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ORDER-CLUSTERED FIXED POINT THEOREMS AND THEIR APPLICATIONS TO PARETO EQUILIBRIUM PROBLEMS

LINSEN XIE*, JINLU LI**, ADRIAN PETRUŞEL*** AND JEN-CHIH YAO****

*Department of Mathematics, Lishui University, Lishui, Zhejiang 323000, China
E-mail: linsenxie@163.com

**Department of Mathematics, Shawnee State University, Portsmouth, Ohio 45662, USA
E-mail: jli@shawnee.edu

***Babeş-Bolyai University, Cluj-Napoca, Romania
E-mail: petrusel@math.ubbcluj.ro

**** Center for General Education, China Medical University, Taichung 40402, Taiwan
E-mail: yaojc@mail.cmu.edu.tw

Abstract. In this paper, we provide some properties of order clusters in preordered sets and we prove some order-clustered fixed point theorems on preordered sets. Then by applying these theorems, we show the existence of ordered Pareto equilibrium and Nash equilibrium for some noncooperative strategic games with incomplete (preordered) preferences.
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1. Introduction

In game theory, economic theory, sciences, engineers, etc., there are some problems, in which, either the inputs, or the outputs, or both, of some considered operations cannot be totally ordered. For example, the payoffs (utilities) of the players in some games with incomplete preferences and the preferences of the agents on the productions or endowments in some economies may not be characterized by real valued functions. In economic theory, such utilities or preferences are called non-normal. In [8], some real life examples of games with non-normal (incomplete) preferences are provided. In [11], the Pareto equilibrium problems in strategic games with partially ordered preferences was studied. Some existence theorems were proved by fixed point theorem on posets.

In this paper, we go a step further to extend the Pareto equilibrium problems with partially ordered preferences studied in [11] to Pareto equilibrium problems with preordered preferences. We consider that, in decision theory or game theory, sometimes the decision makers have indifferent preferences on some distinct elements.

Mathematically, it leads authors to study the properties of preordered sets. For example, when we consider the positions of the objects in the real spaces with a given 3-d rectangular coordinate system, if only the horizontal distances from the origin and the altitudes of the objects matter for consideration in this study, then the decision makers of this problem face a preordered set, which is more precisely described below:

Let $R_3 = \{(p, q, t) : p, q, t \in R)\}$ be the 3-d Euclidean space. A binary relation on R_3, denoted by $\succeq_C$, is defined as below: for any $(p_1, q_1, t_1), (p_2, q_2, t_2) \in R_3$, put

$$(p_2, q_2, t_2) \succeq_C (p_1, q_1, t_1), \text{ if and only if, } p_2^2 + q_2^2 \geq p_1^2 + q_1^2 \text{ and } t_2 \geq t_1.$$

$(R_3, \succeq_C)$ is a preordered set. Then the decision makers' preferences of all objects on a given horizontal circle with center 0 in the plane $t = 0$ are considered as indifferent with respect to the preordered preference relation $\succeq_C$ on R_3.

In noncooperative strategic games, if the set of payoffs for the players are totally (completely) ordered, which can be represented by real valued functions, then fixed point theorems in topological spaces or metric spaces have been the essential tools for the proofs of the existence of Nash equilibria or Pareto equilibria of these games (see [2], [6], [12], [15]). As we further study strategic games with incomplete (partially ordered or preordered) preferences, the concepts of Nash equilibria and Pareto equilibria will be extended to ordered Nash equilibria and ordered Pareto equilibria, respectively. The existence of ordered equilibrium can be similarly proved by applying fixed point theorems on partially ordered or preordered sets (see [1], [3], [7-12], [14]). For this reason, fixed point theorems for both single-valued and set-valued mappings on ordered sets have been developed (see [3], [5], [8-11], and [14]), which will be frequently used in this paper.

This paper is organized as follows: in Section 2, we investigate the properties of order-clusters in preordered sets, which are used for the definitions of order-clustered fixed points in the following sections; in Section 3, we prove several order-clustered fixed point theorems for set-valued mappings on preordered sets, and provide the properties of the collections of the order-clustered fixed points; in Section 4, we apply these results to prove the existence of ordered Pareto equilibria for some noncooperative strategic games with incomplete (preordered) preferences.

2. Order-clusters in preordered sets

Throughout the whole paper, we closely follow the notations and definitions in order theory from [1], [3-4], [8], [12] and [14].

Let $(P, \succeq)$ be a preordered set (It is worthy to mention for clarification that a preordered set $(P, \succeq)$ equipped with the preorder $\succeq$ on P is called a partially ordered system (p.o.s.) in [5]). An $\succeq$ -totally ordered (linear ordered) subset C of P is said to be a chain in P. Let A be a nonempty subset of P. The preordered set $(A, \succeq)$ is said to be inductively ordered or inductive if and only if, every chain of elements of A has an upper bound in A. It is a set fulfilling the assumption of Zorn's lemma. $(A, \succeq)$ is said to be chain-complete, whenever every chain of elements of A has a supremum in A.

Let $(P, \succeq)$ be a preordered set. For any $x, y \in P$, we say that x, y are $\succeq$-order equivalent ($\succeq$-order indifference), which is denoted by $x \sim y$, whenever both $x \succeq y$ and $y \succeq x$ hold. It is clear that $\sim$ is an equivalent relation on P. For any $x \in P$, let $[x]$ denote the order equivalent class (order indifference class) containing x, which is called a $\succeq -$*cluster* (or simply an *order cluster*, or a *cluster*, if there is no confusion caused). Let $P/\sim$ or $\widetilde{P}$ denote the collection of all order clusters in the preordered set $(P, \succeq)$. So $x \in [x] \in \widetilde{P}$, for every $x \in P$.

Then the ordering relation $\succeq$ on P naturally induces an ordering relation $\succeq^{\widetilde{P}}$ on $\widetilde{P}$ as following: for every $[x], [y] \in \widetilde{P}$,

$$[x] \succeq^{\widetilde{P}} [y], \text{ if and only if }, x \succeq y. \tag{2.1}$$

That is equivalent to

$$[x] \succeq^{\widetilde{P}} [y], \text{if and only if, } x' \succeq y', \text{ for any } x' \in [x] \text{ and for any } y' \in [y]. \tag{2.2}$$

Proposition 2.1. *For any given preordered set* $(P, \succeq), (\widetilde{P}, \succeq^{\widetilde{P}})$ *is a poset, which is called the cluster poset of the preordered set* $(P, \succeq)$.

The partial order $\succeq^{\widetilde{P}}$ on $\widetilde{P}$ is said to be induced by the preorder $\succeq$ on P.

In game theory and decision theory, sometimes, the utilities of the decision making on order indifference elements from the input set are also order indifferent in the output set. It implies that, in order theory, there are some useful mappings between two preordered sets which map order indifference elements in the domain to order indifference elements in the range. It leads us to give the following definition.

Definition 2.2. Let $(X, \succeq^X)$ and $(U, \succeq^U)$ be two preordered sets and let $F : X \to 2^U \backslash \{\varnothing\}$ be a set-valued mapping. F is said to be order indifference, whenever, $x \sim^X y$ implies

$$u \sim^U v, \text{ for any } u \in F(x) \text{ and } v \in F(y), \text{ for } x, y \in X. \tag{2.3}$$

In particular, if $F : X \to U$ is a single-valued mapping, then F is order indifference, whenever

$$x \sim^X y \text{ implies } F(x) \sim^U F(y). \tag{2.4}$$

Definition 2.3. Let $(X, \succeq^X)$ and $(U, \succeq^U)$ be two preordered sets and let $F : X \to 2^U \backslash \{\varnothing\}$ be a set-valued mapping. F is said to be order cluster-preserving, whenever, $x \sim^X y$ implies

$$\{[u] : u \in F(x)\} = \{[v] : v \in F(y)\}. \tag{2.5}$$

In particular, if $F : X \to U$ is a single-valued mapping, then F is order cluster-preserving, whenever

$$x \sim^X y \text{ implies } F(x) \sim^U F(y). \tag{2.6}$$

It is worthy to note that for any order cluster-preserving set-valued mapping $F : X \to 2^U \backslash \{\varnothing\}$, for any given $u \in F(x)$, it is not necessary to have $[u] \subseteq F(x)$.

Let $(X, \succeq^X)$ and $(U, \succeq^U)$ be preordered sets and let $F : X \to 2^U \backslash \{\varnothing\}$ be a set-valued mapping. F is said to be isotone or order-increasing upward whenever $x \preceq^X y$ in X implies that, for any $z \in F(x)$, there is an element $w \in F(y)$ such that $z \preceq^U w$. F is said to be order-increasing downward whenever if $x \preceq^X y$ in X implies that, for any $w \in F(y)$, there is an element $z \in F(x)$ such that $z \preceq^U w$. If F is both of order-increasing upward and downward, then F is said to be order-increasing.

In particular, a single valued mapping $F : X \to U$ is said to be order-increasing, whenever, $x \preceq^X y$ in X implies $F(x) \preceq^U F(y)$ in U. An order-increasing single valued mapping $F : X \to U$ is said to be strictly order-increasing whenever $x \prec^X y$ implies $F(x) \prec^U F(y)$.

Lemma 2.4. *Let $(X, \succeq^X)$ and $(U, \succeq^U)$ be two preordered sets and let $F : X \to 2^U \backslash \{\varnothing\}$ be a set-valued mapping, then*

(i) *F is an order indifference mapping $\Longrightarrow$ F is order cluster-preserving. The reverse is not true.*

(ii) *An order cluster-preserving set-valued mapping F is order indifference if and only if the set $\{[u] : u \in F(x)\}$ is a singleton.*

Next we provide some counterexamples to show that there are some order cluster-preserving mappings which are not order indifference, in both cases of set-valued and single-valued.

Example 2.5. Let $R_2 = \{(s, t) : s, t \in R\}$ be the 2-d Euclidean space. We define a binary relation on R_2, denoted by $\succeq_l$, as below: for any $(s_1, t_1), (s_2, t_2) \in R_2$,

$$(s_2, t_2) \succeq_l (s_1, t_1), \text{ if and only if}, s_2 \geq s_1.$$

One can check that the relation $\succeq_l$ is a preordered on R_2; and therefore $(R_2, \succeq_l)$ is a preordered set. Every $\succeq_l$-cluster in $(R_2, \succeq_l)$ is a vertical line in R_2. We define a set-valued mapping $F : X \to 2^{R_2} \backslash \{\varnothing\}$ as: for any point $(s, t) \in R_2$,

$$F(s, t) = \{(s + p, t) : p \geq 0\}. \tag{2.7}$$

Note that for any $(s, t) \in R_2$, the $\succeq_l$-cluster $[(s, t)]$ is the vertical line passing through point (s, t) in R_2. From (2.7), for any $(s, t) \sim_l (s, t')$, we have

$$\begin{aligned} &\{[(u, v)] : (u, v) \in F(s, t)\} = \{[(s + p, t)] : p \geq 0\} \\ =&\{[(s + p, t')] : p \geq 0\} = \{[(u, v)] : (u, v) \in F(s, t')\}. \end{aligned} \tag{2.8}$$

From (2.5) in Definition 2.3, (2.8) implies that F is an order cluster-preserving set-valued mapping on $(R_2, \succeq_l)$. For every $(s, t) \in R_2$, it is clearly to see that $\{[(u, v)] : (u, v) \in F(s, t)\}$ is not a singleton. Then from part 2 in Lemma 2.4, F is not an order indifference set-valued mapping. More precisely, we can directly show that F is not an order indifference set-valued mapping. To this end, we take any $(s, t) \sim_l (s, t')$, and take $p_2 > p_1 > 0$. Then $(s + p_2, t) \in F(s, t)$ and $(s + p_1, t') \in F(s, t')$. It is clear to see that $(s + p_2, t) \not\sim_l (s + p_1, t')$.

The following result is about single-valued mappings.

Lemma 2.6. *Let $(X, \succeq^X)$ and $(U, \succeq^U)$ be two preordered sets and let $F : X \to U$ be a single-valued mapping. Then F is order indifference if and only if, it is order cluster-preserving.*

Proof. The proof follows part (ii) in Lemma 2.4. □

Next we provide two counterexamples to show that the order-increasing property is not a sufficient condition for a set-valued mapping to be order cluster-preserving.

Example 2.7. Let the preordered set $(R_2, \succeq_l)$ be defined as in Example 2.5. We define a set-valued mapping $F : R_2 \to 2^{R_2}\backslash\{\varnothing\}$ as: for any point $(s,t) \in R_2$,

$$F(s,t) = \{(s+2n, t) : n = 0, \pm1, \pm2, \ldots\}, \text{ for } t \geq 0; \tag{2.9}$$

and

$$F(s,t) = \{(s+2n+1, t) : n = 0, \pm1, \pm2, \ldots\}, \text{ for } t < 0.$$

One can show that F is an $\succeq_l$-increasing set-valued mapping (both upward and downward). Note that for every $n = 0, \pm1, \pm2, \ldots$, the $\succeq_l$-cluster $[(s+n, t)]$ is a subset of R_2, which can be represented by the vertical line in R_2 passing through the point $(s+n, t)$, that is,

$$[(s+n,t)] = \{(s+n, p) : p \in R\}. \tag{2.10}$$

Then, for any $t_1 \geq 0$ and $t_2 < 0$, we have $(s, t_1) \sim_l (s, t_2)$. On the other hand, from (2.9)-(2.10), it follows

$$\{[u] : u \in F(s,t_1)\} = \{[(s+2n, t_1] : n = 0, \pm1, \pm2, \ldots\} =$$

$$\{\{(s+2n, p) : p \in R\} : n = 0, \pm1, \pm2, \ldots\}$$

and

$$\{[u] : u \in F(s,t_2)\} = \{[(s+2n+1, t_2] : n = 0, 1, 2, \ldots\} =$$

$$\{\{(s+2n+1, p) : p \in R\} : n = 0, 1, 2, \ldots\}.$$

Since for any given real number s, the equation $s + 2n = s + 2m + 1$ is equivalent to $2n = 2m + 1$, which does not have integral solution for m and n. It implies that $\{[u] : u \in F(s,t_1)\} \neq \{[u] : u \in F(s,t_2)\}$. Hence F is not order cluster-preserving.

Example 2.8. Let the preordered set $(R_2, \succeq_l)$ be defined as in Example 2.5. We define a set-valued mapping $F : R_2 \to 2^{R_2}\backslash\{\varnothing\}$ as: for any point $(s,t) \in R_2$,

$$F(s,t) = \{(s,t), (s+1/3, t), (s+1, t)\}, \text{ for } t \geq 0; \tag{2.11}$$

and

$$F(s,t) = \{(s,t), (s+1/2, t), (s+1, t)\}, \text{ for } t < 0. \tag{2.12}$$

One can show that F is an $\succeq_l$-increasing set-valued mapping (both upward and downward) with values of finite completed ordered subsets in $(R_2, \succeq_l)$. For any $t_1 \geq 0$ and $t_2 < 0$, we have $(s, t_1) \sim_l (s, t_2)$. By (2.11) and (2.12) we get

$$\{[u] : u \in F(s,t_1)\} = \{[(s,t)], [(s+1/3, t)], [(s+1, t)]\},$$

and

$$\{[u] : u \in F(s,t_2)\} = \{[(s,t)], [(s+1/2, t)], [(s+1, t)]\}.$$

It implies $\{[u] : u \in F(s,t_1)\} \neq [u] : u \in F(s,t_2)$. Hence F is not order cluster-preserving.

3. Fixed point and order-clustered fixed point theorems for set-valued mappings on preordered sets

Definition 3.1. Let $(P, \succeq)$ be a preordered set and let $F : P \to 2^P \backslash \{\varnothing\}$ be a set-valued mapping. An element $x \in P$ is called an $\succeq$-clustered fixed point (or simply a clustered fixed point) of F, whenever there is a $u \in [x]$ such that $u \in F(x)$.

In particular for single-valued mappings, Definition 3.1 turns to be

Definition 3.2. Let $(P, \succeq)$ be a preordered set and let $F : P \to P$ be a single-valued mapping. An element $x \in X$ is called an $\succeq$-clustered fixed point of F, whenever there is a $v \in [x]$ such that $v = F(x)$; that is, $x \sim F(x)$.

The set of fixed points of F is denoted by $\mathcal{F}(F)$ and the set of $\succeq$-clustered fixed points of F is denoted by $\mathcal{F}^{\mathcal{O}}(F)$.

Since $x \sim x$, then from Definitions 3.1 and 3.2, we have that, in both cases of set-valued and single-valued mappings, that if x is a fixed point of F, then x is an $\succeq -$clustered fixed point of F.

There are some connections between fixed points and order-clustered fixed points of mappings. We list them as a lemma below.

Lemma 3.3. *Let $(P, \succeq)$ be a preordered set and let F be a mapping on P, single-valued or set-valued. Then F has the following properties:*

(i) $\mathcal{F}(F) \subseteq \mathcal{F}^{\mathcal{O}}(F)$.

(ii) *The order-clustered fixed points of F can be characterized as:*

$$x \in \mathcal{F}^{\mathcal{O}}(F) \text{ if and only if } [x] \cap F(x) \neq \varnothing. \tag{3.1}$$

(iii) *If $(P, \succeq)$ is a poset, then $\mathcal{F}(F) = \mathcal{F}^{\mathcal{O}}(F)$.*

The inverse of the implication in part (i) of Lemma 3.3 does not hold. Hence order-clustered fixed points are generalizations of the fixed points. If F is an order cluster-preserving set-valued mapping, then the order-clustered fixed points of F can be characterized by fixed points as below:

Lemma 3.4. *Let $(P, \succeq)$ be a preordered set with its induced order cluster poset $(\widetilde{P}, \succeq^{\widetilde{P}})$. Let $F : P \to 2^P \backslash \{\varnothing\}$ be an order cluster-preserving set-valued mapping. Let $\widetilde{F} : P \to 2^{\widetilde{P}} \backslash \{\varnothing\}$ be a set-valued mapping defined by*

$$\widetilde{F}([x]) = \{[u] : u \in F(x)\}, \text{ for all } [x] \in \widetilde{P}. \tag{3.2}$$

Then, for any given element $x \in P$,

$$x \in \mathcal{F}^{\mathcal{O}}(F), \text{ if and only if } [x] \in \mathcal{F}(\widetilde{F}).$$

Proof. Since F is an order cluster-preserving set-valued mapping, from (2.5), $\widetilde{F}$ is well defined.

"$\Longrightarrow$" For an arbitrary order-clustered fixed point $x \in \mathcal{F}^{\mathcal{O}}(F)$, there is $u \in [x]$ such that $u \in F(x)$. It implies $[u] \in \widetilde{F}([x])$. Since $[x] = [u]$, it follows that $[x] \in \widetilde{F}([x])$, and therefore, $[x] \in \mathcal{F}(\widetilde{F})$.

"$\Longleftarrow$" Suppose that $[x] \in \mathcal{F}(\widetilde{F}) = \{[u] : u \in F(x)\}$. It implies that there is $u \in F(x)$ with $[u] = [x] \in \widetilde{F}([x])$. That is, $u \sim x$ satisfying $u \in F(x)$. Hence $x \in \mathcal{F}^{\mathcal{O}}(F)$. □

We recall some results from [9].

Corollary 3.2. (see [9]) *Let $(P, \succeq)$ be a chain-complete poset and let $F : P \to 2^P \backslash \{\varnothing\}$ be a set-valued mapping. Suppose that F satisfies the following two conditions:*

A1. *F is order-increasing upward;*

A2. *$(F(x), \succeq)$ is inductive with a finite number of maximal elements, for every $x \in P$;*

A3. *There is an element $y*$ in P with $y* \preceq u*$, for some $u* \in F(y*)$.*

Then F has a fixed point. Moreover, we have

(i) *$(\mathcal{F}(F), \succeq)$ is a nonempty inductive poset;*

(ii) *$(\mathcal{F}(F) \cap [y*), \succeq)$ is a nonempty inductive poset.*

By Proposition 2.1, and by applying Corollary 3.2 in [9], we can prove the following theorem.

Theorem 3.5. *Let $(P, \succeq)$ be a chain-complete preordered set with its cluster poset $(\widetilde{P}, \succeq^{\widetilde{P}})$ and let $F : P \to 2^P \backslash \{\varnothing\}$ be a set-valued mapping satisfying the following conditions:*

A0. *F is order cluster-preserving;*

A1. *F is order-increasing upward;*

A2. *$(F(x), \succeq)$ is inductive with a finite number of maximal $\succeq$-clusters, for every $x \in P$;*

A3. *There is an element $y*$ in P with $y* \preceq u*$, for some $u* \in F(y*)$.*

Then F has an $\succeq$-clustered fixed point. Moreover, we have:

(i) *$(\mathcal{F}^{\mathcal{O}}(F), \succeq)$ is a nonempty inductive preordered set;*

(ii) *$(\mathcal{F}^{\mathcal{O}}(F) \cap [y*), \succeq)$ is a nonempty inductive preordered set;*

(iii) *$x \in \mathcal{F}^{\mathcal{O}}(F)$ implies $[x] \subseteq \mathcal{F}^{\mathcal{O}}(F)$.*

Proof. Since $F : P \to 2^P \backslash \{\varnothing\}$ is order cluster-preserving, then similarly to Lemma 3.4, we define the corresponding set-valued mapping $\widetilde{F} : \widetilde{P} \to 2^{\widetilde{P}} \backslash \{\varnothing\}$ as given by (3.2). From the condition A0, Definition 2.3 guarantees that $\widetilde{F}$ is well-defined on the poset $(\widetilde{P}, \succeq^{\widetilde{P}})$. By using the conditions A1, A2, and A3 in this theorem, one can check that $\widetilde{F}$ satisfies all conditions of Corollary 3.2 in [9]; and therefore, we have:

1. $(\mathcal{F}(\widetilde{F}), \succeq^{\widetilde{P}})$ is a nonempty inductive preordered set;

2. $(\mathcal{F}(\widetilde{F}) \cap [[y*]), \succeq^{\widetilde{P}})$ is a nonempty inductive preordered set; and therefore, $\widetilde{F}$ has an $\succeq^{\widetilde{P}}$-maximal fixed point $[x*]$ with $[x*] \succeq^{\widetilde{P}} [y*]$.

From Lemma 3.4, a point $[x] \in \widetilde{P}$ is a fixed point of the mapping $\widetilde{F}$ on the cluster poset $(\widetilde{P}, \succeq^{\widetilde{P}})$, if and only if, $x \in P$ is an $\succeq$-clustered fixed point of F on the preordered set $(P, \succeq)$. It implies that $\mathcal{F}^{\mathcal{O}}(F) \neq \varnothing$.

Before we prove parts (i) and (ii) of this theorem, we first prove part (iii). Suppose $x \in \mathcal{F}^{\mathcal{O}}(F)$. It follows that there is an element w with $x \sim w$ such that $w \in F(x)$. Then we need to show that, for any $y \in [x]$, we have $y \in \mathcal{F}^{\mathcal{O}}(F)$. Since $F : P \to 2^P \backslash \{\varnothing\}$ is order cluster-preserving, then, for an arbitrary element $y \sim x$, from (2.5) and for the given w with $x \sim w$, we have

$$[w] \in \{[u] : u \in F(x)\} = \{[v] : v \in F(y)\}.$$

Hence there is $z \in F(y)$ with $[w] = [z]$, that is, $w \sim z$. Then, from $y \sim x \sim w \sim z$ satisfying $z \in F(y)$, it implies $y \in \mathcal{F}^{\mathcal{O}}(F)$. It proves part (iii) of this theorem.

As $[x]$ is considered as an element in $\widetilde{P}$, we have

$$F(\widetilde{F}) = \{[x] \in \widetilde{P} : [x] \in \widetilde{F}([x])\},$$

By Lemma 3.4 and the proved part (iii), it yields that

$$\mathcal{F}^{\mathcal{O}}(F) = \cup\{[x] \in 2^P : [x] \in \widetilde{F}([x])\},$$

and

$$\mathcal{F}^{\mathcal{O}}(F) \cap [y*) = \cup\{[x] \in 2^P : [x] \in \widetilde{F}([x]) \text{ and } [x] \succeq^{\widetilde{P}} [y*]\},$$

where $[x]$ is considered as a subset of P. Notice that, for any $[x], [z] \in \widetilde{P}, [z] \succeq^{\widetilde{P}} [x]$, if and only if $w \succeq u$, for all $w \in [z]$ and $u \in [x]$. It follows that the inductive property of $(\mathcal{F}(\widetilde{F}), \succeq^{\widetilde{P}})$ implies that $(\mathcal{F}^{\mathcal{O}}(F), \succeq)$ is also inductive. Then parts (i) and (ii) in this theorem immediately follow from parts (i) and (ii) in the conclusions of Corollary 3.2 in [9] and the above two equations, respectively.

In the proof of Theorem 3.5, Corollary 3.2 in [9] is directly applied. If the condition A0 is moved away from Theorem 3.5, then the Definition 3.2 will not be valid and then the results of Corollary 3.2 in [9] cannot be appropriately applied. We will next prove the main theorem in this paper without the condition A0 given in Theorem 3.5. The proof is similar to the proof of Theorem 3.1 [10], where the underlying space is a poset.

Theorem 3.6. *Let $(P, \succeq)$ be a chain-complete preordered set and let $F : P \to 2^P \backslash \{\varnothing\}$ be a set-valued mapping. Suppose that F satisfies the following two conditions:*

A1. *F is order-increasing upward;*

A2. *$(F(x), \succeq)$ is inductive with a finite number of maximal $\succeq$-clusters, for every $x \in P$;*

A3. *There is an element $y*$ in P with $y* \preceq u*$, for some $u* \in F(y*)$.*

Then F has an $\succeq$-clustered fixed point. Moreover, we have

(i) *$(\mathcal{F}^{\mathcal{O}}(F), \succeq)$ is a nonempty inductive preordered set;*

(ii) *$(\mathcal{F}^{\mathcal{O}}(F) \cap [y*), \succeq)$ is a nonempty inductive preordered set.*

Proof. Define a subset A of P as below:

$$A = \{z \in P : \text{ there is } v \in F(z) \text{ with } z \preceq v\}.$$

From condition A3 in this theorem, $y* \in A$; and therefore, $A \neq \varnothing$. We claim that if $z \in A$, then $[z] \subseteq A$. In fact, from $z \in A$, there is $v \in F(z)$ with $z \preceq v$. For any $y \in [z]$, from $z \preceq y$ (and $y \preceq z$), since F is order-increasing upward, there is $u \in F(y)$ with $v \preceq u$. Then, from $y \preceq z \preceq v \preceq u$, it follows that $y \preceq u \in F(y)$. Hence $y \in A$.

It will next be shown that $(A, \succeq)$ is an inductive poset. To this end, taking any arbitrary chain $\{z_\gamma\} \subseteq A$, since $(P, \succeq)$ is chain-complete, then the supremum $\vee\{z_\gamma\}$ exists and it is an $\succeq$-cluster in P. We take $z_0 \in \vee\{z_\gamma\}$. Then we need to show that $\{z_\gamma\}$ has an upper bound in A. It is sufficient to show $z_0 \in A$ (That is, $(A, \succeq)$ is a chain-complete poset).

Since F is order-increasing upward, for any z_γ, from $z_\gamma \preceq z_0$, and $z_\gamma \in A$, there is an element $v_\gamma \in F(z_\gamma)$ such that $z_\gamma \preceq v_\gamma$. Since $z_\gamma \preceq z_0$ and $v_\gamma \in F(z_\gamma)$, from condition A1, there is $u_\gamma \in F(z_0)$ such that $v_\gamma \preceq u_\gamma$. Hence for every index γ, there is $u_\gamma \in F(z_0)$ such that $z_\gamma \preceq u_\gamma \in F(z_0)$.

By condition A2, $(F(z_0), \succeq)$ is inductive with a finite number of maximal $\succeq$-clusters. Similarly to the proof of Theorem 3.1 in [8], we can show that there is an maximal $\succeq$-clusters $[v]$ of $(F(z_0), \succeq)$ with v being selected from $F(z_0)$ such that

$$z\gamma \preceq v, \text{ for all } z_\gamma \text{ in the chain } \{z_\gamma\}.$$

Since $z_0 \in \vee\{z_\gamma\}$, it implies $z_0 \preceq v \in F(z_0)$; and therefore $z_0 \subseteq A$. From the above claim, we have $\vee\{z_\gamma\} = [z_0] \subseteq A$. It follows that $(A, \succeq)$ is a chain-complete preordered set; and therefore, $(A, \succeq)$ is an inductive preordered set. Hence, $(A, \succeq)$ has $\succeq$-maximal element. Take an arbitrary $\succeq$-maximal element x^* of $(A, \succeq)$. Since $x^* \in A$, then there is $v^* \in F(x^*)$ with $x^* \preceq v^*$. Applying condition A1, we can get $v^* \in A$. From $x^* \preceq v^*$ and since x^* is an $\succeq$-maximal element of $(A, \succeq)$, it implies $x^* \sim v^*$, that is $v^* \in [x^*]$ and $v^* \in F(x^*)$. It follows that x^* is an $\succeq$-clustered fixed point of F. Hence $\mathcal{F}^{\mathcal{O}}(F) \neq \varnothing$. (Here notice that x^* may not be in $F(x^*)$. So we only showed $x^* \in \mathcal{F}^{\mathcal{O}}(F)$. It does not imply $\mathcal{F}(F) \neq \varnothing$).

We will next show that $(\mathcal{F}^{\mathcal{O}}(F), \succeq)$ is an inductive preordered set, which is contained in the chain-complete preordered set $(A, \succeq)$. Taking any arbitrary chain $\{x_\alpha\} \subseteq \mathcal{F}^{\mathcal{O}}(F) \subseteq A$, since $(A, \succeq)$ is chain-complete, then the supremum$\vee\{x_\alpha\}$ exists which is an $\succeq$-cluster contained in A. We take an arbitrary point $x \in \vee\{x_\alpha\} \subseteq A$. It follows that $[x) \cap A \neq \varnothing$. Then we divide rest of the proof for $(\mathcal{F}^{\mathcal{O}}(F), \succeq)$ to be an inductive preordered set into two parts:

Suppose $[x) \cap A = \{[x]\} = \vee\{x_\alpha\}$. It implies that x is an $\succeq$-maximal element in $(A, \succeq)$. Since $x \in A$, then from above proof, there is an $\succeq$-maximal $v \in F(x)$ such that $x \preceq v$. From the order-increasing upward property of F, for the given $v \in F(x)$ with $x \preceq v$, there is $w \in F(v)$ such that $v \preceq w$. It implies that $v \in A$. Since x is an $\succeq$-maximal element in $(A, \succeq)$, and $x \preceq v$, it follows that $x \sim v \in F(x)$. Hence $x \in \mathcal{F}^{\mathcal{O}}(F)$; and therefore, x is an upper bound of the chain $\{x_\alpha\}$ in $\mathcal{F}^{\mathcal{O}}(F)$. Since $x \in \vee\{x_\alpha\}$ is arbitrarily taken, it follows that

$$\vee\{x_\alpha\} \subseteq \mathcal{F}^{\mathcal{O}}(F).$$

Suppose $[x) \cap A$ contains more than one $\succeq$-clusters. Since $([x) \cap A, \succeq)$ is also a preordered set. From the Hausdorff maximality theorem, $([x) \cap A, \succeq)$ contains a maximal chain $\{x_\beta\}$ in $([x) \cap A, \succeq)$ (with respect to the sets inclusion partial order $\supseteq$). Since x is the $\succeq$-smallest element in $[x) \cap A$, it implies that $\{x_\beta\}$ contains all elements in $[x]$ as its smallest elements. So $\{x_\alpha\} \cup \{x_\beta\}$ is a chain in $(A, \succeq)$. Since $(A, \succeq)$ has been proved to be chain-complete, then $\vee(\{x_\alpha\} \cup \{x_\beta\})$ exists, which is also an $\succeq$-cluster contained in denoted in A. Take any poiny $y \in \vee(\{x_\alpha\} \cup \{x_\beta\})$. Since $\{x_\alpha\} \subseteq \mathcal{F}^{\mathcal{O}}(F)$, then for every α, as the above proof, there is $v_\alpha \in F(y)$ such that $x_\alpha \preceq v_\alpha$. From $\{x_\beta\} \subseteq A$, for every β, there is $u_\beta \in F(x_\beta)$ such that $x_\beta \preceq u_\beta$. From the order-increasing upward property of F, there is $v_\beta \in F(y)$ such that $x_\beta \preceq u_\beta \preceq v_\beta$. By condition A2, $(F(y), \succeq)$ is inductive with a finite number of $\succeq$maximal clusters. Similarly to the above proof, there is an $\succeq$-maximal clusters $[u]$, with u being selected in $F(y)$ such that $x_\alpha \preceq u$, and $x_\beta \preceq u$, for all x_α in the chain $\{x_\alpha\}$ and for all x_β in the chain $\{x_\beta\}$. Since $y \in \vee(\{x_\alpha\} \cup \{x_\beta\})$, it implies $y \preceq u \in F(y)$, which implies that $y \in A$. From

$x \preceq y$, it follows that $y \in [x) \cap A$. Since $\{x_\beta\}$ is a maximal chain $\{x_\beta\}$ in $([x) \cap A, \succeq)$ (with respect to the sets inclusion partial order $\supseteq$), then we must have $y \in \{x_\beta\}$.

From the order-increasing upward property of F and $y \preceq u \in F(y)$, there is $w \in F(u)$ such that $u \preceq w$. So we have $u \in A$. From $x \preceq y \preceq u \preceq w$, it implies that $u \in [x) \cap A$. Since $\{x_\beta\}$ is a maximal chain in $([x) \cap A, \succeq)$ and $y \in \vee(\{x_\alpha\} \cup \{x_\beta\}) \in \{x_\beta\}$, from $y \preceq u \in [x) \cap A$, it follows that $y \sim u \in F(y)$. It implies that $y \in \mathcal{F}^{\mathcal{O}}(F)$; and therefore, from $y \in \vee(\{x_\alpha\} \cup \{x_\beta\})$, y is an upper bound of the chain $\{x_\alpha\}$ in $\mathcal{F}^{\mathcal{O}}(F)$.

Hence we proved that the given arbitrary chain $\{x_\alpha\} \subseteq \mathcal{F}^{\mathcal{O}}(F)$ has an upper bound in $(\mathcal{F}^{\mathcal{O}}(F), \succeq)$. It follows that $(\mathcal{F}^{\mathcal{O}}(F), \succeq)$ is inductive.

Next we prove part (ii). From condition A3, $u* \in [y*)$; and therefore, $[y*) \neq \varnothing$. Define a set-valued mapping $F_{y*} : [y*) \to 2^{[y*)}$ as $F_{y*}(x) = F(x) \cap [y*)$, for all $x \in [y*)$.

From $u* \succeq y*$, and $u* \in F(y*)$, it follows that $F_{y*}(y*) \neq \varnothing$. By the increasing upward property of F, it can be shown that, for all $x \in [y*)$, $F_{y*}(x) = F(x) \cap [y*) \neq \varnothing$. Then similarly to the proof of Corollary 3.2 in [9], one can show that the mapping from F_{y*} to $2^{[y*)} \backslash \{\varnothing\}$ satisfies all conditions A1, A2, and A3 in this theorem. Hence part (ii) of this theorem immediately follows from the proved result in part (i).

As a consequence of Theorem 3.6 for single-valued mappings, we have:

Corollary 3.7. *Let $(P, \succeq)$ be a chain-complete preordered set and let $F : P \to P$ be a single-valued mapping. Suppose that F satisfies the following two conditions:*

A1. *F is order-increasing;*

A3. *There is an element y in P with $y \preceq F(y)$.*

Then F has an $\succeq$-clustered fixed point. Moreover, we have:

(i) *$(\mathcal{F}^{\mathcal{O}}(F), \succeq)$ is a nonempty inductive preordered set;*

(ii) *$(\mathcal{F}^{\mathcal{O}}(F) \cap [y*), \succeq)$ is a nonempty inductive preordered set; and therefore, F has an maximal $\succeq$-clustered fixed point in $[y*)$.*

In Lemma 3.3, we note that if $(X, \succeq^X)$ is a poset, then order-clustered fixed point coincides with fixed point, that is, $\mathcal{F}(F) = \mathcal{F}^{\mathcal{O}}(F)$. Then, in case if the underlying space $(P, \succeq)$ is a poset, which is considered as a special case of preordered sets, the Corollary 3.2 in [9] follows from Theorem 3.6 immediately.

In [9], some counterexamples for Corollary 3.2 in [9] are provided to demonstrate that the condition in A2 that the number of maximal elements is finite is necessary for F to have a fixed point. We similarly provide some counterexamples for Theorem 3.6 below.

Example 3.8. Let $R_3 = \{(u, v, w) : u, v, w \in R)\}$ be the 3-d Euclidean space. We define a binary relation on R_3, denoted by $\succeq_L$, as below: for any (u_1, v_1, w_1), $(u_2, v_2, w_2) \in R_3$, we write

$$(u_2, v_2, w_2) \succeq_L (u_1, v_1, w_1), \text{ if and only if, } u_2 + v_2 \geq u_1 + v_1 \text{ and } w_2 \geq w_1.$$

One can check that the relation $\succeq_L$ is a preordered on R_3; and therefore $(R_3, \succeq_L)$ is a preordered set. For every real numbers a, b, the subset

$$\{(u, v, w) \in R_3 : u + v = a \text{ and } w = b\}$$

is an $\succeq_L$-cluster of elements in $(R_3, \succeq_L)$.

Let S be the closed tetrahedron in R_3 with vertexes $(1,0,0),(0,1,0),(0,0,1)$ and $(2,2,2)$. The base of S is denoted by B_1 that is the closed triangle R_3 with vertexes $(1,0,0),(0,1,0)$ and $(0,0,1)$. Let T be the closed triangle R_3 with vertexes $(6,0,0),(0,6,0)$ and $(2,2,2)$. Take P to be the union of S and L, that is, $P = S \cup L$.

One can show that $(P, \succeq_L)$ is indeed a chain-complete preordered set. (It is not a poset.)

For any given number $c \in [1,6)$, let B_c be the intersection of S and the plane $u+v+w=c$. For any given number $d \in [0,2]$, let T_d be the intersection of T and the plane $w=d$. Then we have

$$P = \left(\bigcup_{1 \le c < 6} B_c \right) \cup \left(\bigcup_{0 \le d \le 2} T_d \right).$$

All subsets T_d, for $0 \le d \le 2$, are not $\succeq_L$-comparable $\succeq_L$-clusters of $(P, \succeq_L)$.
$A_2 = [(2,2,2)]$ is the unique singleton $\succeq_L$-cluster, which only contains the element $(2,2,2)$, for this chain-complete preordered set $(P, \succeq_L)$.

For $0 \le d < 2$, let A_d be the subset of T above the segment T_d without the point $(2,2,2)$, which is the topless triangle with the missed top vertex $(2,2,2)$ and base T_d, that is,

$$A_d = \bigcup_{d \le \lambda < 2} T_\lambda, \text{ for every } 0 \le d < 2.$$

Then every $\succ_L$-cluster in any given B_c, for $c \in [1,6)$, or in any given A_d, for every $0 \le d < 2$ is an $\succeq_L$-maximal element in B_c, or in A_d, respectively. Hence B_c and A_d both contain infinitely many $\succeq_L$-maximal elements. Then P can be rewritten as

$$P = \left(\bigcup_{1 \le c < 6} B_c \right) \cup \left(\bigcup_{0 \le d < 2} T_d \right) \cup \{(2,2,2)\}.$$

We define a set-valued mapping $F : P \to 2^P \backslash \{\varnothing\}$as below:

1. For any point $(u,v,w) \in B_c$, with a given number $c \in [1,6)$, let
$$F(u,v,w) = B_{3(1+(c-1)/5)};$$
2. For any point $(u,v,w) \in T_d$, for some $0 \le d < 2$, define
$$F(u,v,w) = A_{1+d/2}, \text{ for every } 0 \le d < 2;$$
3. $F(2,2,2) = T \backslash \{(2,2,2)\}$

One can check that F satisfies all conditions in Theorems 3.1 [8], excepting the condition that the set of the maximal elements of the inductive set $(F(x), \succeq_L)$ is not finite, for every $x \in P$. Next we show that F does not have $\succeq_L$-clustered fixed point.

Notice that, except the singleton $\succeq_L$-cluster (2, 2, 2) in $(P, \succeq_L)$, every $\succeq_L$-cluster in S is a closed segment contained in B_c, for some $c \in [1,6)$; and every $\succeq^L$-cluster in T is T_d, for some $0 \le d < 2$. On the other hand, we have

$$B_c \cap F(B_c) = B_c \cap B_{3(1+(c-1)/5)} = \varnothing, \text{ for every } c \in [1,6);$$

$$T_d \cap F(T_d) = T_d \cap A_{1+d/2} = \varnothing, \text{ for every } d \in [0, 2);$$

and

$$\{(2,2,2)\} \cap F(2,2,2) = \{(2,2,2)\} \cap (T \backslash \{(2,2,2)\}) = \varnothing.$$

It implies that for any given $x, y \in P$ with $x \sim_L y$, it is impossible to have $y \in F(x)$ (Or we can apply (3.1) in Lemma 3.3). It follows that F does not have $\succeq_L$-clustered fixed point.

Example 3.9. Let $(R_3, \succeq_L)$, T and T_d, for $d \in [0, 2]$, be defined as in Example 3.8. Let E be the closed tetrahedron in R_3 with vertexes $(1,0,0), (0,1,0)$, and $(2,2,2)$. Take P to be the union of E and T, that is, $P = E \cup T$. One can show that $(P, \succeq_L)$ is indeed a chain-complete preordered set. (It is not a poset.)

For any given number $a \in [0, 2)$, let E_a be the intersection of E and the plane $w = a$. Then we have

$$P = \left(\bigcup_{0 \le a < 2} E_a\right) \cup \left(\bigcup_{0 \le d < 2} T_d\right) \cup \{(2,2,2)\}.$$

All subsets E_a, T_d, for $0 \le a, d < 2$, are $\succeq_L$-clusters of $(P, \succeq_L)$. $[(2,2,2)]$ is the unique singleton $\succeq_L$-cluster, which only contains the element $(2,2,2)$.

We define a set-valued mapping $F : P \to 2^P \backslash \{\varnothing\}$ as below:

1. For any point $(u, v, w) \in E_a$, with a given number $a \in [0, 2)$, let

$$F(u, v, w) = \left(\frac{3a+10}{8}, \frac{3a+10}{8}, 1 + \frac{a}{2}\right);$$

2. For any point $(u, v, w) \in T_d$, for some $0 \le d < 2$, define

$$F(u, v, w) = \left(\frac{10-a}{4}, \frac{10-a}{4}, 1 + \frac{a}{2}\right), \text{ for every } 0 \le d < 2;$$

3. $F(2,2,2) = T \backslash \{(2,2,2)\}$

Notice that, $F(x)$ is a singleton, for all $x \in P$, except the point $(2,2,2)$, at which $F(2,2,2)$ is an infinite set that contains infinitely many $\succeq_L$-maximal element. One can check that F satisfies all conditions in Theorems 3.4, except the condition that the set of the maximal elements of the inductive set $(F(2,2,2), \succeq_L)$ is not finite. We can similarly show that F does not have $\succeq_L$-clustered fixed point.

Next we give a counterexample to show that the chain-completeness of the preordered set $(P, \succeq)$ in Theorem 3.6 is necessary for F to have a fixed point.

Example 3.10. Let $R_2 = \{(u, v) : u, v \in R)\}$ be the 2-d Euclidean space. We define a binary relation on R_2, denoted by $\succeq_C$, as below: for any $(u_1, v_1), (u_2, v_2) \in R_2$,

$$(u_2, v_2) \succeq_C (u_1, v_1), \text{ if and only if }, u_2^2 + v_2^2 \ge u_1^2 + v_1^2.$$

One can check that the relation $\succeq_C$ is a preordered on R_2; and therefore $(R_2, \succeq_C)$ is a preordered set. For every nonnegative numbers r, let C_r denote the circle with radius r and at center $(0,0)$

$$C_r = \{(u, v) \in R_2 : u_2 + v_2 = r_2\}.$$

Let P be the desk in R_2 with radius 2 and at center $(0,0)$ not including the unit circle, that is,

$$P = \bigcup_{0\leq r\leq 2, r\neq 1} C_r.$$

All C_r, for $r \in [0,1)\cup(1,2]$, are the $\succeq_C$-clusters in $(P,\succeq_C)$. We can see that $(P,\succeq_C)$ is an inductive preordered set, which is not chain-complete. We define a set-valued mapping $F : P \to 2^P\backslash\{\varnothing\}$ as: for any point $(u,v) \in C_r$, with a given number $r \in [0,1)\cup(1,2]$, let

$$F(u,v) = C_{(1+r)/2}.$$

One can check that F satisfies all conditions in Theorem 2.6. $(P,\succeq_C)$ is not chain-complete. It is clearly to see

$$C_r \cap F(C_r) = C_r \cap C_{(1+r)/2} = \varnothing, \text{ for every } r \in [0,1)\cup(1,2].$$

Then from (3.1) in Lemma 3.3, it follows that F does not have order-clustered fixed point.

Let $(P,\succeq)$ be a preordered set. Define the reversed ordering relation $\succeq^-$ of the preorder order $\succeq$ on P by, for $x,y \in P$,

$$x \succeq^- y, \text{ if and only if }, x \preceq y.$$

Then $\succeq^-$ is also a preorder on P and it is called the reversed preorder of $\succeq$.

A preordered set $(P,\succeq)$ is said to be reversed inductive, simply denoted by re-inductive, if every totally ordered subset (chain) in P has a lower bound in P (with respect to the order $\succeq$). $(P,\succeq)$ is said to be reversed chain-complete, simply denoted by re-chain-complete, if every totally ordered subset (chain) in P has an infimum (with respect to the order $\succeq$). There are some connections between the original preorder $\succeq$ and its reversed preorder $\succeq^-$:

1.$(P,\succeq)$ is re-inductive, if and only if $(P,\succeq^-)$ is inductive;

2. $(P,\succeq)$ is re-chain-complete if and only if $(P,\succeq^-)$ is chain-complete;

3. A mapping $F : P \to 2^P\backslash\{\varnothing\}$ is $\succeq$-increasing downward (upward), if and only if, it is $\succeq^-$-increasing upward (downward).

The above property 3 can be shown as follows. Assume that $F : P \to 2^P\backslash\{\varnothing\}$ is $\succeq$-increasing downward. Then, for any $x,y \in P$ with $x \preceq y$ (if and only if $y \preceq^- x$) and for any $u \in F(y)$, there is $w \in F(x)$ such that $w \preceq u$ (if and only if $u \preceq^- w$). It implies that $F : P \to 2^P\backslash\{\varnothing\}$ is $\succeq^-$-increasing upward with respect to the partial order $\succeq^-$. We can similarly show that $F : P \to 2^P\backslash\{\varnothing\}$ is $\succeq$-increasing upward, if and only if, it is $\succeq^-$-increasing downward.

Considering that Pareto equilibrium problem is a minimization problem, we modify Theorem 3.6 based on the above properties of reversed preorders to accomplish a different version of fixed point theorem with respect to order-minimizing conditions. It will be applied to solve generalized ordered Pareto equilibrium problems in next section.

Theorems 3.11. *Let $(P,\succeq)$ be a re-chain-complete preordered set and let $F : P \to 2^P\backslash\{\varnothing\}$ be a set-valued mapping. Suppose that F satisfies the following three conditions:*

B1. *F is order-increasing downward;*

B2. *$(F(x), \succeq)$ is re-inductive with a finite number of minimal $\succeq$-clusters, for every $x \in P$;*

B3. *There are elements $z*$ in P and $u* \in F(z*)$ such that $u* \preceq z*$.*

Then

(i) *$(\mathcal{F}^{\mathcal{O}}(F), \succeq)$ is a nonempty re-inductive preordered set;*

(ii) *$(\mathcal{F}^{\mathcal{O}}(F) \cap (z*], \succeq)$ is a nonempty re-inductive preordered set.*

4. The existence of Pareto equilibrium of games with preordered preferences

We recall the concept of n-person noncooperative strategic games.

Definition 4.1. Let n be a positive integer greater than 1. An n-person noncooperative strategic game with preordered preferences consists of the following elements:

1. A set of n players, which is denoted by $N = \{1, 2, \ldots, n\}$;
2. For every player $i = 1, 2, \ldots, n$, his set of strategies $(S_i, \succeq_i)$ is a preordered set;
3. An outcome space $(U; \succeq^U)$ that is a preordered set;
4. For every player $i = 1, 2, \ldots, n$, his utility function (payoff) f_i is a mapping from $S_1 \times S_2 \times \ldots \times S_n$ to $(U; \succeq^U)$.

As usual, the collection of profiles of strategies is denoted by $S = S_1 \times S_2 \times \ldots \times S_n$ and we write the profile function $f = \{f_1, f_2, \ldots, f_n\}$. This game is denoted by $G = (N, S, f, U)$.

In an n-person noncooperative strategic game $G = (N, S, f, U)$, the players follow the performing rules: when all n players simultaneously and independently choose their own strategies $x_1, x_2, \ldots, x_n$, to act, respectively, where $x_i \in S_i$, for $i = 1, 2, \ldots, n$, player i will receive his utility (payoff) $f_i(x_1, x_2, \ldots, x_n) \in U$.

For any $x = (x_1, x_2, \ldots, x_n) \in S$, and for every $i = 1, 2, \ldots, n$, as usual, we denote

$$x_{-i} := (x_1, x_2, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n),$$

$$S_{-i} := S_1 \times S_2 \times \ldots \times S_{i-1} \times S_{i+1} \times \ldots \times S_n.$$

Then $x \in S$ can be simply written as $x = (x_i, x_{-i})$. Moreover, for all $x_{-i} \in S_{-i}$, we denote

$$f_i(S_i, x_{-i}) := \{f_i(t_i, x_{-i}) : t_i \in S_i\}.$$

For convenience, we write $S_{-0} := S = S_1 \times S_2 \times \ldots \times S_n$.

For $i = 0, 1, 2, \ldots, n$, as usual, we denote the component-wise ordering relation $\succeq^{-i}$ on the product poset $S_{-i} = S_1 \times S_2 \times \ldots \times S_{i-1} \times S_{i+1} \times \ldots \times S_n$ as: for any $x_{-i}, y_{-i} \in S_{-i}$ with $x_{-i} = (x_1, x_2, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$ and $y_{-i} = (y_1, y_2, \ldots, y_{i-1}, y_{i+1}, \ldots, y_n)$

$$x_{-i} \succeq^{-i} y_{-i} \text{ if and only if } x_j \succeq_j y_j, \text{ for all } j = 1, 2, \ldots, i-1, i+1, \ldots, n. \tag{4.1}$$

It can be seen that $(S_{-i}, \succeq^{-i})$ is a preordered set. Furthermore, if, for every $j = 1, 2, \ldots, i-1, i+1, \ldots, n$, the set $(S_j, \succeq_j)$ is a chain complete (an inductive) preordered set, then $(S_{-i}, \succeq^{-i})$ is also a chain complete (an inductive) preordered set. For convenience, $(S, \succeq^S)$ is written as $(S_{-0}, \succeq^{-0})$.

Now we extend the concept of Pareto equilibrium of noncooperative strategic games with normal utilities to generalized Pareto equilibrium of noncooperative strategic games with preordered preferences.

Definition 4.2. In an n-person noncooperative strategic game $G = (N, S, f, U)$ with preordered preferences, a profile of strategies $(\hat{x_1}, \hat{x_2}, \ldots, \hat{x_n}) \in S_1 \times S_2 \times \ldots \times S_n$ is called a Pareto equilibrium of this game if, for every $i = 1, 2, \ldots, n$, there is no $z_i \in S_i$ such that

$$f_i(z_i, \hat{x_{-i}}) \prec^U f_i(\hat{x_i}, \hat{x_{-i}}).$$

Let $\mathcal{P}(G)$ denote the collection of all Pareto equilibria of this game G. In [9], it has been shown that the concept of Pareto equilibrium for n-person noncooperative strategic game with partially ordered preferences is indeed a significant generalization of the concept of Pareto equilibrium of noncooperative strategic games with vector utilities. It is clear to see that Pareto equilibrium for n-person strategic game with preordered preferences widely generalizes the concept of Pareto equilibrium with partially ordered preferences.

In an n-person noncooperative strategic game $G = (N, S, f, U)$ with preordered preferences, for all $i = 1, \ldots, n$, we define a set-valued mapping $\gamma_i : S_{-i} \to 2^{S_i}$ by

$$\gamma_i(x_{-i}) = \{z_i \in S_i : f_i(z_i, x_{-i}) \text{ is an } \succeq^U -\text{minimal element of } f_i(S_i, x_{-i})\},$$

for all $x_{-i} \in S_{-i}$, where γ_i is called the $\succeq^S$-minimal response function for player i.

In case if $\gamma_i(x_{-i}) \neq \varnothing$, for all $x_{-i} \in S_{-i}$, then the set-valued mapping γ_i is order-increasing downward on S_{-i}. That is, whenever, for any $x_{-i}, y_{-i} \in S_{-i}$ with $x_{-i} \preceq^{-i} y_{-i}$ and for any $w_i \in \gamma_i(y_{-i})$, there is $z_i \in \gamma_i(x_{-i})$ such that $z_i \preceq_i w_i$.

Theorem 4.3. *Let $G = (N, S, f, U)$ be an n-person noncooperative strategic game with preordered preferences such that $(S_i, \succeq_i)$ is a re-chain-complete preordered set, for $i = 1, 2, \ldots, n$. Suppose that, for every player $i = 1, 2, 3, \ldots, n$, $f_i : S \to U$ is an order indifference single-valued mapping and, for any $x \in S$, the following conditions hold:*

G1. *$f_i(S_i, x_{-i})$ is a re-inductive subset of $(U, \succeq^U)$, for every $x_{-i} \in S_{-i}$;*

G2. *$\gamma_i : S_{-i} \to 2^{S_i} \backslash \{\varnothing\}$ is order-increasing downward on $(S_{-i}, \succeq^{-i})$ with re-inductive values that have a finite number of $\succeq^S$-minimal elements;*

G3. *There are elements $a = (a_1, a_2, \ldots, a_n)$ and $b = (b_1, b_2, \ldots, b_n)$ in S with $b \preceq^S a$ satisfying $b_i \in \gamma_i(a_{-i})$, for $i = 1, 2, \ldots, n$.*

Then there is an $\succeq^S$-cluster in S in which every element is a Pareto equilibrium of G. Moreover, we have

(i) *$(\mathcal{P}(G), \succeq^S)$ is a nonempty re-inductive preordered set;*

(ii) *$(\mathcal{P}(G) \cap (a], \succeq^S)$ is a nonempty re-inductive preordered set.*

Proof. The first part of the proof is similar to the proof of Corollary 3.2 in [9]. By applying Zorn's lemma and condition G1 in this theorem, for every $x_{-i} \in S_{-i}$, $f_i(S_i, x_{-i})$ has at least one minimal element in the set $f_i(S_i, x_{-i})$; and therefore, $\gamma_i(x_{-i}) \neq \varnothing$, for all $x_{-i} \in S_{-i}$.

For every $x = (x_1, x_2, \ldots, x_n) \in S$, and for every $i = 1, 2, \ldots, n$, x can be denoted as $x = (x_i, x_{-i})$. So the profile space S can be rewritten as $S = (S_i, S_{-i})$. Then, for each player i, the $\succeq^S$-minimal response function $\gamma_i : S_{-i} \to 2^{S_i} \backslash \{\varnothing\}$ can be considered as a set-valued mapping from S to $2^{S_i}\backslash\{\varnothing\}$ defined as:

$$\gamma_i(x) = \gamma_i(x_{-i}),\ \text{for any}\ x = (x_i, x_{-i}) \in S,\ \text{for every}\ i = 1, 2, \ldots, n.$$

Then we define $F : S \to 2^S\backslash\{\varnothing\}$ by $F = \gamma_1 \times \gamma_2 \times \ldots \times \gamma_n$; that is,

$$F(x) = (\gamma_1(x), \gamma_2(x), \ldots, \gamma_n(x)),\ \text{for any}\ x \in S.$$

Similarly to the proof of Theorem 3.3 in [9], by the conditions G1, G2 and G3 in this theorem, we can show that F satisfies all condition B1, B2, and B3 in Theorem 3.11. Then it follows that F has an order-clustered fixed point, say $\hat{x} = (\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_n) \in S$. Hence there is $\hat{t} \in [\hat{x}]$, such that $\hat{t} \in F(\hat{x})$.

We will next show that any order-clustered fixed point $\hat{x}$ of F is Pareto equilibrium for this game $G = (N, S, f, U)$. To this end, from $\hat{t} \in [\hat{x}]$ and $\hat{t} \in F(\hat{x})$, it implies $\hat{t}_i \in \gamma_i(\hat{x})$; that is, $f_i(\hat{t}_i, \hat{x}_{-i})$ is an $\succeq^U$ −minimal element of $f_i(S_i, \hat{x}_{-i})$, for every fixed $i = 1, 2, \ldots, n$.

For every $i = 1, 2, 3, \ldots, n$, it is equivalent to that there is no $x_i \in S_i$ such that

$$f_i(x_i, \hat{x}_{-i}) \prec^U f_i(\hat{t}_i, \hat{x}_{-i}). \tag{4.2}$$

The following is from the definition (4.1) of the product preorder $\succeq^S$ on $(S, \succeq^S)$:

$$\hat{t} \in [\hat{x}] \Rightarrow \hat{t} \sim^S \hat{x} \Rightarrow \hat{t}_i \sim_i \hat{x}_i \Rightarrow (\hat{t}_i, \hat{x}_{-i}) \sim^S (\hat{x}_i, \hat{x}_{-i}) = \hat{x},\ \text{for}\ i = 1, 2, \ldots, n.$$

Since the utility function $f_i : S \to U$ is an order indifference single-valued mapping, from $(\hat{t}_i, \hat{x}_{-i}) \sim^S (\hat{x}_i, \hat{x}_{-i})$ and (2.6) in Definition 2.3, it yields that

$$f_i(\hat{t}_i, \hat{x}_{-i}) \sim^U f_i(\hat{x}_i, \hat{x}_{-i}). \tag{4.3}$$

By (4.3) and (4.2), it implies that there is no $x_i \in S_i$ such that

$$f_i(x_i, \hat{x}_{-i}) \prec^U f_i(\hat{x}_i, \hat{x}_{-i}). \tag{4.4}$$

This shows that $\hat{x} = (\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_n)$ is a Pareto equilibrium of this game G. Next we show $[\hat{x}] \subseteq \mathcal{P}(G)$. By applying the order indifference properties of the utility functions f_i and from (4.3), for every $y \in [\hat{x}]$, we must have $f_i(y) \sim^U f_i(\hat{x}_i, \hat{x}_{-i})$.

On the other hand, since $(x_i, y_{-i}) \sim^S (x_i, \hat{x}_{-i})$, it yields

$$f_i(x_i, \hat{x}_{-i}) \sim^U f_i(x_i, y_{-i}). \tag{4.5}$$

From (4.3) - (4.5), it implies that there is no $x_i \in S_i$ such that

$$f_i(x_i, y_{-i}) \prec^U f_i(y). \tag{4.6}$$

It follows that, for every $y \sim^S \hat{x} = (\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_n)$, y is a Pareto equilibrium of this game G. Then the conclusion immediately follows from Theorem 3.11.

Definition 4.4. In an n-person noncooperative strategic game $G = (N, S, f, U)$, with preordered preferences, a profile of strategies $(\check{x}_1, \check{x}_2, \ldots, \check{x}_n) \in S_1 \times S_2 \times \ldots \times S_n$ is called a generalized Nash equilibrium of this game, if, for every $i = 1, 2, 3, \ldots, n$, the following order inequality holds

$$f_i(x_i, \check{x}_{-i}) \preceq^U f_i(\check{x}_i, \check{x}_{-i}),\ \text{for all}\ x_i \in S_i.$$

The collection of all Nash equilibrium of game G is denoted by $\mathcal{N}(G)$. In an n-person noncooperative strategic game $G = (N, S, f, U)$ with preordered preferences, for every $i = 1, 2, \ldots, n$, we define a set-valued mapping $\beta_i : S_{-i} \to 2^{S_i}$by

$$\beta_i(x_{-i}) = \{z_i \in S_i : f_i(z_i, x_{-i}) \text{ is an } \succeq^U -\text{maximal element of } f_i(S_i, x_{-i})\},$$

for all $x_{-i} \in S_{-i}$ where β_i is called the $\succeq^S$-maximal response function for player i.

Theorem 4.5. *Let $G = (N, S, f, U)$ be an n-person noncooperative strategic game with preordered preferences. Suppose that, for every player $i = 1, 2, 3, \ldots, n$, $f_i : S \to U$ is an order indifference single-valued mapping and, for any $x \in S$, the following conditions hold:*

G1. *$f_i(S_i, x_{-i})$ is an inductive subset of the preordered set $(U, \succeq^U)$;*

G2. *$\beta_i : S_{-i} \to 2^{S_i}\backslash\{\varnothing\}$ is order-increasing upward on $(S_{-i}, \succeq^{-i})$ with inductive values that have a finite number of $\succeq^{-i}$-maximal clusters;*

G3. *There are $p = (p_i, p_{-i}), q = (q_i, q_{-i}) \in S$ with $p \preceq^S q$ satisfying $q_i \in \beta_i(p_{-i})$, for $i = 1, 2, 3, \ldots, n$.*

Then

(i) *$(\mathcal{N}(G), \succeq^S)$ is a nonempty inductive preordered set;*

(ii) *$(\mathcal{N}(G) \cap [p), \succeq^S)$ is a nonempty inductive preordered set.*

Proof. Based on the $\succeq^S$-maximal response functions β_i, we define a set-valued mapping $F : S \to 2^S\backslash\{\varnothing\}$ by $F(x) = (\beta_1(x), \beta_2(x), \ldots, \beta_n(x))$, for all $x \in S$. The rest of the proof is similar to the proof of Theorem 4.3 by applying Theorem 3.6.

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