Fixed Point Theory, 19(2018), No. 1, 141-166
DOI 10.24193/fpt-ro.2018.1.11
http://www.math.ubbcluj.ro/~nodeacj/sfptcj.html

M-CONSTANTS IN ORLICZ-LORENTZ SEQUENCE SPACES WITH APPLICATIONS TO FIXED POINT THEORY

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Abstract. In this paper some estimates of M-constants in Orlicz-Lorentz sequence spaces for both, the Luxemburg and the Amemiya norms are given. Since degenerated Orlicz functions φ and degenerated weighted sequences ω are also admitted, this investigations concern the most possible wide class of Orlicz-Lorentz sequence spaces. M-constants were defined in 1969 by E.A. Lifshits, and used in the study of lattice structures on Banach spaces, as well as in the fixed point theory, by a number of authors. In the last section of the paper an application of our results to the fixed point property is presented.

Key Words and Phrases: M-constants, L-constants, lattice structures on Banach spaces, Riesz angle, fixed point property, symmetric spaces, Orlicz-Lorentz spaces, Orlicz functions, Luxemburg norm, Amemiya norm.
2010 Mathematics Subject Classification: 46B20, 46B45, 46A45, 46B42, 46A80, 47H10.

1. Introduction

Recall that M-constants were introduced by Lifshits [27]. His investigations were continued by Abramovich, Lozanovski and Tsekrekos (see [1] and [37], respectively). Let $E = (E, \leq, \|\cdot\|)$ be a normed lattice. Then M-constants in E are defined by formula

$$\mu_n(E) = \sup\left\{\left\|\bigvee_{i=1}^{n} x_i\right\| : 0 \leq x_i \in E,\ \|x_i\| \leq 1 \ \text{ for } \ i = 1, \ldots, n\right\}$$

The first author is supported by NFC of Heilong Jiang Province A2015018.
The second and the third authors are supported partially by National Science Centre (NCN), Poland, Grant No. DEC – 2012/07/B/ST1/03360.

or, equivalently

$$\mu_n(E)=\sup\left\{\left\|\sum_{i=1}^n x_i\right\| : 0\le x_i\in E, \|x_i\|\le 1 \text{ for } i=1,\dots,n \text{ and } x_i\wedge x_j=0 \text{ for } i\neq j\right\}$$
$$=\sup\left\{\left\|\sum_{i=1}^n x_i\right\| : 0\le x_i\in E, \|x_i\|=1 \text{ for } i=1,\dots,n \text{ and } x_i\wedge x_j=0 \text{ for } i\neq j\right\}.$$

In [27], there were also defined the following L-constants

$$\lambda_n(E)=\sup\left\{\frac{\|x_1\|+\ldots+\|x_n\|}{\|x_1+\ldots+x_n\|} : 0\le x_i\in E \text{ for } i=1,\dots,n,\ \sum_{i=1}^n x_i\neq 0\right\}.$$

Denoting by E^* the dual of E, we have $\lambda_n(E^*)=\mu_n(E)$ and $\mu_n(E^*)=\lambda_n(E)$ (see [37]*Proposition 2). Moreover, if E is infinite dimensional, then $\lambda_n(E)\mu_n(E)\geq n$ for all $n\in\mathbb{N}$. Generally, the numbers $\mu_n(E)$ and $\lambda_n(E)$ are useful in the lattice-isomorphic classification of special type of Banach lattices. For instance, [37] contains a characterization of l^1 and c_0 (by a behavior of M-constants and L-constants) in the family of Banach lattices with unconditional basis.

The above constants were also considered by Kalton in [16]. In particular, relationships between lattice structure on E and the facts that $\liminf_{n\to\infty}\frac{\lambda_n(E)}{\sqrt{\log n}}$ or $\liminf_{n\to\infty}\frac{\mu_n(E)}{\sqrt{\log n}}$ are equal to zero were shown (see [16]*Theorems 9.4-6). Next, Mastyło obtained an analogous result in Calderón-Lozanovskiĭ spaces (see Theorem 2.5 in [32]). In this paper, he also find estimates for the constants μ_n and λ_n for Calderón-Lozanovskiĭ spaces. In consequence, the estimate for L-constant for Orlicz-Lorentz spaces are found.

Borwein and Sims in [4] showed that the constant $\mu_2(E)$, called the Riesz angle, plays an important role in the fixed point theory. More precisely, they showed that if E is a weakly orthogonal Banach lattice such that $\mu_2(E)<2$, then E has the weak fixed point property (that is, every non-expansive mapping from a non-empty weakly compact convex subset of E into itself has a fixed point). More about the fixed point theory can be found in the handbook [24]. The estimates of the Riesz angle for Orlicz spaces were given by Jincai Wang and Yaqiang Yan (see [38] and [40, 41], respectively). Also the papers [13] and [23] deal with the fixed point property in Orlicz spaces.

At the end, we note that a lot of important information on M-constants, can be found in the Wnuk paper [39].

The paper is organized as follows. In the next section we will recall some basic definitions. In the third section we will present our results concerning the estimates of the M-constants in Orlicz-Lorentz sequence spaces. In the last section we prove that in Köthe sequence spaces X with the semi-Fatou property the notion of weak orthogonality of X in the sense of Borwein and Sims coincide with its order continuity. We also present some sufficient conditions for the fixed point property of Orlicz-Lorentz sequence spaces $\lambda_{\varphi,\omega}$ and we illustrate this result with a non-expansive operator from some Orlicz-Lorentz sequence spaces into itself that has a fixed point in the unit ball of $\lambda_{\varphi,\omega}$.

2. Preliminaries

Let l^0 be the space of all real sequences $x : \mathbb{N} \to (-\infty, \infty)$ and e_i, $i \in \mathbb{N}$, be the unit basic vectors in c_0. Given any $x \in l^0$ we define its distribution function $\mu_x : [0, +\infty) \to \{0, \infty\} \cup \mathbb{N}$ by

$$\mu_x(\lambda) = \mathrm{m}\{i \in \mathbb{N} : |x(i)| > \lambda\},$$

where m is the counting measure on $2^{\mathbb{N}}$ (see [3, 25, 29]), and its non-increasing rearrangement $x^* = (x^*(i))_{i=1}^{\infty}$ as

$$x^*(i) = \inf\{\lambda \geq 0 : \mu_x(\lambda) < i\}$$

(under the convention $\inf \emptyset = \infty$). We say that two sequences $x, y \in l^0$ are equi-measurable if $\mu_x(\lambda) = \mu_y(\lambda)$ for all $\lambda \geq 0$. It is obvious that equi-measurability of x and y gives $x^* = y^*$.

A Banach sequence space $E = (E, \leq, \|\cdot\|)$, where $E \subset l^0$, is said to be a Köthe sequence space if the following conditions are satisfied:

(i) if $x \in E$, $y \in l^0$ and $|y| \leq |x|$, then $y \in E$ and $\|y\| \leq \|x\|$,

(ii) there exists a sequence x in E that is strictly positive on the whole $\mathbb{N}$.

Recall that the Köthe sequence space E is called a symmetric space if E is rearrangement invariant in the sense that if $x \in E$, $y \in l^0$ and $x^* = y^*$, then $y \in E$ and $\|x\| = \|y\|$. For basic properties of symmetric spaces we refer to [3, 25, 29].

In the whole paper φ denotes an Orlicz function (see [5, 31, 35]), that is, $\varphi : [-\infty, \infty] \to [0, \infty]$ (our definition is extended from $\mathbb{R}$ into $\mathbb{R}^e = [-\infty, \infty]$ by assuming that $\varphi(-\infty) = \varphi(\infty) = \infty$) and φ is convex, even, vanishing and continuous at zero, left continuous on $(0, \infty)$ and not identically equal to zero on $(-\infty, \infty)$. Let us denote

$$\begin{aligned} a_\varphi &= \sup\{u \geq 0 : \varphi(u) = 0\}, \\ b_\varphi &= \sup\{u \geq 0 : \varphi(u) < \infty\}. \end{aligned}$$

Note that the left continuity of φ on $(0, \infty)$ is equivalent to the fact that $\lim_{u \to (b_\varphi)^-} \varphi(u) = \varphi(b_\varphi)$.

The inverse function of the function φ restricted to the interval $[a_\varphi, b_\varphi)$ or $[a_\varphi, b_\varphi]$, according to the situation when $b_\varphi = \infty$ or when $b_\varphi < \infty$ and $\varphi(b_\varphi) < \infty$, respectively, is denoted by φ^{-1}.

Recall that an Orlicz function φ satisfies condition $\Delta_2(0)$ if there exist constants $u_0 > 0$ and $K > 0$ such that $\varphi(u_0) > 0$ and the inequality

$$\varphi(2u) \leq K\varphi(u) \tag{2.1}$$

holds for any $u \in [0, u_0]$ (then we also have $a_\varphi = 0$). Analogously, we say that an Orlicz function φ satisfies condition $\Delta_2(\infty)$ if there exist constants $u_0 > 0$ and $K > 0$ such that $\varphi(u_0) < \infty$ and inequality (2.1) holds for any $u \geq u_0$ (then we obtain $b_\varphi = \infty$). Finally, we say that an Orlicz function φ satisfies condition $\Delta_2(\mathbb{R}_+)$ if φ satisfies simultaneously conditions $\Delta_2(0)$ and $\Delta_2(\infty)$, that is, there exists a constant $K > 0$ such that inequality (2.1) holds for any $u \in \mathbb{R}_+$ (obviously, then $a_\varphi = 0$ and $b_\varphi = \infty$).

For any Orlicz function φ we define its complementary function in the sense of Young by the formula

$$\psi(u) = \sup_{v>0}\{|u|v - \varphi(v)\} \tag{2.2}$$

for all $u \in \mathbb{R}$. It is easy to show that ψ is also an Orlicz function.

Let $\omega : \mathbb{N} \to \mathbb{R}_+$ be a non-increasing and non-negative sequence, called a weighted sequence. In the whole paper we will assume that $\omega(1) > 0$.

If $\varphi(b_\varphi) \geq \frac{1}{\omega(1)}$, then we define $u_1 > 0$ by the equality $\varphi(u_1) = \frac{1}{\omega(1)}$. We also define

$$\gamma_{\varphi,\omega} = \begin{cases} u_1 & \text{if } \varphi(b_\varphi) \geq \frac{1}{\omega(1)}, \\ b_\varphi & \text{otherwise.} \end{cases} \tag{2.3}$$

Now we will recall the definition of Orlicz-Lorentz spaces, which are a natural generalization of Orlicz and Lorentz spaces. They were introduced at the beginning of nineties [30, 17, 18, 19, 33, 34]. These investigations gave an impulse to further investigations of the spaces, results of which have been published among others in the papers [2, 8, 7, 9, 10, 11, 12, 14, 15, 20, 21, 22, 26].

Given any Orlicz function φ and a weighted sequence ω, we define on l^0 the convex modular $I_{\varphi,\omega} : l^0 \to \mathbb{R}_+^e = [0,\infty]$ by the formula

$$I_{\varphi,\omega}(x) := \sum_{i=1}^{\infty} \varphi(x^*(i))\omega(i) = \sup_{\pi} \sum_{i=1}^{\infty} \varphi(x(\pi(i)))\omega(i), \tag{2.4}$$

where π denotes a permutation of the set $\mathbb{N}$ and the supremum is extended over all permutation of $\mathbb{N}$. The modular space

$$\lambda_{\varphi,\omega} = \{x \in l^0 : I_{\varphi,\omega}(\beta x) < \infty \ \text{ for some } \ \beta > 0\},$$

generated by the modular $I_{\varphi,\omega}$ is called the Orlicz-Lorentz sequence space. Since the modular unit ball $\{x \in l^0 : I_{\varphi,\omega}(x) \leq 1\}$ is an absolutely convex and absorbing subset of $\lambda_{\varphi,\omega}$, its Minkowski functional $\|\cdot\|_{\varphi,\omega}$ defined by

$$\|x\|_{\varphi,\omega} = \inf\{\beta > 0 : I_{\varphi,\omega}(x/\beta) \leq 1\}.$$

is a seminorm in $\lambda_{\varphi,\omega}$. It is easy to check that, thanks to the condition $\varphi(u) \to \infty$ as $u \to \infty$ it is a norm in $\lambda_{\varphi,\omega}$, which is called the Luxemburg norm.

We will also consider in the space $\lambda_{\varphi,\omega}$ another norm $\|\cdot\|^A_{\varphi,\omega}$, called the Amemiya norm. It is defined by the formula

$$\|x\|^A_{\varphi,\omega} = \inf_{k>0} \frac{1}{k}\{1 + I_{\varphi,\omega}(kx)\}$$

for any $x \in \lambda_{\varphi,\omega}$. The norms $\|\cdot\|_{\varphi,\omega}$ and $\|\cdot\|^A_{\varphi,\omega}$ are equivalent and the inequality $\|x\|_{\varphi,\omega} \leq \|x\|^A_{\varphi,\omega} \leq 2\|x\|_{\varphi,\omega}$ holds for all $x \in \lambda_{\varphi,\omega}$ which follows by the formula

$$\|x\|_{\varphi,\omega} = \inf_{k>0} \frac{1}{k} \max(1, I_{\varphi,\omega}(kx)).$$

Both spaces $\lambda_{\varphi,\omega} = (\lambda_{\varphi,\omega}, \|\cdot\|_{\varphi,\omega})$ and $\lambda^A_{\varphi,\omega} = (\lambda_{\varphi,\omega}, \|\cdot\|^A_{\varphi,\omega})$ are symmetric Banach spaces.

3. Results

3.1. **The case of the Luxemburg norm.** First, we will present results concerning Orlicz-Lorentz sequence spaces equipped with the Luxemburg norm. At the beginning note that if $a_\varphi = b_\varphi$ then $(\lambda_{\varphi,\omega}, \|\cdot\|_{\varphi,\omega}) = (l^\infty, \|\cdot\|_\infty/b_\varphi)$ and $\mu_n(\lambda_{\varphi,\omega}) = 1$ for any $n \in \mathbb{N}$. Assuming in the following that $a_\varphi < b_\varphi$ we start with some definitions.

For any triple (φ, ω, n), $(n \in \mathbb{N})$ we define three indexes $\alpha_{\varphi,\omega,n}$, $\alpha'_{\varphi,\omega,n}$ and $\alpha^0_{\varphi,n}$ by the formula

$$\alpha_{\varphi,\omega,n} = \inf\left\{\frac{\varphi^{-1}\left(\frac{\varphi(u)}{n}\right)}{u} : a_\varphi < u \le \gamma_{\varphi,\omega}\right\} = \inf\left\{\frac{\varphi^{-1}(v)}{\varphi^{-1}(nv)} : 0 < v \le \frac{\varphi(\gamma_{\varphi,\omega})}{n}\right\},$$

$$\alpha'_{\varphi,\omega,n} = \inf\left\{\frac{\varphi^{-1}\left(\frac{1}{\sum_{i=1}^{nk}\omega(i)}\right)}{\varphi^{-1}\left(\frac{1}{\sum_{i=1}^{k}\omega(i)}\right)} : k = 1, 2, \ldots\right\},$$

$$\alpha^0_{\varphi,n} = \liminf_{u\to a_\varphi} \frac{\varphi^{-1}\left(\frac{\varphi(u)}{n}\right)}{u} = \liminf_{v\to 0} \frac{\varphi^{-1}(v)}{\varphi^{-1}(nv)},$$

where $\gamma_{\varphi,\omega}$ is defined by formula (2.3). Moreover, in the definition of the index $\alpha'_{\varphi,\omega,n}$ we put $\varphi(b_\varphi)$ in place $1/\sum_{i=1}^m \omega(i)$, whenever $\varphi(b_\varphi) < 1/\sum_{i=1}^m \omega(i)$.

Now, using the indexes $\alpha_{\varphi,\omega,n}$, $\alpha'_{\varphi,\omega,n}$ and $\alpha^0_{\varphi,n}$, we give in the first three theorems upper and lower estimates for $\mu_n(\lambda_{\varphi,\omega})$, respectively.

Theorem 3.1. *For any Orlicz-Lorentz sequence space $\lambda_{\varphi,\omega}$ and any natural number n, there holds the following upper estimate*

$$\mu_n(\lambda_{\varphi,\omega}) \le k(n) \le \frac{1}{\alpha_{\varphi,\omega,n}}, \tag{3.1}$$

where

$$k(n) := \sup\{k_x(n) \colon x \in S_+(\lambda_{\varphi,\omega})\}$$

and

$$k_x(n) := \inf\left\{k > 0 \colon I_{\varphi,\omega}\left(\frac{x}{k}\right) \le \frac{1}{n}\right\}.$$

Remark 3.1. Recall that, by the definition of the Luxemburg norm, $I_{\varphi,\omega}(x) \le 1$ for any $x \in S_+(\lambda_{\varphi,\omega})$. Let $n \in \mathbb{N}$ and $x \in S_+(\lambda_{\varphi,\omega})$. If $\frac{1}{n} \le I_{\varphi,\omega}(x)$, then there exists exactly one number $k_x(n) > 1$ such that $I_{\varphi,\omega}\left(\frac{x}{k_x(n)}\right) = \frac{1}{n}$. In the opposite case, that is, when $\frac{1}{n} > I_{\varphi,\omega}(x)$, we have $k_x(n) = 1$. Finally, recall that if φ satisfies condition $\Delta_2(0)$ and $\varphi(b_\varphi) > 1/\omega(1)$, then the equality $I_{\varphi,\omega}(x) = 1$ holds for any $x \in S_+(\lambda_{\varphi,\omega})$.

Proof of Theorem 3.1. At the beginning we will show the first inequality from inequalities (3.1). Let $x_1, \ldots, x_n$ be a sequence from $S_+(\lambda_{\varphi,\omega})$ such that $x_i \perp x_j$ for

$i \neq j$, $i, j = 1, \ldots, n$. Then we have

$$\begin{aligned}
I_{\varphi,\omega}\left(\frac{\sum_{j=1}^{n} x_j}{k(n)}\right) &= \sum_{i=1}^{\infty} \varphi\left(\left(\frac{\sum_{j=1}^{n} x_j}{k(n)}\right)^{*}(i)\right)\omega(i) \\
&= \sup_{\pi} \sum_{i=1}^{\infty} \varphi\left(\left(\frac{\sum_{j=1}^{n} x_j}{k(n)}\right)(\pi(i))\right)\omega(i) \\
&= \sup_{\pi} \sum_{j=1}^{n}\left(\sum_{i=1}^{\infty} \varphi\left(\left(\frac{x_j}{k(n)}\right)(\pi(i))\right)\omega(i)\right) \\
&\le \sum_{j=1}^{n}\left(\sup_{\pi} \sum_{i=1}^{\infty} \varphi\left(\left(\frac{x_j}{k(n)}\right)(\pi(i))\right)\omega(i)\right) \\
&= \sum_{j=1}^{n}\left(\sum_{i=1}^{\infty} \varphi\left(\left(\frac{x_j}{k(n)}\right)^{*}(i)\right)\omega(i)\right) \\
&= \sum_{j=1}^{n} I_{\varphi,\omega}\left(\frac{x_j}{k(n)}\right) \le \sum_{j=1}^{n} I_{\varphi,\omega}\left(\frac{x_j}{k_{x_j(n)}}\right) \le n \cdot \frac{1}{n} = 1.
\end{aligned}$$

(see (2.4)). Thus $\| \sum_{j=1}^{n} x_i \|_{\varphi,\omega} \le k(n)$, whence by the arbitrariness of the sequence $x_1, \ldots, x_n$ with the properties listed above, we obtain $\mu_n(\lambda_{\varphi,\omega}) \le k(n)$.

Now we will show the second inequality from inequalities (3.1). Let $x \in S_+(\lambda_{\varphi,\omega})$ be arbitrary. Then for any natural number i and any permutation π there is exactly one of two possibilities. Either $x(\pi(i)) \le a_\varphi$ and then

$$\varphi(\alpha_{\varphi,\omega,n} \cdot x(\pi(i))) = 0 = \frac{\varphi(x(\pi(i)))}{n},$$

or $x(\pi(i)) > a_\varphi$, and then, by the definition of $\alpha_{\varphi,\omega,n}$,

$$\varphi(\alpha_{\varphi,\omega,n} \cdot x(\pi(i))) \le \varphi\left(\frac{\varphi^{-1}\left(\frac{\varphi(x(\pi(i)))}{n}\right)}{x(\pi(i))} \cdot x(\pi(i))\right) = \frac{\varphi(x(\pi(i)))}{n}.$$

Therefore,

$$\begin{aligned}
I_{\varphi,\omega}\left(\frac{x}{\frac{1}{\alpha_{\varphi,\omega,n}}}\right) &= I_{\varphi,\omega}\left(\alpha_{\varphi,\omega,n} \cdot x\right) = \sup_{\pi} \sum_{i=1}^{\infty} \varphi(\alpha_{\varphi,\omega,n} \cdot x(\pi(i)))\omega(i) \\
&\le \sup_{\pi} \sum_{i=1}^{\infty} \frac{\varphi(x(\pi(i)))}{n} = \frac{1}{n} \cdot I_{\varphi,\omega}(x) \le \frac{1}{n}.
\end{aligned}$$

Thus, $k_x(n) \le \frac{1}{\alpha_{\varphi,\omega,n}}$ for any x from $S_+(\lambda_{\varphi,\omega})$, which finishes the proof. □

Remark 3.2. In the case when $\omega(i) = \omega(1)$ for any $i \in \mathbb{N}$, that is, when the Orlicz-Lorentz space $\lambda_{\varphi,\omega}$ is the classical Orlicz space l^{ϕ}, where $\phi(u) = \omega(1) \cdot \varphi(u)$ for any $u \ge 0$ (see [28]), we have $\mu_n(\lambda_{\varphi,\omega}) = \mu_n(l^{\phi}) = k(n)$ for any Orlicz function φ and any

$n \in \mathbb{N}$ (for $n = 2$ see [41], for $n > 2$ we can proceed analogously as for $n = 2$). As we will show in Example 3.2 below, the equality

$$\mu_n(\lambda_{\varphi,\omega}) = k(n) \tag{3.2}$$

is not true for any Orlicz-Lorentz space $\lambda_{\varphi,\omega}$. However, since there are Orlicz-Lorentz spaces for which equality (3.2) is satisfied and which are not Orlicz spaces (see Example 3.1), the question about the weakest possible condition which guarantees this equation in these space arises.

Theorem 3.2. *For any Orlicz-Lorentz sequence space $\lambda_{\varphi,\omega}$ and any natural number n there holds the lower estimate*

$$\frac{1}{\alpha'_{\varphi,\omega,n}} = \sup\left\{\frac{\varphi^{-1}\left(\frac{1}{\sum_{i=1}^{k}\omega(i)}\right)}{\varphi^{-1}\left(\frac{1}{\sum_{i=1}^{nk}\omega(i)}\right)} : k = 1, 2, \ldots\right\} \le \mu_n(\lambda_{\varphi,\omega}). \tag{3.3}$$

Proof. We can assume without loss of generality that $\varphi(b_\varphi) \ge \frac{1}{\omega(1)}$ and then for any natural k we define a sequence $x_1^k, x_2^k, \ldots, x_n^k$ of orthogonal elements of $S_+(\lambda_{\varphi,\omega})$, by formulas

$$\begin{aligned}
x_1^k &= \varphi^{-1}\left(\frac{1}{\sum_{i=1}^{k}\omega(i)}\right)e_1 + \ldots + \varphi^{-1}\left(\frac{1}{\sum_{i=1}^{k}\omega(i)}\right)e_k,\\
x_2^k &= \varphi^{-1}\left(\frac{1}{\sum_{i=1}^{k}\omega(i)}\right)e_{k+1} + \ldots + \varphi^{-1}\left(\frac{1}{\sum_{i=1}^{k}\omega(i)}\right)e_{2k},\\
&\vdots\\
x_n^k &= \varphi^{-1}\left(\frac{1}{\sum_{i=1}^{k}\omega(i)}\right)e_{(n-1)k+1} + \ldots + \varphi^{-1}\left(\frac{1}{\sum_{i=1}^{k}\omega(i)}\right)e_{nk}.
\end{aligned}$$

For any natural k we have $I_{\varphi,\omega}\left(x_1^k\right) = I_{\varphi,\omega}\left(x_2^k\right) = \ldots = I_{\varphi,\omega}\left(x_n^k\right) = 1$, so

$$\|x_1^k\|_{\varphi,\omega} = \|x_2^k\|_{\varphi,\omega} = \ldots = \|x_n^k\|_{\varphi,\omega} = 1$$

and

$$I_{\varphi,\omega}\left(\frac{x_1^k + x_2^k + \ldots + x_n^k}{\frac{\varphi^{-1}\left(\frac{1}{\sum_{i=1}^{k}\omega(i)}\right)}{\varphi^{-1}\left(\frac{1}{\sum_{i=1}^{nk}\omega(i)}\right)}}\right) = \sum_{j=1}^{nk}\varphi\left(\varphi^{-1}\left(\frac{1}{\sum_{i=1}^{nk}\omega(i)}\right)\right)\omega(j) = 1,$$

whence

$$\|x_1^k + x_2^k + \ldots + x_n^k\|_{\varphi,\omega} = \frac{\varphi^{-1}\left(\frac{1}{\sum_{i=1}^{k}\omega(i)}\right)}{\varphi^{-1}\left(\frac{1}{\sum_{i=1}^{nk}\omega(i)}\right)}.$$

Finally, by the arbitrariness of natural k, we obtain inequality (3.3). $\square$

Since in many cases the index $\alpha'_{\varphi,\omega,n}$ can be difficult to calculate, the next theorem is very important.

Theorem 3.3. *Let $\sum_{i=1}^{\infty}\omega(i)=\infty$. Then for any Orlicz-Lorentz sequence space $\lambda_{\varphi,\omega}$ and any natural number n there holds the lower estimate*

$$\frac{1}{\alpha^0_{\varphi,n}} \le \mu_n(\lambda_{\varphi,\omega}). \tag{3.4}$$

Proof. Let us take any $\varepsilon>0$. First note that, by the assumption that $\omega=(\omega(i))_{i=1}^{\infty}$ is non-increasing and $\sum_{i=1}^{\infty}\omega(i)=\infty$, we can find $l_1\in\mathbb{N}$ such that

$$(1-\varepsilon)\frac{1}{\sum_{i=1}^{m}\omega(i)} \le \frac{1}{\sum_{i=1}^{m+1}\omega(i)} \tag{3.5}$$

for any $m\ge l_1$. Indeed, denoting by l_1 the smallest natural number such that $\omega(1)\le \varepsilon\sum_{i=1}^{l_1}\omega(i)$, we get

$$\frac{\sum_{i=1}^{m}\omega(i)}{\sum_{i=1}^{m+1}\omega(i)} = 1-\frac{\omega(i+1)}{\sum_{i=1}^{m+1}\omega(i)} \ge 1-\frac{\omega(1)}{\sum_{i=1}^{l_1}\omega(i)} \ge 1-\varepsilon$$

for any $m\ge l_1$. We can also assume, without loss of generality, that $1/\sum_{i=1}^{l_1}\omega(i)\le \varphi(\gamma_{\varphi,\omega})/n$ (for the definition of $\gamma_{\varphi,\omega}$ see formula (2.3)).
By the definition of $\alpha^0_{\varphi,n}$, there exists a sequence $(u_k)_{k=1}^{\infty}$ in $\mathbb{R}$ such that $u_k\searrow 0$ and

$$\frac{\varphi^{-1}(u_k)}{\varphi^{-1}(nu_k)} \le \alpha^0_{\varphi,n}+\varepsilon \tag{3.6}$$

for any $k\in\mathbb{N}$. Since $nu_k\searrow 0$, we can find k_1 such that

$$nu_{k_1} \le \frac{1}{\sum_{i=1}^{l_1}\omega(i)}.$$

Next, we can define $m_1\ge l_1$ as the biggest natural number among these $m\in\mathbb{N}$ for which the equality

$$nu_{k_1} \le \frac{1}{\sum_{i=1}^{m}\omega(i)}$$

is satisfied. By inequality (3.5), we have

$$(1-\varepsilon)\frac{1}{\sum_{i=1}^{m_1}\omega(i)} \le \frac{1}{\sum_{i=1}^{m_1+1}\omega(i)} < nu_{k_1} \le \frac{1}{\sum_{i=1}^{m_1}\omega(i)}.$$

Since $\sum_{i=1}^{\infty}\omega(i)=\infty$, there exists $l_2\ge m_1+1$ such that

$$\sum_{i=m_1+1}^{m}\omega(i) \ge (1-\varepsilon)\sum_{i=1}^{m}\omega(i) \tag{3.7}$$

for any $m\ge l_2$. Next, by $nu_k\searrow 0$, we can find k_2 such that

$$nu_{k_2} \le \frac{1}{\sum_{i=1}^{l_2}\omega(i)}.$$

Analogously as above, by $m_2\ge l_2$ we denote the biggest natural number among these $m\in\mathbb{N}$ for which the equality

$$nu_{k_2} \le \frac{1}{\sum_{i=1}^{m}\omega(i)}$$

is satisfied. Note that, by inequalities (3.5) and (3.7), we get inequalities (3.8) and (3.9) for $j = 2$. Applying again condition $\sum_{i=1}^{\infty} \omega(i) = \infty$, we can find $l_3 \geq m_2 + 1$ such that

$$\sum_{i=m_1+m_2+1}^{m} \omega(i) \geq (1-\varepsilon) \sum_{i=1}^{m} \omega(i)$$

for any $m \geq l_3$. In consequence, in a finite number of steps we can built the sequences $(k_j)_{j=1}^n$ and $(m_j)_{j=1}^n$ such that

$$(1-\varepsilon) \frac{1}{\sum_{i=1}^{m_j} \omega(i)} \leq \frac{1}{\sum_{i=1}^{m_j+1} \omega(i)} < n u_{k_j} \leq \frac{1}{\sum_{i=1}^{m_j} \omega(i)} \tag{3.8}$$

for $j = 1, \ldots, n$ and

$$\sum_{i=m_1+\ldots+m_{j-1}+1}^{m_1+\ldots+m_{j-1}+m_j} \omega(i) \geq (1-\varepsilon) \sum_{i=1}^{m_1+\ldots+m_{j-1}+m_j} \omega(i) \geq (1-\varepsilon) \sum_{i=1}^{m_j} \omega(i) \tag{3.9}$$

for $j = 2, \ldots, n$.
Now we define

$$\begin{aligned}
x_1 &:= \varphi^{-1}\left(\frac{1}{\sum_{i=1}^{m_1} \omega(i)}\right) e_1 + \ldots + \varphi^{-1}\left(\frac{1}{\sum_{i=1}^{m_1} \omega(i)}\right) e_{m_1}, \\
x_2 &:= \varphi^{-1}\left(\frac{1}{\sum_{i=1}^{m_2} \omega(i)}\right) e_{m_1+1} + \ldots + \varphi^{-1}\left(\frac{1}{\sum_{i=1}^{m_2} \omega(i)}\right) e_{m_1+m_2}, \\
&\ \ \vdots \\
x_n &:= \varphi^{-1}\left(\frac{1}{\sum_{i=1}^{m_n} \omega(i)}\right) e_{m_1+\ldots+m_{n-1}+1} + \ldots + \varphi^{-1}\left(\frac{1}{\sum_{i=1}^{m_n} \omega(i)}\right) e_{m_1+\ldots+m_{n-1}+m_n}.
\end{aligned}$$

We have $x_j \geq 0$ and $I_{\varphi,\omega}(x_j) = 1$ for $j = 1, \ldots, n$, whence $x_j \in S_+(\lambda_{\varphi,\omega})$ for the same j. Moreover, by the definition of $x_1, \ldots, x_n$, we get $x_i \perp x_j$ for $i, j = 1, \ldots, n$, $i \neq j$, and $(\sum_{j=1}^n x_j)^* = (\sum_{j=1}^n x_j)$. Hence, by inequality (3.6), we get

$$\begin{aligned}
I_{\varphi,\omega}\left((\alpha_{\varphi,n}^0 + \varepsilon) \sum_{j=1}^n x_j\right) &= \sum_{i=1}^{m_1} \varphi\left((\alpha_{\varphi,n}^0 + \varepsilon) \cdot \varphi^{-1}\left(\frac{1}{\sum_{j=1}^{m_1} \omega(j)}\right)\right) \omega(i) + \ldots \\
&\quad + \sum_{i=m_1+\ldots+m_{n-1}+1}^{m_1+\ldots+m_{n-1}+m_n} \varphi\left((\alpha_{\varphi,n}^0 + \varepsilon) \cdot \varphi^{-1}\left(\frac{1}{\sum_{j=1}^{m_n} \omega(j)}\right)\right) \omega(i) \\
&\geq \sum_{i=1}^{m_1} \varphi\left(\frac{\varphi^{-1}(u_{k_1})}{\varphi^{-1}(n u_{k_1})} \cdot \varphi^{-1}\left(\frac{1}{\sum_{j=1}^{m_1} \omega(j)}\right)\right) \omega(i) + \ldots \\
&\quad + \sum_{i=m_1+\ldots+m_{n-1}+1}^{m_1+\ldots+m_{n-1}+m_n} \varphi\left(\frac{\varphi^{-1}(u_{k_n})}{\varphi^{-1}(n u_{k_n})} \cdot \varphi^{-1}\left(\frac{1}{\sum_{j=1}^{m_n} \omega(j)}\right)\right) \omega(i)
\end{aligned}$$

Since φ^{-1} is an increasing function, we have

$$\varphi^{-1}(nu_{k_i}) \le \varphi^{-1}\left(\frac{1}{\sum_{j=1}^{m_i}\omega(j)}\right)$$

for $i = 1,\dots,n$. Therefore, continuing the previous inequalities, by using the last inequality and inequalities (3.8) and (3.9), we get

$$\begin{aligned}
I_{\varphi,\omega}\left((\alpha^0_{\varphi,n}+\varepsilon)\sum_{j=1}^{n}x_j\right) &\ge \sum_{i=1}^{m_1}\varphi\left(\varphi^{-1}(u_{k_1})\right)\omega(i) \\
&\quad + \dots + \sum_{i=m_1+\dots+m_{n-1}+1}^{m_1+\dots+m_{n-1}+m_n}\varphi\left(\varphi^{-1}(u_{k_n})\right)\omega(i) \\
&= \sum_{i=1}^{m_1}u_{k_1}\cdot\omega(i) + \dots + \sum_{i=m_1+\dots+m_{n-1}+1}^{m_1+\dots+m_{n-1}+m_n}u_{k_n}\cdot\omega(i) \\
&\ge \frac{1-\varepsilon}{n}\left(\sum_{i=1}^{m_1}\frac{1}{\sum_{j=1}^{m_1}\omega(j)}\omega(i) + \dots \right. \\
&\quad \left. + \sum_{i=m_1+\dots+m_{n-1}+1}^{m_1+\dots+m_{n-1}+m_n}\frac{1}{\sum_{j=1}^{m_n}\omega(j)}\omega(i)\right) \\
&\ge \frac{1-\varepsilon}{n}\cdot n\cdot(1-\varepsilon) = (1-\varepsilon)^2.
\end{aligned}$$

So,

$$\left\|\sum_{j=1}^{n}x_j\right\|_{\varphi,\omega} \ge \frac{(1-\varepsilon)^2}{\alpha^0_{\varphi,n}+\varepsilon},$$

whence by the arbitrariness of $\varepsilon > 0$, we get the estimate

$$\mu_n(\lambda_{\varphi,\omega}) \ge \frac{1}{\alpha^0_{\varphi,n}}. \qquad \square$$

Remark 3.3. As we will show in Example 3.1 below, the condition $\sum_{i=1}^{\infty}\omega(i) = \infty$ is not necessary for inequality (3.4) to be satisfied. On the other hand, for any $\varepsilon > 0$ we can find a weighted sequence ω such that for any Orlicz function φ, we have $\mu_n(\lambda_{\varphi,\omega}) \le 1+\varepsilon$ for any $n \in \mathbb{N}$ (see Example 3.2), and then the estimate (3.4) is not satisfied whenever $1+\varepsilon < \frac{1}{\alpha^0_{\varphi,n}}$.

Example 3.1. Let $\varphi(u) = u^p$ for $u \ge 0$, where $1 \le p < \infty$. It is easy to show that

$$\frac{1}{\alpha^0_{\varphi,n}} = \sqrt[p]{n} \quad \text{and} \quad \frac{1}{\alpha_{\varphi,\omega,n}} = \sqrt[p]{n}.$$

Therefore, by Theorem 3.1 we have $\mu_n(\lambda_{\varphi,\omega}) \le \sqrt[p]{n}$. Additionally, if $\sum_{i=1}^{\infty}\omega(i) = \infty$, by Theorem 3.3, we get

$$\mu_n(\lambda_{\varphi,\omega}) = \sqrt[p]{n}. \tag{3.10}$$

The above equality holds also in the case when $\omega(1) = \ldots = \omega(n)$. Indeed, under this assumption, we have

$$\frac{1}{\alpha'_{\varphi,\omega,n}} \geq \frac{\varphi^{-1}\left(\frac{1}{\sum_{i=1}^{1}\omega(i)}\right)}{\varphi^{-1}\left(\frac{1}{\sum_{i=1}^{n}\omega(i)}\right)} = \frac{\frac{1}{\sqrt[p]{\omega(1)}}}{\frac{1}{\sqrt[p]{n\omega(1)}}} = \sqrt[p]{n},$$

whence by Theorem 3.2, we get equality (3.10). In general, equality (3.10) need not occur, as the next example shows.

Example 3.2. Now let ε be arbitrary positive number and let us take a weighted sequence $(\omega(i))$ such that $\sum_{i=1}^{\infty}\omega(i) \leq (1+\varepsilon)\omega(1)$, for example $\omega(i) = (\varepsilon \cdot \omega(1))/2^i$ for $i \geq 2$. Then for any Orlicz function φ and any sequence $x_1, \ldots, x_n$ from $S_+(\lambda_{\varphi,\omega})$, $x_i \perp x_j$ for $i \neq j$, $i, j = 1, \ldots, n$, we obtain

$$\begin{aligned} I_{\varphi,\omega}\left(\frac{\sum_{j=1}^{n} x_j}{1+\varepsilon}\right) &= \sum_{i=1}^{\infty}\varphi\left(\left(\frac{\sum_{j=1}^{n} x_j}{1+\varepsilon}\right)^{*}(i)\right)\omega(i) \leq \sum_{i=1}^{\infty}\varphi\left(\frac{\gamma_{\varphi,\omega}}{1+\varepsilon}\right)\omega(i) \\ &\leq \frac{\varphi(\gamma_{\varphi,\omega})}{1+\varepsilon}\sum_{i=1}^{\infty}\omega(i) \leq \frac{\varphi(\gamma_{\varphi,\omega})}{1+\varepsilon}(1+\varepsilon)\omega(1) \leq 1, \end{aligned}$$

whence $\mu_n(\lambda_{\varphi,\omega}) \leq 1+\varepsilon$. This example shows that for any Orlicz function φ there is a weighted sequence $(\omega(i))$ such that $\mu_n(\lambda_{\varphi,\omega})$ is arbitrarily close to 1.

In the next example we will construct an Orlicz function for which $\alpha^0_{\varphi,n} > \alpha_{\varphi,\omega,n}$.

Example 3.3. For the Orlicz function φ defined by

$$\varphi(u) = \begin{cases} u^2 & \text{for } u \in [0,1], \\ 2u-1 & \text{for } u > 1, \end{cases}$$

we have

$$\varphi^{-1}(v) = \begin{cases} \sqrt{v} & \text{for } v \in [0,1], \\ \frac{v+1}{2} & \text{for } v > 1, \end{cases}$$

and, in consequence,

$$g(v) := \frac{\varphi^{-1}(v)}{\varphi^{-1}(nv)} = \begin{cases} \frac{1}{\sqrt{n}} & \text{for } v \in (0, \frac{1}{n}], \\ \frac{2\sqrt{v}}{nv+1} & \text{for } v \in (\frac{1}{n}, 1), \\ \frac{v+1}{nv+1} & \text{for } v \geq 1. \end{cases}$$

Note that the function g is continuous and non-increasing (decreasing for $v \geq \frac{1}{n}$). We have $\alpha^0_{\varphi,n} = \frac{1}{\sqrt{n}}$ and simultaneously, by the fact that the function g in non-increasing,

$$\alpha_{\varphi,\omega,n} = \frac{\varphi^{-1}\left(\frac{\varphi(\gamma_{\varphi,\omega})}{n}\right)}{\gamma_{\varphi,\omega}},$$

where $\gamma_{\varphi,\omega}$ is defined by formula (2.3). Since $b_\varphi = \infty$, so $\gamma_{\varphi,\omega}$ is just the number satisfying the equality $\varphi(\gamma_{\varphi,\omega}) = \frac{1}{\omega(1)}$. Therefore, if $\omega(1) \leq \frac{1}{n}$, then

$$\varphi(\gamma_{\varphi,\omega}) \geq n,\ \gamma_{\varphi,\omega} = \frac{(1/\omega((1))+1}{2} \geq \frac{n+1}{2} \text{ and } \alpha_{\varphi,\omega,n} = \frac{2\gamma_{\varphi,\omega}+n-1}{2n\gamma_{\varphi,\omega}}.$$

In the case when $\omega(1) \in (\frac{1}{n}, 1)$, we have $\varphi(\gamma_{\varphi,\omega}) \in (1, n)$, $\gamma_{\varphi,\omega} = \frac{(1/\omega((1))+1}{2} \in (1, \frac{n+1}{2})$ and $\alpha_{\varphi,\omega,n} = \frac{\sqrt{2\gamma_{\varphi,\omega}-1}}{\gamma_{\varphi,\omega}\sqrt{n}}$. Finally, for $\omega(1) \geq 1$, we get $\varphi(\gamma_{\varphi,\omega}) \leq 1$, $\gamma_{\varphi,\omega} = \frac{1}{\sqrt{\omega(1)}} \leq 1$ and $\alpha_{\varphi,\omega,n} = \frac{1}{\sqrt{n}}$. Therefore, for $\omega(1) < 1$, we get $\alpha_{\varphi,\omega,n} < \alpha^0_{\varphi,n}$.

In the next theorem we will use the following

Lemma 3.1 ([6] Lemma 1.1)**.** *For a given Orlicz function φ, the following assertions are equivalent:*

(i) *The function ψ, complementary to φ in the sense of Young* (*see formula* (2.2)), *satisfies condition $\Delta_2(\mathbb{R}_+)$.*

(ii) *There exist $a > 1$ and $k \in (0, 1)$ such that*

$$\varphi\left(\frac{u}{a}\right) \leq \frac{k}{a}\varphi(u)$$

for any $u \in \mathbb{R}$.

(iii) *For any $a > 1$ there exists $\xi > 1$ such that*

$$\varphi\left(\frac{\xi u}{a}\right) \leq \frac{1}{\xi a}\varphi(u) \tag{3.11}$$

for any $u \in \mathbb{R}$.

Theorem 3.4. *Let ψ denote the function complementary in the sense of Young to an Orlicz function φ* (*see formula* (2.2))*. Then the following statements hold true:*

(i) *If ψ satisfies condition $\Delta_2(0)$, then $\mu_n(\lambda_{\varphi,\omega}) < n$ for any $n \in \mathbb{N}$.*

(ii) *Let $\sum_{i=1}^{\infty} \omega(i) = \infty$. If ψ does not satisfy condition $\Delta_2(0)$, then $\mu_n(\lambda_{\varphi,\omega}) = n$ for any $n \in \mathbb{N}$.*

Proof. (*i*). Let $n \geq 2$ be any fixed natural number. If $a_\varphi > 0$, then we define on the interval $[0, \gamma_{\varphi,\omega} - a_\varphi]$ a new convex and increasing function φ_1, by $\varphi_1(u) = \varphi(u + a_\varphi)$. Since $\varphi^{-1}(v) = \varphi_1^{-1}(v) + a_\varphi$ for any $v \in [0, \varphi(\gamma_{\varphi,\omega})]$, we obtain

$$\begin{aligned}
\frac{\varphi^{-1}(v)}{\varphi^{-1}(nv)} &= \frac{\varphi_1^{-1}(v) + a_\varphi}{\varphi_1^{-1}(nv) + a_\varphi} \geq \frac{\frac{1}{n} \cdot \varphi_1^{-1}(nv) + a_\varphi}{\varphi_1^{-1}(nv) + a_\varphi} \\
&= \frac{1}{n} \cdot \frac{\varphi_1^{-1}(nv) + a_\varphi}{\varphi_1^{-1}(nv) + a_\varphi} + \frac{n-1}{n} \cdot \frac{a_\varphi}{\varphi_1^{-1}(nv) + a_\varphi} \geq \frac{1}{n} + \frac{n-1}{n} \cdot \frac{a_\varphi}{\gamma_{\varphi,\omega}}
\end{aligned}$$

for any $v \in [0, \varphi(\gamma_{\varphi,\omega})/n]$. In consequence, by Theorem 3.1, $\mu_n(\lambda_{\varphi,\omega}) < n$.

Now let $a_\varphi = 0$. First we will show that if ψ satisfies condition $\Delta_2(0)$, then there exists $b = b(n) \in (0, 1)$ such that the inequality

$$\varphi\left(\frac{1}{nb}u\right) \leq \frac{1}{n}\varphi(u) \tag{3.12}$$

holds for any $u \in [0, \gamma_{\varphi,\omega}]$.

Note that if ψ satisfies condition $\Delta_2(\mathbb{R}_+)$, then putting $a = n$ and $\xi = \frac{1}{b}$ in inequality (3.11), we get inequality (3.12).

Assume now that ψ does not satisfy condition $\Delta_2(\mathbb{R}_+)$. For any fixed $0 < v_0 < b_\psi$, we have $0 < r(v_0) < \infty$, where r denotes the right derivative of ψ. We define a new Orlicz function $\bar{\psi}$ by the formulas

$$\bar{r}(s) = \begin{cases} r(s) & \text{for } s \in [0, v_0], \\ \frac{r(v_0)}{v_0} \cdot s & \text{for } s \in (v_0, \infty) \end{cases} \qquad \text{and} \qquad \bar{\psi}(v) = \int_0^v \bar{r}(s)ds.$$

Since $\bar{\psi}(v) = \psi(v)$ for $v \in [0, v_0]$ and ψ satisfies condition $\Delta_2(0)$, the function $\bar{\psi}$ satisfies condition $\Delta_2(\mathbb{R}_+)$. So, analogously as above we get that the function $\bar{\varphi}$, where $\bar{\varphi}(u) = \int_0^u \bar{p}(t)dt$ and $\bar{p}(t) = \sup\{s : \bar{r}(s) \le t\}$, satisfies inequality (3.12) for some $\bar{b} = \bar{b}(n) \in (0, 1)$. Since $r(s) = \bar{r}(s)$ for $u \in [0, v_0]$, we get $p(t) = \bar{p}(t)$ for $t \in [0, r(v_0))$ and, in consequence, $\varphi(u) = \bar{\varphi}(u)$ for $u \in [0, r(v_0)]$. Therefore, if $r(v_0) \ge \gamma_{\varphi,\omega}$, then φ satisfies inequality (3.12) with $b = \bar{b}$.

Let now $r(v_0) < \gamma_{\varphi,\omega}$. Note that the function f defined on the interval $[\frac{1}{2n}, 1 - \frac{1}{2n}]$, by the formula

$$f(c) = \sup_{u \in [r(v_0), \gamma_{\varphi,\omega}]} \frac{\varphi(cu)}{c\varphi(u)},$$

is continuous on this interval (more precisely the function $\sup_{u \in [r(v_0), \gamma_{\varphi,\omega}]} \frac{\varphi(cu)}{\varphi(u)}$ is convex and has finite values, so in consequence, the function f(c) is continuous) and $f(c) < 1$ for any $c \in [\frac{1}{2n}, 1 - \frac{1}{2n}]$. Thus, there exists $\hat{b} \in (\frac{2}{3}, 1)$ such that $f(c) \le \hat{b}$ for any $c \in [\frac{1}{2n}, 1 - \frac{1}{2n}]$. Hence, we have

$$\varphi(cu) \le \hat{b}c\varphi(u)$$

for any $u \in [r(v_0), \gamma_{\varphi,\omega}]$ and any $c \in [\frac{1}{2n}, 1 - \frac{1}{2n}]$. In particular, for $\hat{c} = \frac{1}{\hat{b}n}$, we have $\hat{c} \in (\frac{1}{n}, 1 - \frac{1}{2n})$ and

$$\varphi\left(\frac{1}{\hat{b}n}u\right) = \varphi(\hat{c}u) \le \hat{b}\hat{c}\varphi(u) = \frac{1}{n}\varphi(u)$$

for any $u \in [r(v_0), \gamma_{\varphi,\omega}]$. So, inequality (3.12) holds for any $u \in [0, \gamma_{\varphi,\omega}]$ for $b = \max(\bar{b}, \hat{b})$.

Finally, we will prove the inequality $\mu_n(\lambda_{\varphi,\omega}) < n$. Substituting in (3.12) $w = \varphi(u)$, we get

$$\varphi\left(\frac{1}{nb}\varphi^{-1}(w)\right) \le \frac{1}{n}w$$

for $w \in [0, \varphi(\gamma_{\varphi,\omega})]$, whence

$$\frac{1}{nb}\varphi^{-1}(w) \le \varphi^{-1}\left(\frac{1}{n}w\right),$$

for the same w. In turn, denoting $v = \frac{w}{n}$, we have

$$\frac{1}{nb}\varphi^{-1}(nv) \le \varphi^{-1}(v)$$

for $v \in [0, \varphi(\gamma_{\varphi,\omega})/n]$. Hence $\alpha_{\varphi,\omega,n} \ge \frac{1}{nb}$ and, by Theorem 3.1, $\mu_n(\lambda_{\varphi,\omega}) \le nb < n$.

(ii) Let $n \ge 2$ be a fixed natural number. In order to show the equality $\mu_n(\lambda_{\varphi,\omega}) = n$, by Theorem 3.3, we need only to prove that $\alpha^0_{\varphi,n} = \frac{1}{n}$.

By concavity of φ^{-1}, we have $\varphi^{-1}(v) \geq \frac{1}{n}\varphi^{-1}(nv)$ for any $v \in [0, b_\varphi)$, whence

$$\alpha^0_{\varphi,n} = \liminf_{v\to 0} \frac{\varphi^{-1}(v)}{\varphi^{-1}(nv)} \geq \frac{1}{n}. \tag{3.13}$$

Since the function ψ does not satisfy condition $\Delta_2(0)$, we have $a_\varphi = 0$ (it is easy to show that if $a_\varphi > 0$, then ψ satisfies condition $\Delta_2(0)$). Moreover, by Lemma 3.1, we can find a sequence of positive numbers $(u_m)_{m=3}^\infty$ such that $u_m < b_\varphi$, $\lim_{m\to\infty} u_m = 0$ and

$$\varphi\left(\frac{u_m}{n\left(1-\frac{1}{m}\right)}\right) \geq \frac{1-\frac{1}{m}}{n\left(1-\frac{1}{m}\right)}\varphi(u_m) = \frac{\varphi(u_m)}{n}$$

for any $m \geq 3$. Putting $w_m = \varphi(u_m)$, $m = 2, 3, \ldots$, we obtain

$$\varphi\left(\frac{\varphi^{-1}(w_m)}{n\left(1-\frac{1}{m}\right)}\right) \geq \frac{w_m}{n},$$

whence

$$\frac{\varphi^{-1}(w_m)}{n\left(1-\frac{1}{m}\right)} \geq \varphi^{-1}\left(\frac{w_m}{n}\right).$$

Denoting $v_m = \frac{w_m}{n}$, $m = 3, 4, \ldots$, we have

$$\frac{\varphi^{-1}(v_m)}{\varphi^{-1}(nv_m)} \leq \frac{1}{n\left(1-\frac{1}{m}\right)}. \tag{3.14}$$

By inequalities (3.13) and (3.14), we get $\alpha^0_{\varphi,n} = \frac{1}{n}$. $\square$

3.2. **The case of the Amemiya norm.** Now, we will work on estimates of $\mu_n(X)$ in case, when X is the Orlicz-Lorentz sequence space equipped with the Amemiya norm, that is, $X = \lambda^A_{\varphi,\omega} = (\lambda_{\varphi,\omega}, \|\cdot\|^A_{\varphi,\omega})$.

In order to get possibly precise upper and lower estimates of the numbers $\mu_n(\lambda^A_{\varphi,\omega})$, we define for any triple (φ, ω, n), $(n \in \mathbb{N})$ another indices, namely

$$\begin{aligned} \beta'_{\psi,\omega,n} &= \sup\left\{\frac{\sum_{i=1}^{nk}\omega(i)}{\sum_{i=1}^{k}\omega(i)} \cdot \frac{\psi^{-1}\left(\frac{1}{\sum_{i=1}^{nk}\omega(i)}\right)}{\psi^{-1}\left(\frac{1}{\sum_{i=1}^{k}\omega(i)}\right)} : k = 1, 2, \ldots\right\}, \\ \beta^0_{\psi,n} &= \limsup_{u\to 0} \frac{\psi^{-1}\left(\frac{\psi(u)}{n}\right)}{u} = \limsup_{v\to 0} \frac{\psi^{-1}(v)}{\psi^{-1}(nv)}. \end{aligned}$$

where ψ denotes as before the complementary function of φ in the sense of Young (see formula (2.2)). It is easy to show that for any $n \in \mathbb{N}$ we have

$$\frac{1}{\alpha^0_{\varphi,n}} = n\beta^0_{\psi,n}.$$

Recall that an Orlicz function φ is an N-function at 0 if $a_\varphi = 0$ and $\lim_{u\to 0} \varphi(u)/u = 0$. Analogously, an Orlicz function φ is an N-function at ∞ if $b_\varphi = \infty$ and $\lim_{u\to\infty} \varphi(u)/u = \infty$. Finally, we say that φ is an N-function, if it is simultaneously an N-function at 0 and an N-function at ∞. We start with the following

Lemma 3.2. *Let φ be an N-function at ∞. Then for any $x \in \lambda_{\varphi,\omega}^{A}$ we have the following formula*

$$\|x\|_{\varphi,\omega}^{A} = \inf_{k>0} \frac{1}{k}(1 + I_{\varphi,\omega}(kx)) = \sup\left\{\sum_{i=1}^{\infty} x^*(i)\, y^*(i)\, \omega(i) \colon I_{\psi,\omega}(y) \leq 1\right\}.$$

Proof. Let us take any $y \in \lambda_{\psi,\omega}$ with $I_{\psi,\omega}(y) \leq 1$. Then for any $k > 0$ we have

$$\begin{aligned} \sum_{i=1}^{\infty} x^*(i)y^*(i)\omega(i) &= \frac{1}{k}\sum_{i=1}^{\infty} k \cdot x^*(i)y^*(i)\omega(i) \\ &\leq \frac{1}{k}\sum_{i=1}^{\infty} \left(\varphi(kx^*(i)) + \psi(y^*(i))\right)\omega(i) \\ &\leq \frac{1}{k}\left(1 + I_{\varphi,\omega}(kx)\right), \end{aligned}$$

whence

$$\sup\left\{\sum_{i=1}^{\infty} x^*(i)\, y^*(i)\, \omega(i) \colon I_{\psi,\omega}(y) \leq 1\right\} \leq \|x\|_{\varphi,\omega}^{A}.$$

Now, we will prove the opposite inequality. By the assumption that φ is N-function at ∞, there exist a constant $k_1 > 0$ and a non-increasing sequence $s = (s(i))_{i=1}^{\infty}$ with $s(i) \in [l(k_1 x^*(i)), p(k_1 x^*(i))]$ for $i \in \mathbb{N}$ (where l and p denote the left and the right derivatives of φ, respectively) satisfying $I_{\psi,\omega}(s) = 1$. Then

$$\begin{aligned} \sum_{i=1}^{\infty} x^*(i)s(i)\omega(i) &= \frac{1}{k_1}\sum_{i=1}^{\infty} k_1 \cdot x^*(i)s(i)\omega(i) \\ &= \frac{1}{k_1}\sum_{i=1}^{\infty} \left(\varphi(k_1x^*(i)) + \psi(s(i))\right)\omega(i) \\ &= \frac{1}{k_1}\left(1 + I_{\varphi,\omega}(k_1x)\right) \geq \|x\|_{\varphi,\omega}^{A}. \end{aligned}$$

Therefore, the desired equality has been proved. □

Lemma 3.3. *Let φ be an N-function and $n \in \mathbb{N}$ be arbitrary and fixed. Then for any $B \subset \mathbb{N}$ such that* $\mathrm{m(B) = n}$ (*where* m *denotes the counting measure*), *we obtain*

$$\|\chi_B\|_{\varphi,\omega}^{A} = \sum_{i=1}^{n} \psi^{-1}\left(\frac{1}{\sum_{j=1}^{n}\omega(j)}\right)\omega(i). \tag{3.15}$$

Proof. First we assume additionally that φ is differentiable. Then we have

$$\begin{aligned} \|\chi_B\|_{\varphi,\omega}^{A} &= \inf_{k>0}\frac{1}{k}(1 + I_{\psi,\omega}(k\chi_B)) = \inf_{k>0}\frac{1}{k}\left(1 + \sum_{i=1}^{\infty}\varphi(k(\chi_B)^*(i))\omega(i)\right) \\ &= \inf_{k>0}\frac{1}{k}\left(1 + \sum_{i=1}^{n}\varphi(k)\omega(i)\right) = \inf_{k>0}\frac{1}{k}\left(1 + \varphi(k)\sum_{i=1}^{n}\omega(i)\right). \end{aligned}$$

Let us define the function $f(0,\infty) \to (0,\infty)$ by

$$f(k) = \frac{1}{k}\left(1 + \varphi(k)\sum_{i=1}^{n}\omega(i)\right)$$

We need to find its infimum. We have

$$f'(k) = \frac{k \cdot \varphi'(k)\sum_{i=1}^{n}\omega(i) - (1 + \varphi(k)\sum_{i=1}^{n}\omega(i))}{k^2},$$

so $f'(k_0) = 0$ if and only if

$$k_0 \cdot \varphi'(k_0)\sum_{i=1}^{n}\omega(i) = 1 + \varphi(k_0)\sum_{i=1}^{n}\omega(i) = 1 + (k_0\varphi'(k_0) - \psi(\varphi'(k_0)))\sum_{i=1}^{n}\omega(i),$$

which is equivalent to the equation $\psi(\varphi'(k_0))\sum_{i=1}^{n}\omega(i) = 1$, that is,

$$\varphi'(k_0) = \psi^{-1}\left(\frac{1}{\sum_{i=1}^{n}\omega(i)}\right).$$

Hence

$$\begin{aligned}
\|\chi_B\|_{\varphi,\omega}^{A} &= \frac{1}{k_0}\left(1 + \varphi(k_0)\sum_{i=1}^{n}\omega(i)\right) = \frac{1}{k_0}\left(1 + (k_0\varphi'(k_0) - \psi(\varphi'(k_0)))\sum_{i=1}^{n}\omega(i)\right) \\
&= \frac{1}{k_0}\left(1 + k_0\varphi'(k_0)\sum_{i=1}^{n}\omega(i) - 1\right) = \varphi'(k_0)\sum_{i=1}^{n}\omega(i) \\
&= \psi^{-1}\left(\frac{1}{\sum_{i=1}^{n}\omega(i)}\right)\sum_{i=1}^{n}\omega(i).
\end{aligned}$$

which finishes this part of the proof.

Assume now that φ is an arbitrary N-function. Then for any $\varepsilon \in (0,1)$, we can find an Orlicz function φ_ε such that φ_ε is smooth and whole $\mathbb{R}$ and

$$\varphi_\varepsilon((1-\varepsilon)u) \le \varphi(u) \le \varphi_\varepsilon((1+\varepsilon)u)$$

for any $u \in \mathbb{R}$. Consequently

$$(1-\varepsilon)\|x\|_{\varphi_\varepsilon,\omega}^{A} \le \|x\|_{\varphi,\omega}^{A} \le (1+\varepsilon)\|x\|_{\varphi_\varepsilon,\omega}^{A}$$

for any $x \in \lambda_{\varphi,\omega}^{A}$. In particulary, for any $\varepsilon \in (0,1)$ we obtain

$$\begin{aligned}
&\frac{1-\varepsilon}{1+\varepsilon}\sum_{i=1}^{n}\psi^{-1}\left(\frac{1}{\sum_{j=1}^{n}\omega(j)}\right)\omega(i) \le \|\chi_B\|_{\varphi,\omega}^{A} \\
&\le \frac{1+\varepsilon}{1-\varepsilon}\sum_{i=1}^{n}\psi^{-1}\left(\frac{1}{\sum_{j=1}^{n}\omega(j)}\right)\omega(i)
\end{aligned}$$

Finally, by the arbitrarines of $\varepsilon \in (0,1)$, we get the equality (3.15) $\square$

Theorem 3.5. *For any Orlicz-Lorentz sequence space $\lambda^A_{\varphi,\omega}$ and any natural number n, there holds the following upper estimate*

$$\mu_n(\lambda^A_{\varphi,\omega}) \le d(n), \tag{3.16}$$

where

$$d(n) := \inf\{d_k(n)\colon k > 1\}, \quad d_k(n) := \sup\left\{d_{k,x}(n)\colon x \in S_+(\lambda^A_{\varphi,\omega})\right\}$$

and

$$d_{k,x}(n) := \inf\left\{d > 0\colon I_{\varphi,\omega}\left(\frac{kx}{d}\right) \le \frac{k-1}{n}\right\}.$$

Proof. Let $x_1, \ldots, x_n$ be a sequence from $S_+(\lambda^A_{\varphi,\omega})$ such that $x_i \perp x_j$ for $i \neq j$, $i, j = 1, \ldots, n$. Then for any $k > 1$, we have

$$\begin{aligned}
\left\|\frac{\sum_{i=1}^n x_i}{d_k(n)}\right\|^A_{\varphi,\omega} &\le \frac{1}{k}\left(1 + I_{\varphi,\omega}\left(\frac{k\sum_{i=1}^n x_i}{d_k(n)}\right)\right)\\
&\le \frac{1}{k}\left(1 + \sum_{i=1}^n I_{\varphi,\omega}\left(\frac{k\cdot x_i}{d_k(n)}\right)\right)\\
&\le \frac{1}{k}\left(1 + \sum_{i=1}^n I_{\varphi,\omega}\left(\frac{k\cdot x_i}{d_{k,x}(n)}\right)\right)\\
&\le \frac{1}{k}\left(1 + n\cdot\frac{k-1}{n}\right) = 1.
\end{aligned}$$

Therefore, $\|\sum_{i=1}^n x_i\|^A_{\varphi,\omega} \le d_k(n)$ for any $k > 1$, whence $\|\sum_{i=1}^n x_i\|^A_{\varphi,\omega} \le d(n)$. Consequently, by the arbitrariness of the sequence $x_1, \ldots, x_n$, we obtain $\mu_n(\lambda^A_{\varphi,\omega}) \le d(n)$. $\square$

Theorem 3.6. *Let φ be an N-function. Then for any Orlicz-Lorentz sequence space $\lambda^A_{\varphi,\omega}$ and any natural number n there holds the lower estimate*

$$\beta'_{\psi,\omega,n} \le \mu_n(\lambda^A_{\varphi,\omega}). \tag{3.17}$$

Proof. Let $k \in \mathbb{N}$ be arbitrary and let a sequence $x_1^k, x_2^k, \ldots, x_n^k$ of orthogonal elements be defined by the formulas

$$\begin{aligned}
x_1^k &= \psi^{-1}\left(\frac{1}{\sum_{i=1}^k \omega(i)}\right)e_1 + \ldots + \psi^{-1}\left(\frac{1}{\sum_{i=1}^k \omega(i)}\right)e_k,\\
x_2^k &= \psi^{-1}\left(\frac{1}{\sum_{i=1}^k \omega(i)}\right)e_{k+1} + \ldots + \psi^{-1}\left(\frac{1}{\sum_{i=1}^k \omega(i)}\right)e_{2k},\\
&\vdots\\
x_n^k &= \psi^{-1}\left(\frac{1}{\sum_{i=1}^k \omega(i)}\right)e_{(n-1)k+1} + \ldots + \psi^{-1}\left(\frac{1}{\sum_{i=1}^k \omega(i)}\right)e_{nk}.
\end{aligned}$$

We have $I_{\psi,\omega}(x_j^k) = 1$ for $j = 1, 2, \ldots, n$ and

$$I_{\psi,\omega}\left(\frac{\psi^{-1}\left(\frac{1}{\sum_{i=1}^{nk}\omega(i)}\right)}{\psi^{-1}\left(\frac{1}{\sum_{i=1}^{k}\omega(i)}\right)}(x_1^k + x_2^k + \ldots + x_n^k)\right) = \sum_{j=1}^{nk}\psi\left(\psi^{-1}\left(\frac{1}{\sum_{i=1}^{nk}\omega(i)}\right)\right)\omega(j) = 1.$$

Next, we define

$$\begin{aligned} y_1^k &:= \frac{1}{\sum_{i=1}^{k}\psi^{-1}\left(\frac{1}{\sum_{j=1}^{k}\omega(j)}\right)\omega(i)}(e_1 + \ldots + e_k), \\ y_2^k &:= \frac{1}{\sum_{i=1}^{k}\psi^{-1}\left(\frac{1}{\sum_{j=1}^{k}\omega(j)}\right)\omega(i)}(e_{k+1} + \ldots + e_{2k}), \\ &\vdots \\ y_n^k &:= \frac{1}{\sum_{i=1}^{k}\psi^{-1}\left(\frac{1}{\sum_{j=1}^{k}\omega(j)}\right)\omega(i)}(e_{(n-1)k+1} + \ldots + e_{nk}). \end{aligned}$$

By Lemma 3.3 we have $\|y_j^k\|_{\varphi,\omega}^A = 1$ for any $j = 1, \ldots, n$. Moreover, $y_i^k \perp y_j^k$ for $i, j = 1, \ldots, n$, $i \neq j$ and $(\sum_{j=1}^n y_j^k)^* = (\sum_{j=1}^n y_j^k)$. Thus, by Lemma 3.2, we get

$$\begin{aligned} \left\|\sum_{j=1}^n y_j^k\right\|_{\varphi,\omega}^A &\geq \sum_{i=1}^{nk}\left(\sum_{j=1}^n y_j^k\right)(i)\cdot\left(\frac{\psi^{-1}\left(\frac{1}{\sum_{l=1}^{nk}\omega(l)}\right)}{\psi^{-1}\left(\frac{1}{\sum_{l=1}^{k}\omega(l)}\right)}\sum_{j=1}^n x_j^k\right)(i)\cdot\omega(i) \\ &= \sum_{i=1}^{nk}\frac{1}{\sum_{l=1}^{k}\psi^{-1}\left(\frac{1}{\sum_{j=1}^{k}\omega(j)}\right)\omega(l)}\cdot\frac{\psi^{-1}\left(\frac{1}{\sum_{l=1}^{nk}\omega(l)}\right)}{\psi^{-1}\left(\frac{1}{\sum_{l=1}^{k}\omega(l)}\right)}\cdot\psi^{-1}\left(\frac{1}{\sum_{j=1}^{k}\omega(j)}\right)\omega(i) \\ &= \frac{\psi^{-1}\left(\frac{1}{\sum_{l=1}^{nk}\omega(l)}\right)}{\psi^{-1}\left(\frac{1}{\sum_{l=1}^{k}\omega(l)}\right)}\cdot\frac{\sum_{i=1}^{nk}\omega(i)}{\sum_{i=1}^{k}\omega(i)}. \end{aligned}$$

Finally, by the arbitrariness of $k \in \mathbb{N}$, we obtain inequality (3.17). □

Theorem 3.7. *Let φ be an N-function and $\sum_{i=1}^\infty \omega(i) = \infty$. Then for any Orlicz-Lorentz sequence space $\lambda_{\varphi,\omega}^A$ and any natural number n there holds the lower estimate*

$$\frac{1}{\alpha_{\varphi,n}^0} = n\beta_{\psi,n}^0 \leq \mu_n(\lambda_{\varphi,\omega}^A). \tag{3.18}$$

Proof. The proof of this theorem is based on some ideas from the proof of Theorem 3.3. Since they are applied in a different context it is fully presented. Let us take any $\varepsilon > 0$. Analogously as in Theorem 3.3, by the assumption that $\omega = (\omega(i))_{i=1}^\infty$ is non-increasing and $\sum_{i=1}^\infty \omega(i) = \infty$, we can find $l_1 \in \mathbb{N}$ such that

$$(1-\varepsilon)\frac{1}{\sum_{i=1}^m \omega(i)} \leq \frac{1}{\sum_{i=1}^{m+1}\omega(i)} \tag{3.19}$$

for any $m \geq l_1$. By the definition of $\beta^0_{\psi,n}$ there exists a sequence $(v_k)_{k=1}^{\infty}$ in $\mathbb{R}_+$ such that $v_k \searrow 0$ and

$$\beta^0_{\psi,n} - \varepsilon \leq \frac{\psi^{-1}(v_k)}{\psi^{-1}(nv_k)} \tag{3.20}$$

for any $k \in \mathbb{N}$. Since $nv_k \searrow 0$, we can find k_1 such that

$$nv_{k_1} \leq \frac{1}{\sum_{i=1}^{l_1} \omega(i)}.$$

Analogously as in the proof of Theorem 3.3, we denote by m_1 the biggest natural number among these $m \in \mathbb{N}$ for which $m \geq l_1$ and the equality

$$nu_{k_1} \leq \frac{1}{\sum_{i=1}^{m} \omega(i)}$$

is satisfied. By inequality (3.19), we have

$$(1-\varepsilon)\frac{1}{\sum_{i=1}^{m_1} \omega(i)} \leq \frac{1}{\sum_{i=1}^{m_1+1} \omega(i)} < nv_{k_1} \leq \frac{1}{\sum_{i=1}^{m_1} \omega(i)}.$$

By $\sum_{i=1}^{\infty} \omega(i) = \infty$, there exists $l_2 > h_1 := m_1 + 1$ such that

$$\sum_{i=h_1+1}^{m} \omega(i) \geq (1-\varepsilon)\sum_{i=1}^{m} \omega(i)$$

for any $m \geq l_2$. Next, by $nv_k \searrow 0$, we can find k_2 such that

$$nv_{k_2} \leq \frac{1}{\sum_{i=1}^{l_2} \omega(i)}.$$

Analogously as above, we denote by m_2 the biggest natural number among these $m \in \mathbb{N}$ for which $m \geq l_2$ and the equality

$$nv_{k_2} \leq \frac{1}{\sum_{i=1}^{m} \omega(i)}$$

is satisfied. Applying again condition $\sum_{i=1}^{\infty} \omega(i) = \infty$, we can find $l_3 > h_2 := m_2 + 1$ such that

$$\sum_{i=h_1+h_2+1}^{m} \omega(i) \geq (1-\varepsilon)\sum_{i=1}^{m} \omega(i)$$

for any $m \geq l_3$. Finally, in a finite number of steps we can built the sequences $(k_j)_{j=1}^{n}$, $(m_j)_{j=1}^{n}$ and $(h_j)_{j=1}^{n}$ such that $h_j := m_j + 1$ and

$$(1-\varepsilon)\frac{1}{\sum_{i=1}^{m_j} \omega(i)} \leq \frac{1}{\sum_{i=1}^{m_j+1} \omega(i)} < nu_{k_j} \leq \frac{1}{\sum_{i=1}^{m_j} \omega(i)} \tag{3.21}$$

for $j = 1, \ldots, n$ and

$$\sum_{i=h_1+\ldots+h_{j-1}+1}^{h_1+\ldots+h_{j-1}+h_j} \omega(i) \geq (1-\varepsilon) \sum_{i=1}^{h_1+\ldots+h_{j-1}+h_j} \omega(i) \geq (1-\varepsilon)\sum_{i=1}^{h_j} \omega(i) \tag{3.22}$$

for $j = 2, \ldots, n$. Now, we define

$$\begin{aligned}
x_1 &:= \psi^{-1}\left(\frac{1}{\sum_{i=1}^{h_1}\omega(i)}\right)e_1 + \ldots + \psi^{-1}\left(\frac{1}{\sum_{i=1}^{h_1}\omega(i)}\right)e_{h_1},\\
x_2 &:= \psi^{-1}\left(\frac{1}{\sum_{i=1}^{h_2}\omega(i)}\right)e_{h_1+1} + \ldots + \psi^{-1}\left(\frac{1}{\sum_{i=1}^{h_2}\omega(i)}\right)e_{h_1+h_2},\\
&\vdots\\
x_n &:= \psi^{-1}\left(\frac{1}{\sum_{i=1}^{h_n}\omega(i)}\right)e_{h_1+\ldots+h_{n-1}+1} + \ldots + \psi^{-1}\left(\frac{1}{\sum_{i=1}^{h_n}\omega(i)}\right)e_{h_1+\ldots+h_{n-1}+h_n}.
\end{aligned}$$

Then $I_{\psi,\omega}(x_j) = 1$ for any $j = 1, \ldots, n$, $x_i \perp x_j$ for $i, j = 1, \ldots, n$, $i \neq j$ and $(\sum_{j=1}^n x_j)^* = (\sum_{j=1}^n x_j)$. So, by inequalities (3.20) and (3.21), we have

$$\begin{aligned}
&I_{\psi,\omega}\left((1-\varepsilon)(\beta^0_{\psi,n}-\varepsilon)\sum_{j=1}^n x_j\right)\\
=\ &\sum_{i=1}^{h_1}\psi\left((1-\varepsilon)(\beta^0_{\psi,n}-\varepsilon)\cdot\psi^{-1}\left(\frac{1}{\sum_{j=1}^{h_1}\omega(j)}\right)\right)\omega(i)+\ldots\\
&+\sum_{i=h_1+\ldots+h_{n-1}+1}^{h_1+\ldots+h_{n-1}+h_n}\psi\left((1-\varepsilon)(\beta^0_{\psi,n}-\varepsilon)\cdot\psi^{-1}\left(\frac{1}{\sum_{j=1}^{h_n}\omega(j)}\right)\right)\omega(i)\\
\leq\ &\sum_{i=1}^{h_1}\psi\left((1-\varepsilon)\cdot\frac{\psi^{-1}(v_{k_1})}{\psi^{-1}(nv_{k_1})}\cdot\psi^{-1}\left(\frac{1}{\sum_{j=1}^{h_1}\omega(j)}\right)\right)\omega(i)+\ldots\\
&+\sum_{i=h_1+\ldots+h_{n-1}+1}^{h_1+\ldots+h_{n-1}+h_n}\psi\left((1-\varepsilon)\cdot\frac{\psi^{-1}(v_{k_n})}{\psi^{-1}(nv_{k_n})}\cdot\psi^{-1}\left(\frac{1}{\sum_{j=1}^{h_n}\omega(j)}\right)\right)\omega(i)\\
\leq\ &\sum_{i=1}^{h_1}\psi\left((1-\varepsilon)\psi^{-1}(v_{k_1})\right)\omega(i)+\ldots+\sum_{i=h_1+\ldots+h_{n-1}+1}^{h_1+\ldots+h_{n-1}+h_n}\psi\left((1-\varepsilon)\psi^{-1}(v_{k_n})\right)\omega(i)\\
\leq\ &\sum_{i=1}^{h_1}(1-\varepsilon)\psi\left(\psi^{-1}(v_{k_1})\right)\omega(i)+\ldots+\sum_{i=h_1+\ldots+h_{n-1}+1}^{h_1+\ldots+h_{n-1}+h_n}(1-\varepsilon)\psi\left(\psi^{-1}(v_{k_n})\right)\omega(i)\\
\leq\ &\sum_{i=1}^{h_1}(1-\varepsilon)v_{k_1}\omega(i)+\ldots+\sum_{i=h_1+\ldots+h_{n-1}+1}^{h_1+\ldots+h_{n-1}+h_n}(1-\varepsilon)v_{k_n}\omega(i)\\
\leq\ &\sum_{i=1}^{h_1}\left(\frac{1}{n\sum_{j=1}^{h_1}\omega(j)}\right)\omega(i)+\ldots+\sum_{i=h_1+\ldots+h_{n-1}+1}^{h_1+\ldots+h_{n-1}+h_n}\left(\frac{1}{n\sum_{j=1}^{h_n}\omega(i)}\right)\omega(j)\leq 1.
\end{aligned}$$

Let now

$$
\begin{aligned}
y_1 &:= \frac{1}{\sum_{i=1}^{h_1} \psi^{-1}\left(\frac{1}{\sum_{j=1}^{h_1}\omega(j)}\right)\omega(i)}(e_1+\ldots+e_{h_1}),\\
y_2 &:= \frac{1}{\sum_{i=1}^{h_2} \psi^{-1}\left(\frac{1}{\sum_{j=1}^{h_2}\omega(j)}\right)\omega(i)}(e_{h_1+1}+\ldots+e_{h_1+h_2}),\\
&\vdots\\
y_n &:= \frac{1}{\sum_{i=1}^{h_n} \psi^{-1}\left(\frac{1}{\sum_{j=1}^{h_n}\omega(j)}\right)\omega(i)}(e_{h_1+\ldots+h_{n-1}+1}+\ldots+e_{h_1+\ldots+h_{n-1}+h_n}).
\end{aligned}
$$

By Lemma 3.3 we have $\|y_j\|^A_{\varphi,\omega} = 1$ for any $j = 1,\ldots,n$. Moreover, $y_i \perp y_j$ for $i,j = 1,\ldots,n$, $i \neq j$ and $(\sum_{j=1}^n y_j)^* = (\sum_{j=1}^n y_j)$. Thus, by Lemma 3.2 and inequality (3.22), we get

$$
\begin{aligned}
\left\|\sum_{j=1}^n y_j\right\|^A_{\varphi,\omega} &\geq \sum_{i=1}^{h_1+\ldots+h_n}\left(\sum_{j=1}^n y_j\right)(i)\cdot\left((1-\varepsilon)(\beta^0_{\psi,n}-\varepsilon)\sum_{j=1}^n x_j\right)(i)\cdot\omega(i)\\
&= \sum_{i=1}^{h_1} \frac{(1-\varepsilon)(\beta^0_{\psi,n}-\varepsilon)\cdot\psi^{-1}\left(\frac{1}{\sum_{j=1}^{h_1}\omega(j)}\right)}{\sum_{i=1}^{h_1}\psi^{-1}\left(\frac{1}{\sum_{j=1}^{h_1}\omega(j)}\right)\omega(i)}\ \omega(i)+\ldots\\
&\quad+\sum_{i=h_1+\ldots+h_{n-1}+1}^{h_1+\ldots+h_{n-1}+h_n} \frac{(1-\varepsilon)(\beta^0_{\psi,n}-\varepsilon)\cdot\psi^{-1}\left(\frac{1}{\sum_{j=1}^{h_n}\omega(j)}\right)}{\sum_{i=1}^{h_n}\psi^{-1}\left(\frac{1}{\sum_{j=1}^{h_n}\omega(j)}\right)\omega(i)}\ \omega(i)\\
&\geq (1-\varepsilon)^2(\beta^0_{\psi,n}-\varepsilon)\left(\frac{\sum_{i=1}^{h_1}\psi^{-1}\left(\frac{1}{\sum_{j=1}^{h_1}\omega(j)}\right)\omega(i)}{\sum_{i=1}^{h_1}\psi^{-1}\left(\frac{1}{\sum_{j=1}^{h_1}\omega(j)}\right)\omega(i)}+\ldots\right.\\
&\quad\left.+\frac{\sum_{i=1}^{h_n}\psi^{-1}\left(\frac{1}{\sum_{j=1}^{h_n}\omega(j)}\right)\omega(i)}{\sum_{i=1}^{h_n}\psi^{-1}\left(\frac{1}{\sum_{j=1}^{h_n}\omega(j)}\right)\omega(i)}\right)\\
&= n(1-\varepsilon)^2(\beta^0_{\psi,n}-\varepsilon).
\end{aligned}
$$

Finally, by the arbitrariness of $\varepsilon > 0$, we obtain inequality (3.18). $\square$

In Theorem 3.4 it was shown that if ψ does not satisfy condition $\Delta_2(0)$, then $\alpha^0_{\varphi,n} = \frac{1}{n}$. Therefore, by Theorem 3.7, we obtain the following

Theorem 3.8. *Let* $\sum_{i=1}^{\infty}\omega(i)=\infty$. *If* ψ *does not satisfy condition* $\Delta_2(0)$, *then* $\mu_n(\lambda_{\varphi,\omega}^A)=n$ *for any* $n\in\mathbb{N}$.

4. Some application to the fixed point theory

Recall, that in [4] Borwein and Sims proved that if a Banach lattice $X=(X,\|\cdot\|_X)$ is weakly orthogonal, that is,

$$\lim_{n\to\infty}\||x_n|\wedge|x|\|_X=0$$

for any weakly null sequence (x_n) in X and any $x\in X$, and if $\mu_2(X)<2$ ($\mu_2(X)$ is called the Riesz angle of X), then X has the weak fixed point property. We also refer the readers to the paper [36], where more about various weak orthogonalities in Banach lattices and their relationships to the fixed point property can be found.

Since under the assumption that $\sum_{i=1}^{\infty}\omega(i)=\infty$ the necessary and sufficient condition for $\mu_2(\lambda_{\varphi,\omega})<2$ are known (see Theorem 3.4), we need only to know when the spaces $\lambda_{\varphi,\omega}$ are weakly orthogonal. We even prove more general result for Köthe sequence spaces with the semi Fatou property. Recall that a Köthe sequence space X has the semi Fatou property if $0\le x_n\le x\in X$ and $x_n\nearrow x$ coordinatewise, then $\|x_n\|_X\nearrow\|x\|_X$. An element $x\in X$ is said to be order continuous if for any sequence (x_n) in X_+ (the positive cone in X) with $0\le x_n\le|x|$ and $x_n\to 0$ coordinatewise there holds $\|x_n\|\to 0$. The subspace X_a of all order continuous elements in X is an order ideal in X. The space X is called order continuous if $X_a=X$ (see [29]).

Theorem 4.1. *A Köthe sequence spaces* X *with the semi-Fatou property is weakly orthogonal if and only if it is order continuous.*

Proof. Sufficiency. Take any $x\in X$ and any weakly null sequence (x_n) in X and choose arbitrary $\varepsilon>0$. Since X is order continuous there exists $i_0\in\mathbb{N}$ such that $\|\sum_{i=i_0+1}^{\infty}|x(i)|e_i\|_X<\varepsilon$.

Since weakly null sequences in Köthe sequence spaces are also coordinatewise null sequences, there exists $n_0\in\mathbb{N}$ such that $\|\sum_{i=1}^{i_0}|x_n(i)|e_i\|_X<\varepsilon$ whenever $n\ge n_0$. Hence

$$\begin{aligned}\||x_n|\wedge|x|\|_X &\le \left\|\sum_{i=1}^{i_0}|x_n(i)|e_i+\sum_{i=i_0+1}^{\infty}|x(i)|e_i\right\|_X\\ &\le \left\|\sum_{i=1}^{i_0}|x_n(i)|e_i\right\|_X+\left\|\sum_{i=i_0+1}^{\infty}|x(i)|e_i\right\|_X\\ &< 2\varepsilon\end{aligned}$$

for all $n\ge n_0$. By the arbitrariness of $\varepsilon>0$ we have $\lim_{n\to\infty}\||x_n|\wedge|x|\|_X=0$.

Necessity. Assume that X is not order continuous. The Riesz lemma says that for any $0<\varepsilon<1$ there exists $x\in S(X)$ such that

$$\|[x]\|_{X\setminus X_a}=d(x,X_a)>1-\varepsilon.$$

Let us apply this lemma for $\varepsilon = \frac{1}{3}$ and let $x \in S(X)$ be such that $d(x, X_a) > \frac{2}{3}$. By the semi-Fatou property of X, there exists $i_1 \in \mathbb{N}$ such that

$$\frac{1}{2} < \left\| \sum_{i=1}^{i_1} x(i)e_i \right\|_X \leq 1.$$

Set $x_1 = \sum_{i=1}^{i_1} x(i)e_i$. Since $x_1 \in X_a$, we have that $d(x - x_1, X_a) = d(x, X_a) > \frac{2}{3}$. So there exists $i_2 \in \mathbb{N}$, $i_2 > i_1$ such that

$$\frac{1}{2} < \left\| \sum_{i=i_1+1}^{i_2} x(i)e_i \right\|_X \leq 1.$$

Continuing this process by induction, one can find a strictly increasing sequence (i_n) of natural numbers such that

$$\frac{1}{2} < \left\| \sum_{i=i_{n-1}+1}^{i_n} x(i)e_i \right\|_X \leq 1,$$

for any $n \in \mathbb{N}$ with $i_0 = 0$. Denoting

$$x_n = \sum_{i=i_{n-1}+1}^{i_n} x(i)e_i$$

we can easily prove that (x_n) is a weakly null sequence in X. Let us take arbitrary $x^* \in X^*$. Then there exist a sequence $y \in (X_a)'$ (the Köthe dual of X_a) and a singular functional $\varrho \in X^*$, that is, $\varrho(z) = 0$ for any $z \in X_a$, such that

$$x^* = \xi_y + \varrho,$$

where

$$\xi_y(z) = \sum_{i=i}^{\infty} y(i)z(i)$$

for any $z \in X$. Since $x = \sum_{n=1}^{\infty} x_n$, we have

$$|\xi_y(x)| = \left| \sum_{n=1}^{\infty} \xi_y(x_n) \right| \leq \sum_{n=1}^{\infty} \xi_{|y|}(|x_n|) = \xi_{|y|}(|x|) < \infty,$$

whence we conclude that the series $\sum_{n=1}^{\infty} \xi_y(x_n)$ converges. Therefore $\xi_y(x_n) \to 0$ as $n \to \infty$. In consequence (x_n) is a weakly null sequence. However, we have

$$\||x_n| \wedge |x|\|_X = \||x_n|\|_X > \frac{1}{2}$$

for any $n \in \mathbb{N}$. Therefore, X is not weakly orthogonal. □

Recall, that a weight sequence ω is said to be regular, if there exists $\eta > 0$ such that $\sum_{i=1}^{2n} \omega(i) \geq (1 + \eta) \sum_{i=1}^{n} \omega(i)$ for any $n \in \mathbb{N}$ (see [19]). It is easy to show that if ω is regular, then $\sum_{i=1}^{\infty} \omega(i) = \infty$.

Theorem 4.2. *If φ is an Orlicz function such that both φ and ψ satisfy condition $\Delta_2(0)$ and the weighted sequence $\omega = (\omega(i))$ is regular, then the space $\lambda_{\varphi,\omega}$ has the fixed point property.*

Proof. Under the assumptions of the theorem, we have reflexivity of the space $\lambda_{\varphi,\omega}$ (see [10]), so it is enough to prove that the assumptions imply the weak fixed point property of $\lambda_{\varphi,\omega}$. Let us recall that by Theorem 3.4, the assumptions imply that $\mu_2(\lambda_{\varphi,\omega}) < 2$. By Theorem 2.4 from [17] and Theorem 4.1 the assumptions give that the space $\lambda_{\varphi,\omega}$ is weakly orthogonal. In consequence, by virtue of the result of Borwein and Sims, recalled at the begining of this section, the space $\lambda_{\varphi,\omega}$ has the weak fixed point property. $\square$

Example 4.1. Let $\varphi(u) = u^2 \ln(|u| + 1)$ for all $u \in \mathbb{R}$ and ω be regular. Then

$$\lim_{u\to 0} \frac{\varphi(2u)}{\varphi(u)} = 8 \ \text{ and } \ \lim_{u\to 0} \frac{\varphi(\frac{u}{2})}{\varphi(u)} = \frac{1}{8}.$$

Hence we get that $\varphi \in \Delta_2(0)$ and $\psi \in \Delta_2(0)$, whence by Theorem 4.2 the Orlicz-Lorentz space $\lambda_{\varphi,\omega}$ has the fixed point property.

Now we define $T : \lambda_{\varphi,\omega} \to \lambda_{\varphi,\omega}$ by

$$T(x) = (\ln(1 + |x(1)|), \ln(1 + |x(2)|), \ldots)$$

for any $x = (x(i)) \in \lambda_{\varphi,\omega}$. Since $\ln(1 + u) \le u$ for any $u \ge 0$, by properties of the rearrangement and the fact that the Luxemburg norm is monotone, we have that $T : B(\lambda_{\varphi,\omega}) \to B(\lambda_{\varphi,\omega})$, where $B(\lambda_{\varphi,\omega})$ is the unit ball of $\lambda_{\varphi,\omega}$. Moreover, for any $0 < x_1 < x_2$, we have

$$\ln(1 + x_2) - \ln(1 + x_1) = \frac{1}{1 + \xi}(x_2 - x_1)$$

for some $\xi \in (x_1, x_2)$. Thus, for any $x_1, x_2 \in (0, \infty)$ we get

$$|\ln(1 + x_2) - \ln(1 + x_1)| \le |x_2 - x_1|,$$

whence using properties of the rearrangement and the Luxemburg norm again, we obtain

$$\|T(x) - T(y)\|_{\varphi,\omega} \le \|x - y\|_{\varphi,\omega}$$

for any $B(\lambda_{\varphi,\omega})$. Therefore T is a non-expansive operator, whence it has a fixed point in $B(\lambda_{\varphi,\omega})$.

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Received: July 18, 2016; Accepted: December 22, 2016.