n+1} = P_C[(I - \beta_n A) G_n x_n + \beta_n y_n], \quad \forall n \ge 0.
\end{cases}
$$

Assume that the sequences $\{\alpha_n\}$ and $\{\beta_n\}$ satisfy the conditions (C1) and (C2) in Theorem 3.3. If $\{x_n\}$ is weakly asymptotically regular, then $\{x_n\}$ converges strongly to a fixed point $\tilde{x}$ of G which is the unique solution of the VIP (3.2).

Remark 3.1. If $\{\alpha_n\}, \{\beta_n\}$ in Corollary 3.2 and $\{\nu_{j,n}\}_{j=1}^2$ in G_n satisfy conditions (C2) and

(C3) $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$ and $\sum_{n=0}^{\infty} |\beta_{n+1} - \beta_n| < \infty$; or

(C4) $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$ and $\lim_{n\to\infty} \frac{\beta_n}{\beta_{n+1}} = 1$ or, equivalently, $\lim_{n\to\infty} \frac{\alpha_n - \alpha_{n+1}}{\alpha_{n+1}} = 0$ and $\lim_{n\to\infty} \frac{\beta_n - \beta_{n+1}}{\beta_{n+1}} = 0$; or,

(C5) $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$ and $|\beta_{n+1} - \beta_n| \le o(\beta_{n+1}) + \sigma_n$, $\sum_{n=0}^{\infty} \sigma_n < \infty$ (the perturbed control condition);

(C6) $\sum_{n=0}^{\infty} |\nu_{j,n+1} - \nu_{j,n}| < \infty$ for $j = 1, 2$,

then the sequence $\{x_n\}$ generated by (3.3) is asymptotically regular.

In view of the above remark, we have the following corollary.

Corollary 3.4. *Let $\{x_n\}$ be the sequence generated by the explicit scheme (3.3), where the sequences $\{\alpha_n\}, \{\beta_n\}$ and $\{\nu_{j,n}\}_{j=1}^2$ satisfy the conditions (C1), (C2), (C5) and (C6) (or the conditions (C1), (C2), (C3) and (C6), or the conditions (C1), (C2), (C4) and (C6)). Then $\{x_n\}$ converges strongly to a fixed point $\tilde{x}$ of G (i.e., $\tilde{x} \in \Xi$), which is the unique solution of the VIP (3.2).*

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