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SATURATED FIBRE CONTRACTION PRINCIPLE

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Abstract. For a triangular operator $A : X \times Y \to X \times Y$, $A = (B, C)$, where $B : X \to X$ and $C : X \times Y \to Y$ we study in which conditions on operators $B : X \to X$ and $C : X \times Y \to Y$ we have that:

(1) the fixed point problem for triangular operator $A = (B, C)$ is well posed
(2) the operator $A = (B, C)$ has the Ostrowski property
(3) the fixed point equation $(x, y) = A(x, y)$ is generalized Ulam-Hyers stable.

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1. Introduction

In this paper we shall use the terminologies and notations from [21] and [27]. For the convenience of the reader we shall recall some of them.

Let $(X, \to)$ be an L-space and $f : X \to X$ an operator. We denote by $f^0 := 1_X$, $f^1 := f$, $f^{n+1} := f \circ f^n$, $n \in \mathbb{N}$ the iterate operators of the operator A. Also:

$$P(X) := \{Y \subseteq X \mid Y \neq \emptyset\} \text{ and } F_f := \{x \in X \mid f(x) = x\}$$

By $(X, \to)$ we will denote an L-space. Actually, an L-space is any set endowed with a structure implying a notion of convergence for sequences. For examples of L-spaces see Fréchet [10], Blumenthal [7] and I. A. Rus [21].

Let $(X, \to)$ be an L-space.

Definition 1.1. $f : X \to X$ is said to be a weakly Picard operator (briefly WPO) if the sequence $(f^n(x))_{n \in N}$ converges for all $x \in X$ and the limit (which may depend on x) is a fixed point of f. If additionally, $F_f = \{x^*\}$, then f is called a Picard operator (PO).

If $f : X \to X$ is a WPO, then we may define the operator $f^\infty : X \to X$ by

$$f^\infty(x) := \lim_{n \to \infty} f^n(x).$$

Obviously $f^{\infty}(X) = F_f$. Moreover, if f is a PO and we denote by x^* its unique fixed point, then $f^{\infty}(x) = x^*$, for each $x \in X$.

Let (X, d) be a metric space.

Definition 1.2. (F.S. De Blasi and J. Myjak (see [28] p.42, see also [26]))The fixed point problem for an operator $f : X \to X$ is well posed iff:

(a) $F_f = \{x^*\}$;

(b) if $x_n \in X$, $n \in \mathbb{N}$ and $d(x_n, f(x_n)) \to 0$ as $n \to +\infty$, then $d(x_n, x^*) \to 0$ as $n \to +\infty$.

Definition 1.3. An operator $f : X \to X$ has the Ostrowski property iff:

(a) $F_f = \{x^*\}$;

(b) $x_n \in X$, $n \in \mathbb{N}$, and $d(x_{n+1}, f(x_n)) \to 0$ as $n \to \infty$ imply that $d(x_n, x^*) \to 0$ as $n \to \infty$.

Some authors refer to the above property as the "limit shadowing property" (see [15] and the references in, [11], [20], [14], [12], [17], [29], ...).

An important result used in the proof of the Ostrowski property, also in the proof of fiber contraction principle, is the Cauchy Lemma. For details and generalizations see [16], [30].

Lemma 1.1. (Cauchy Lemma). Let a_n, $b_n \in \mathbb{R}_+$, $n \in \mathbb{N}$. We suppose that:

(i) $\sum\limits_{k=0}^{\infty} a_k < +\infty$;

(ii) $b_n \to 0$ as $n \to \infty$.

Then

$$\sum_{k=0}^{n} a_{n-k} b_k \to 0 \text{ as } n \to \infty.$$

Definition 1.4. Let (X, d) be a metric space and $f : X \to X$ such that $F_f = \{x^*\}$. By definition, f is an l-quasicontraction iff $l \in [0; 1[$ and

$$d(f(x), x^*) \leq l d(x, x^*), \ \forall x \in X.$$

Theorem 1.1. *Let (X, d) be a metric space and $f : X \to X$ be such that $F_f = \{x^*\}$. If the operator f is an l-quasicontraction then f has the Ostrowski property.*

Proof. Let $(x_n)_{n \in \mathbb{N}} \subset X$ such that $d(x_{n+1}, f(x_n)) \to 0$ as $n \to +\infty$. Then, we have:

$$\begin{aligned} d(x_{n+1}, x^*) &\leq d(x_{n+1}, f(x_n)) + d(f(x_n), x^*) \\ &\leq d(x_{n+1}, f(x_n)) + l d(x_n, x^*) \leq ... \\ &\leq \sum_{j=0}^{n} l^j \cdot d(x_{n+1-j}, f(x_{n-j})) + l^n \cdot d(x_0, x^*). \end{aligned}$$

Making $n \to \infty$ and applying the Cauchy Lemma 1.1 for $a_n = l^n$ and $b_n = d(x_{n+1}, f(x_n))$ we get the conclusion. □

Let (X, d) be a metric space, $f : X \to X$ and we consider the fixed point equation

$$x = f(x). \tag{1.1}$$

Definition 1.5. By definition, the fixed point equation (1.1) is Ulam-Hyers stable if there exists a constant $c_f > 0$ such that: for each $\varepsilon > 0$ and each solution $y^* \in X$ of the inequation

$$d(y, f(y)) \leq \varepsilon \tag{1.2}$$

there exists a solution x^* of the equation (1.1) such that

$$d(y^*, x^*) \leq c_f \varepsilon.$$

Definition 1.6. By definition, the equation (1.1) is generalized Ulam-Hyers stable if there exists $\theta : \mathbb{R}_+ \to \mathbb{R}_+$ increasing and continuous in 0 with $\theta(0) = 0$ such that: for each $\varepsilon > 0$ and for each solution y^* of (1.2) there exists a solution x^* of (1.1) such that

$$d(y^*, x^*) \leq \theta(\varepsilon).$$

Definition 1.7 ([6])**.** Let (X, d) be a metric space and $f : X \to X$ be an operator so that its fixed point set F_f is nonempty. Let $r : X \to F_f$ be a set retraction. Then, by definition, f satisfies the (ψ, r) retraction-displacement condition (ψ-condition in [9], (ψ, r)-operator in [5], ψ-weakly Picard operator in the case of Picard iterations in [21], the collage condition in [3]) if:

(i) $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ is increasing, continuous at 0 and $\psi(0) = 0$;
(ii) $d(x, r(x)) \leq \psi(d(x, f(x))$, for every $x \in X$.

Remark 1.1. If $F_f = \{x^*\}$, then the (ψ, r) retraction-displacement condition takes the following form:

(i) $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ is increasing, continuous at 0 and $\psi(0) = 0$;
(ii) $d(x, x^*) \leq \psi(d(x, f(x))$, for every $x \in X$.

We will call it the (x^*, ψ) retraction-displacement condition.

Remark 1.2. Let (X, d) be a metric space and $f : X \to X$ such that $F_f = \{x^*\}$. If the operator f is an l-quasicontraction then f satisfies (x^*, ψ) retraction-displacement condition with $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ given by $\psi(t) = \frac{1}{1-l}t$.

Proof. For all $x \in X$ we have:

$$\begin{aligned} d(x, x^*) &\leq d(x, f(x)) + d(f(x), x^*) \\ &\leq d(x, f(x)) + l \cdot d(x, x^*). \ \square \end{aligned}$$

Theorem 1.2. *Let (X, d) be a metric space and $f : X \to X$ such that $F_f = \{x^*\}$. If the operator f satisfies an (x^*, ψ) retraction-displacement condition, then the fixed point problem for f is well-posed.*

Proof. Let $(x_n)_{n\in\mathbb{N}} \subset X$ such that $d(x_n, f(x_n)) \to 0$ as $n \to +\infty$. Then, we have:

$$d(x_n, x^*) \leq \psi(d(x_n, f(x_n)) \to 0 \text{ as } n \to \infty.$$

□

Theorem 1.3. *Let (X, d) be a metric space and $f : X \to X$ such that $F_f = \{x^*\}$. If f satisfies a (x^*, ψ) retraction-displacement condition, then the equation (1.1) is generalized Ulam-Hyers stable.*

Proof. Let $y^* \in X$ be a solution of (1.2). Since f satisfies the (r, ψ) retraction-displacement condition we have:

$$d(y^*, x^*) \leq \psi(d(y^*, f(y^*)) \leq \psi(\varepsilon).$$ □

For more considerations on Ulam stability see I.A. Rus [24].

2. Fibre contraction principle

Let (X, d_X) and (Y, d_Y) be two metric spaces. We consider on $X \times Y$ the following metric

$$d_\infty : X \times Y \to \mathbb{R}_+$$
$$d_\infty \left((x_1, y_1), (x_2, y_2) \right) = \max \left\{ d_X (x_1, x_2), d_Y (y_1, y_2) \right\}$$

Let $B : X \to X$ and $C : X \times Y \to Y$ be two operators and the triangular operator $A : X \times Y \to X \times Y$ be defined by

$$A(x, y) := (B(x), C(x, y)).$$

We have the following result:

Theorem 2.1 (Fibre contraction principle)**.** *(*[31]*,* [18]*,* [19]*) We suppose that:*

(i) (Y, d_Y) *is a complete metric space;*
(ii) B *is a WPO;*
(iii) $C(x, \cdot) : Y \to Y$ *is* $\alpha-$ *contraction for every* $x \in X$*;*
(iv) $C : X \times Y \to Y$ *is continuous.*

Then

(a) A *is a WPO;*
(b) *If* B *is a PO then* A *is a PO.*

For other generalizations of fibre contraction principle see S. András [1], C. Bacoţiu [2], I.A. Rus [16], [18], [19], M.A. Şerban [33], [34].

Following the result of I. A. Rus in [25], saturated contraction principle, the aim of this paper is to give the Fibre contraction principle with a generous conclusions.

We have:

Theorem 2.2 (Saturated fibre contraction principle)**.** *We suppose that:*

(i) (Y, ρ) *is a complete metric space;*
(ii) B *is a PO,* $F_B = \{x^*\}$*;*
(iii) $C(x, \cdot) : Y \to Y$ *is* $\alpha-$ *contraction for every* $x \in X$*;*
(iv) $C(\cdot, y) : X \to X$ *is* $L-$ *lipschitz for every* $y \in Y$*.*

Then:

(a) A *is a PO;*
(b) $F_A = F_{A^n} = \{(x^*, y^*)\}$*, where* $\{y^*\} = F_{C(x^*, \cdot)}$*;*
(c) *If, in addition,* B *satisfies the* (x^*, ψ_B) *retraction-displacement condition then:*
  (c$_1$) A *satisfies the* $((x^*, y^*), \psi_A)$ *retraction-displacement condition, where*

$$\psi_A : \mathbb{R}_+ \to \mathbb{R}_+, \ \psi_A(t) = \max \left\{ \psi_B(t), \frac{1}{1-\alpha} [t + L\psi_B(t)] \right\};$$

(c_2) *the fixed point problem for A is well posed;*
(c_3) *the fixed point equation for A is generalized Ulam-Hyers stable;*

(d) *If , in addition, B is an l_B-quasicontraction then:*
(d_1) *A is an l_A-quasicontraction in $(X \times Y, \rho_\infty)$, where*

$$\rho_\infty\left(\left(x_1, y_1\right), \left(x_2, y_2\right)\right) = \max\left\{r \cdot d_X\left(x_1, x_2\right), d_Y\left(y_1, y_2\right)\right\},$$

with $r > \dfrac{L}{1-\alpha}$ *and* $l_A = \max\left\{l_B, \dfrac{L}{r} + \alpha\right\}$;

(d_2) *A has the Ostrowski property.*

Proof. (*a*) Let $x_0 \in X$ and $y_0 \in Y$. B is a PO then $F_B = \{x^*\}$ and $B^\infty(x) = x^*$ for all $x \in X$. From conditions (i) and (iii) we obtain that the operator $C(x^*, \cdot)$ has a unique fixed point $y^* \in Y$, thus $F_A = \{(x^*, y^*)\}$. We show that

$$A^n\left(x_0, y_0\right) \to \left(x^*, y^*\right) \text{ as } n \to +\infty.$$

It is easy to check that

$$A^n\left(x_0, y_0\right) = \left(x_n, y_n\right)$$

where $x_n = B^n(x_0) \to x^*$as $n \to \infty$ and $y_n = C(x_{n-1}, y_{n-1})$, $n \in \mathbb{N}$. We have:

$$\begin{aligned}
d_Y\left(y_{n+1}, y^*\right) &\le d_Y\left(C\left(x_n, y_n\right), C\left(x_n, y^*\right)\right) + d_Y\left(C\left(x_n, y^*\right), y^*\right)\\
&\le \alpha \cdot d_Y\left(y_n, y^*\right) + d_Y\left(C\left(x_n, y^*\right), y^*\right)\\
&\le \alpha^2 \cdot d_Y\left(y_{n-1}, y^*\right) + \alpha \cdot d_Y\left(C\left(x_{n-1}, y^*\right), y^*\right) + d_Y\left(C\left(x_n, y^*\right), y^*\right)\\
&\le \dots \le\\
&\le \alpha^{n+1} d_Y\left(y_0, y^*\right) + \alpha^n d_Y\left(C\left(x_0, y^*\right), y^*\right) + \dots + d_Y\left(C\left(x_n, y^*\right), y^*\right).
\end{aligned}$$

If we take $b_n = d_Y(C(x_n, y^*), y^*)$, from (iv) we deduce that $b_n \to 0$ as $n \to \infty$, and the conclusion is obtained from Cauchy Lemma 1.1 for $a_n = \alpha^n$ and b_n.

(*b*) Follows from the fact that A is a PO, $F_A = \{(x^*, y^*)\}$, and any PO has no periodic point with period $p > 1$.

(c_1) Let $(x, y) \in X \times Y$. If B is a PO and satisfies the (x^*, ψ_B) retraction-displacement condition then

$$d_X\left(x, x^*\right) \le \psi_B\left(d_X\left(x, B\left(x\right)\right)\right), \ \forall x \in X,$$

where $\psi_B : \mathbb{R}_+ \to \mathbb{R}_+$ is increasing, continuous at 0 with $\psi_B(0) = 0$. From (*a*) we have that A is a PO and $F_A = \{(x^*, y^*)\}$, where $\{y^*\} = F_{C(x^*, \cdot)}$. From (*iii*) and (*iv*) we get

$$\begin{aligned}
d_Y\left(y, y^*\right) &\le d_Y\left(y, C\left(x, y\right)\right) + d_Y\left(C\left(x, y\right), y^*\right)\\
&\le d_Y\left(y, C\left(x, y\right)\right) + L d_X\left(x, x^*\right) + \alpha d_Y\left(y, y^*\right),
\end{aligned}$$

so

$$\begin{aligned}
d_Y\left(y, y^*\right) &\le \frac{1}{1-\alpha}\left[d_Y\left(y, C\left(x, y\right)\right) + L d_X\left(x, x^*\right)\right]\\
&\le \frac{1}{1-\alpha}\left[d_Y\left(y, C\left(x, y\right)\right) + L \psi_B\left(d_X\left(x, B\left(x\right)\right)\right)\right].
\end{aligned}$$

This implies that

$$d_\infty\left((x,y),(x^*,y^*)\right) \leq \max\left\{\psi_B\left(d\left(x,B\left(x\right)\right)\right), \frac{1}{1-\alpha}\left[d_Y\left(y,C\left(x,y\right)\right)+L\psi_B\left(d_X\left(x,B\left(x\right)\right)\right)\right]\right\}$$
$$\leq \psi_A\left(d_\infty\left((x,y),A\left(x,y\right)\right)\right),$$

where $\psi_A:\mathbb{R}_+\to\mathbb{R}_+$

$$\psi_A\left(t\right)=\max\left\{\psi_B\left(t\right),\frac{1}{1-\alpha}\left[t+L\psi_B\left(t\right)\right]\right\}.$$

It is easy to check that $\psi_A:\mathbb{R}_+\to\mathbb{R}_+$ is increasing, continuous at 0 with $\psi_A(0)=0$.

(c_2) Follows from Theorem 1.2.

(c_3) Follows from Theorem 1.3.

(d_1) Let $(x,y)\in X\times Y$ and $r>\frac{L}{1-\alpha}$. If B is an l_B-quasicontraction then

$$\begin{aligned} r\cdot d_X\left(B\left(x\right),x^*\right) &\leq l_B\cdot r\cdot d_X\left(x,x^*\right) \\ &\leq l_B\cdot\rho_\infty\left((x,y),(x^*,y^*)\right),\ \forall\left(x,y\right)\in X\times Y, \end{aligned}$$

$r>\frac{L}{1-\alpha}\Longleftrightarrow\frac{L}{r}+\alpha<1$ and from (iii) and (iv) we have

$$\begin{aligned} d_Y\left(C\left(x,y\right),y^*\right) &\leq \frac{L}{r}\cdot r\cdot d_X\left(x,x^*\right)+\alpha\cdot d_Y\left(y,y^*\right) \\ &\leq \left(\frac{L}{r}+\alpha\right)\rho_\infty\left((x,y),(x^*,y^*)\right),\ \forall\left(x,y\right)\in X\times Y \end{aligned}$$

so

$$\begin{aligned} \rho_\infty\left(A\left(x,y\right),(x^*,y^*)\right) &= \max\left\{r\cdot d_X\left(B\left(x\right),x^*\right),d_Y\left(C\left(x,y\right),y^*\right)\right\} \\ &\leq \max\left\{l_B,\frac{L}{r}+\alpha\right\}\cdot\rho_\infty\left((x,y),(x^*,y^*)\right),\ \forall\left(x,y\right)\in X\times Y. \end{aligned}$$

(d_2) Follows from Theorem 1.1 and from the fact that d_∞ and ρ_∞ are metric equivalent. □

3. Applications

3.1. **System of integral equation.** In what follow we apply fibre contraction principle to study the following system of integral equations:

$$\begin{cases} x\left(t\right)=\int_a^t K\left(t,s,x\left(s\right)\right)ds+k\left(t\right), & t\in[a;b] \\ y\left(t\right)=\int_a^b P\left(t,s,x\left(s\right)\right)ds+\int_a^t Q\left(t,s,x\left(s\right),y\left(s\right)\right)ds+h\left(t\right), & t\in[a;b] \end{cases} \tag{3.1}$$

The system (3.1) is equivalent with the following fixed point problem:

$$(x,y)=A\left(x,y\right), \tag{3.2}$$

where

$$A\left(x,y\right)\left(t\right)=\left(B\left(x\right)\left(t\right),C\left(x,y\right)\left(t\right)\right), \tag{3.3}$$

$$B(x)(t) = \int_a^t K(t, s, x(s))\,ds + k(t),$$

$$C(x,y)(t) = \int_a^b P(t, s, x(s))\,ds + \int_a^t Q(t, s, x(s), y(s))\,ds + h(t).$$

In addition, we consider the following hypothesis:

(H1) K, P, $Q \in C([a;b] \times [a;b] \times \mathbb{R})$ and $k, h \in C[a;b]$;

(H2) there exists $L_K > 0$ such that

$$|K(t, s, u_1) - K(t, s, u_2)| \leq L_K \cdot |u_1 - u_2|$$

for all $t, s \in [a;b]$ and $u_1, u_2 \in \mathbb{R}$;

(H3) there exists $L_Q > 0$, such that

$$|Q(t, s, u, v_1) - Q(t, s, u, v_2)| \leq L_Q \cdot |v_1 - v_2|,$$

for all $t, s \in [a;b]$ and $u, v_1, v_2 \in \mathbb{R}$;

(H3)' there exist $L_P > 0$, $l_Q > 0$, $L_Q > 0$, such that

$$|P(t, s, u_1) - P(t, s, u_2)| \leq L_P \cdot |u_1 - u_2|,$$

$$|Q(t, s, u_1, v_1) - Q(t, s, u_2, v_2)| \leq l_Q \cdot |u_1 - u_2| + L_{H_i} \cdot |v_1 - v_2|,$$

for all $t, s \in [a;b]$ and $u_1, u_2, v_1, v_2 \in \mathbb{R}$, $i = 1, 2$.

We have:

Theorem 3.1.1. *If conditions* $(H1) - (H3)$ *hold then the system (3.1) has a unique solution* $(x^*, y^*) \in C\left([a;b], \mathbb{R}^2\right)$.

Proof. Let $X = Y := C[a;b]$ and $Y = C[a;b]$. We consider on X the Bielecki norm

$$\|x\|_\tau = \max_{t \in [a;b]} \left(\left| x(t) \cdot e^{-\tau(t-a)} \right| \right), \ \tau > 0.$$

From the $(H1)$ we have that A, defined by (3.3), satisfies $A : X \times X \to X \times X$.

From $(H2)$ we have that

$$\|B(x_1) - B(x_2)\|_\tau \leq \frac{L_K}{\tau} \|x_1 - x_2\|_\tau, \ \forall x_1, x_2 \in X$$

Using condition $(H3)$ we get

$$\|C(x, y_1) - C(x, y_2)\|_\tau \leq \frac{L_Q}{\tau} \|y_1 - y_2\|_\tau$$

for all $x, y_1, y_2 \in X$. For a suitable choice of $\tau > \max\{L_K, L_Q\}$ we have that $B : X \to X$ is an α_B-contraction, with $\alpha_B = \frac{L_K}{\tau}$, $C(x, \cdot) : X \to X$ is an α-contraction, with $\alpha = \frac{L_Q}{\tau}$, for all $x \in X$. From fibre contraction principle, Theorem 2.1, we have that A is PO and $F_A = \{(x^*, y^*)\}$. □

Theorem 3.1.2. *If conditions* $(H1)$, $(H2)$ *and* $(H3)'$ *hold then:*

(a) *the equation (3.2), is well posed;*

(b) *the equation (3.2), is Ulam-Hyers stable;*

(c) *the operator A, defined by (3.3), has the Ostrowski property.*

Proof. $(a)-(c)$ From $(H1)-(H2)$ we have B is an α_B-contraction, with $\alpha_B = \frac{L_K}{\tau}$, then B is l_B-quasicontraction, with $l_B = \frac{1}{1-\alpha_B}$. Using condition $(H3)'$ we get

$$|C(x_1,y_1)(t) - C(x_2,y_2)(t)| \leq \left(L_P(b-a) + \frac{l_Q}{\tau}\right)\|x_1-x_2\|_\tau e^{\tau(t-a)} + \frac{L_Q}{\tau}\|y_1-y_2\|_\tau e^{\tau(t-a)},$$

so

$$\|C(x_1,y_1) - C(x_2,y_2)\|_\tau \leq \left(L_P(b-a) + \frac{l_Q}{\tau}\right)\|x_1-x_2\|_\tau + \frac{L_Q}{\tau}\|y_1-y_2\|_\tau .$$

Choosing $\tau > \max\{L_K, L_Q\}$ we have that $C(x,\cdot): X \to X$ is an α-contraction, with $\alpha = \frac{L_Q}{\tau}$ and $C(\cdot,y): X \to X$ is L-lipschitz with $L = \left(L_P(b-a) + \frac{l_Q}{\tau}\right)$. The conclusion follows from the saturated fibre contraction principle, Theorem 2.2. □

3.2. **Differentiability of nonlocal initial value problem solution with respect to a parameter.** We consider the following nonlocal initial value problem for the first order differential equation

$$\begin{cases} x'(t) = f(t, x(t), \lambda), \ t \in [0;1] \\ x(0) + \sum\limits_{k=1}^{m} a_k x(t_k) = 0, \end{cases} \tag{3.4}$$

where $\lambda \in J$, $J \subseteq \mathbb{R}$ a closed interval, t_k are given points with $0 \leq t_1 \leq t_2 \leq ... \leq t_m < 1$ and a_k are real numbers with $1 + \sum\limits_{k=1}^{m} a_k \neq 0$.

We consider the following hypothesis:

(H1) $f \in C([0;1] \times \mathbb{R} \times J)$;

(H2) there exist $l_1 > 0$, $l_2 > 0$ such that

$$|f(t,u_1,\lambda) - f(t,u_2,\lambda)| \leq \begin{cases} l_1|u_1-u_2|, & t \in [0;t_m] \\ l_2|u_1-u_2|, & t \in [t_m;1] \end{cases}$$

for all $t \in [0;1]$, $u_1, u_2 \in \mathbb{R}$, $\lambda \in J$;

(H3) $f \in C^1([0;1] \times \mathbb{R} \times J)$;

(H4) $\left|\frac{\partial f}{\partial u}(t,u,\lambda)\right| \leq l_1$, for all $(t,u,\lambda) \in [0;t_m] \times \mathbb{R} \times J$ and $\left|\frac{\partial f}{\partial u}(t,u,\lambda)\right| \leq l_2$, for all $(t,u,\lambda) \in [t_m;1] \times \mathbb{R} \times J$;

(H5) $l_1 \cdot t_m \cdot \left(1 + |a| \cdot \sum\limits_{k=1}^{m} |a_k|\right) < 1$, where $a = \left(1 + \sum\limits_{k=1}^{m} a_k\right)^{-1}$;

Theorem 3.2.1. *If conditions* $(H1)$, $(H2)$ *and* $(H5)$ *hold then the problem (3.4) has a unique solution* $x^* \in C([0;1] \times J)$.

Proof. Let $X = (C([0;1] \times J), \|\cdot\|)$ where

$$\|x\| = \max\{\|x\|_\infty, \|x\|_\tau\}, \tag{3.5}$$

$$\|x\|_\infty = \max_{(t,\lambda)\in[0;t_m]\times J} |x(t,\lambda)| \ \text{ and } \ \|x\|_\tau = \max_{(t,\lambda)\in[t_m;1]\times J} |x(t,\lambda)| e^{-\tau(x-t_m)}.$$

Following the technique from [8] and [13], the problem (3.4) is equivalent with the following fixed point problem

$$x(t,\lambda) = B(x)(t,\lambda), \tag{3.6}$$

where $B : X \to X$

$$B(x)(t,\lambda) = -a\sum_{k=1}^{m} a_k \int_0^{t_k} f(s,x(s,\lambda),\lambda)\,ds + \int_0^{t} f(s,x(s,\lambda),\lambda)\,ds. \tag{3.7}$$

Actually, the operator B appears as sum of two integral operators, one of Fredholm, whose values depend only on the restrictions of functions to $[0;t_m]$ and second of a Volterra type depending on the restrictions of functions to $[t_m;1]$

$$B = B_F + B_V,$$

where

$$B_F(x)(t,\lambda) = \begin{cases} -a\sum\limits_{k=1}^{m} a_k \int\limits_0^{t_k} f(s,x(s,\lambda),\lambda)\,ds + \int\limits_0^{t} f(s,x(s,\lambda),\lambda)\,ds, & t\in[0;t_m] \\ -a\sum\limits_{k=1}^{m} a_k \int\limits_0^{t_k} f(s,x(s,\lambda),\lambda)\,ds + \int\limits_0^{t_m} f(s,x(s,\lambda),\lambda)\,ds, & t\in[t_m;1] \end{cases} \tag{3.8}$$

and

$$B_V(x)(t,\lambda) = \begin{cases} 0, & t\in[0;t_m] \\ \int\limits_{t_m}^{t} f(s,x(s,\lambda),\lambda)\,ds, & t\in[t_m;1] \end{cases}. \tag{3.9}$$

For $t \in [0;t_m]$, we have

$$\begin{aligned} |B(x_1)(t,\lambda) - B(x_2)(t,\lambda)| &= |B_F(x_1)(t,\lambda) - B_F(x_2)(t,\lambda)| \\ &\le |a|\cdot\sum_{k=1}^{m}|a_k|\int_0^{t_k}|f(s,x_1(s,\lambda),\lambda) - f(s,x_2(s,\lambda),\lambda)|\,ds \\ &\quad + \int_0^{t}|f(s,x_1(s,\lambda),\lambda) - f(s,x_2(s,\lambda),\lambda)|\,ds \\ &\le \left(1+|a|\cdot\sum_{k=1}^{m}|a_k|\right)\int_0^{t_m}|f(s,x_1(s,\lambda),\lambda) - f(s,x_2(s,\lambda),\lambda)|\,ds \\ &\le l_1\cdot t_m\cdot\left(1+|a|\cdot\sum_{k=1}^{m}|a_k|\right)\|x_1-x_2\|_\infty, \end{aligned}$$

therefore

$$\|B(x_1) - B(x_2)\|_\infty \le l_1\cdot\left(1+|a|\cdot\sum_{k=1}^{m}|a_k|\right)\|x_1-x_2\|_\infty. \tag{3.10}$$

For $t\in[t_m;1]$, we have

$$|B(x_1)(t,\lambda) - B(x_2)(t,\lambda)|$$

$$\le |a|\cdot\sum_{k=1}^{m}|a_k|\int_0^{t_k}|f(s,x_1(s,\lambda),\lambda) - f(s,x_2(s,\lambda),\lambda)|\,ds$$

$$+\int_0^{t_m} |f(s,x_1(s,\lambda),\lambda)-f(s,x_2(s,\lambda),\lambda)|\,ds$$

$$+\int_{t_m}^{t} |f(s,x_1(s,\lambda),\lambda)-f(s,x_2(s,\lambda),\lambda)|\,ds$$

$$\leq l_1\cdot t_m\cdot\left(1+|a|\cdot\sum_{k=1}^{m}|a_k|\right)\|x_1-x_2\|_\infty+\frac{l_2}{\tau}\|x_1-x_2\|_\tau e^{\tau(t-t_m)},$$

therefore

$$\|B(x_1)-B(x_2)\|_\tau\leq l_1\cdot t_m\cdot\left(1+|a|\cdot\sum_{k=1}^{m}|a_k|\right)\|x_1-x_2\|_\infty+\frac{l_2}{\tau}\|x_1-x_2\|_\tau. \quad (3.11)$$

From (3.10) and (3.11) we get tat $\|B(x_1)-B(x_2)\|\leq\alpha_B\|x_1-x_2\|$, with $\alpha_B = l_1\cdot t_m\cdot\left(1+|a|\cdot\sum_{k=1}^{m}|a_k|\right)+\frac{l_2}{\tau}$. According to $(H5)$, we can choose $\tau>0$ large enough such that $\alpha_B<1$. Hence B is an α_B-contraction, so we obtain the conclusion. □

Theorem 3.2.2. *If conditions*$(H1)$, $(H3)-(H5)$ *hold then the problem (3.4) has a unique solution* $x^*\in C^1([0;1]\times J)$.

Proof. Let $X=(C([0;1]\times J),\|\cdot\|)$ with the norm defined by (3.5). Condition $(H4)$ implies $(H2)$, thus from Theorem 3.2.1 we have B, defined by (3.7), is an α_B-contraction and $F_B=\{x^*\}$. It is clear that if $f(\cdot,u,\lambda)\in C^1[0;1]$ for all $(u,\lambda)\in\mathbb{R}\times J$ then $x^*(\cdot,\lambda)\in C^1[0;1]$ for all $\lambda\in J$.

If we formally derivate the fixed point equation (3.6) with respect to λ we get

$$\begin{aligned}\frac{\partial x}{\partial\lambda}(t,\lambda) &= -a\sum_{k=1}^{m}a_k\int_0^{t_k}\frac{\partial f}{\partial u}(s,x(s,\lambda),\lambda)\frac{\partial x}{\partial\lambda}(t,\lambda)\,ds-a\sum_{k=1}^{m}a_k\int_0^{t_k}\frac{\partial f}{\partial\lambda}(s,x(s,\lambda),\lambda)\,ds\\&\quad+\int_0^{t}\frac{\partial f}{\partial u}(s,x(s,\lambda),\lambda)\frac{\partial x}{\partial\lambda}(t,\lambda)\,ds+\int_0^{t}\frac{\partial f}{\partial\lambda}(s,x(s,\lambda),\lambda)\,ds.\end{aligned}$$

This suggest us to consider the operator $C: X\times X\to X$ with $(x,y)\longmapsto C(x,y)$, where

$$\begin{aligned}C(x,y)(t,\lambda) &= -a\sum_{k=1}^{m}a_k\int_0^{t_k}\frac{\partial f}{\partial u}(s,x(s,\lambda),\lambda)\,y(t,\lambda)\,ds-a\sum_{k=1}^{m}a_k\int_0^{t_k}\frac{\partial f}{\partial\lambda}(s,x(s,\lambda),\lambda)\,ds\\&\quad+\int_0^{t}\frac{\partial f}{\partial u}(s,x(s,\lambda),\lambda)\,y(t,\lambda)\,ds+\int_0^{t}\frac{\partial f}{\partial\lambda}(s,x(s,\lambda),\lambda)\,ds.\end{aligned}$$

The operator C appears as a sum of two operators $C = C_F + C_V$, one of Fredholm type

$$C_F(x,y)(t,\lambda) = \begin{cases} -a\sum\limits_{k=1}^{m} a_k \int\limits_0^{t_k} \frac{\partial f}{\partial u}(s,x(s,\lambda),\lambda)\, y(t,\lambda)\,ds - a\sum\limits_{k=1}^{m} a_k \int\limits_0^{t_k} \frac{\partial f}{\partial \lambda}(s,x(s,\lambda),\lambda)\,ds + \\ \quad + \int\limits_0^{t} \frac{\partial f}{\partial u}(s,x(s,\lambda),\lambda)\, y(t,\lambda)\,ds + \int\limits_0^{t} \frac{\partial f}{\partial \lambda}(s,x(s,\lambda),\lambda)\,ds, \quad t\in[0;t_m] \\ -a\sum\limits_{k=1}^{m} a_k \int\limits_0^{t_k} \frac{\partial f}{\partial u}(s,x(s,\lambda),\lambda)\, y(t,\lambda)\,ds - a\sum\limits_{k=1}^{m} a_k \int\limits_0^{t_k} \frac{\partial f}{\partial \lambda}(s,x(s,\lambda),\lambda)\,ds + \\ \quad + \int\limits_0^{t_m} \frac{\partial f}{\partial u}(s,x(s,\lambda),\lambda)\, y(t,\lambda)\,ds + \int\limits_0^{t_m} \frac{\partial f}{\partial \lambda}(s,x(s,\lambda),\lambda)\,ds, \quad t\in[t_m;1] \end{cases}$$

and second of Volterra type

$$C_V(x,y)(t,\lambda) = \begin{cases} 0, \ t\in[0;t_m] \\ \int\limits_{t_m}^{t} \frac{\partial f}{\partial u}(s,x(s,\lambda),\lambda)\, y(t,\lambda)\,ds + \int\limits_{t_m}^{t} \frac{\partial f}{\partial \lambda}(s,x(s,\lambda),\lambda)\,ds, \ t\in[t_m;1] \end{cases}.$$

In the same manner as for B we get that $\|C(x,y_1) - C(x,y_2)\| \le \alpha_C \|y_1 - y_2\|$, for all $x, y_1, y_2 \in X$, where $\alpha_C = \alpha_B = l_1 \cdot t_m \cdot \left(1 + |a| \cdot \sum\limits_{k=1}^{m} |a_k|\right) + \frac{l_2}{\tau}$. Thus $C(x,\cdot) : X \to X$ is an α_C-contraction, for all $x \in X$. By Theorem 2.1, we get that the operator $A : X \times X \to X \times X$

$$A(x,y) = (B(x), C(x,y)) \tag{3.12}$$

is a PO with $F_A = \{(x^*, y^*)\}$ and the sequence (x_n, y_n), given by

$$x_{n+1} = B(x_n),\, y_{n+1} = C(x_n, y_n)$$

converge uniformly to (x^*, y^*) for any starting point $(x_0, y_0) \in X \times X$.

If we take $(x_0, y_0) \in X \times X$ such that $y_0 = \frac{\partial x_0}{\partial \lambda}$ then we prove by induction that $y_n = \frac{\partial x_n}{\partial \lambda}$, for all $n \in \mathbb{N}$. Thus,

$$x_n \overset{unif}{\longrightarrow} x^* \text{ and } \frac{\partial x_n}{\partial \lambda} \overset{unif}{\longrightarrow} y^* \text{ as } n \to +\infty.$$

From the above convergences it follows that there exists $\frac{\partial x^*}{\partial \lambda}$ and $\frac{\partial x^*}{\partial \lambda} = y^*$. □

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