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A CONTRACTION PRINCIPLE ON GAUGE SPACES WITH GRAPHS AND APPLICATION TO INFINITE GRAPH-DIRECTED ITERATED FUNCTION SYSTEMS

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Abstract. We consider multi-valued maps defined on a complete gauge space endowed with a directed graph. We establish a fixed point result for maps which send connected points into connected points and satisfy a generalized contraction condition. Then, we study infinite graph-directed iterated function systems (H-IIFS). We give conditions insuring the existence of a unique attractor to an H-IIFS. Finally, we apply our fixed point result for multi-valued contractions on gauge spaces endowed with a graph to obtain more information on the attractor of an H-IIFS. More precisely, we construct a suitable gauge space endowed with a graph G and a suitable multi-valued G-contraction such that its fixed points are sub-attractors of the H-IIFS.

Key Words and Phrases: Fixed point, multi-valued map, contraction, graph, graph-directed iterated function system, infinite system, attractor gauge space.

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1. Introduction

In 2008, Jachymski [13] introduced the notion of single-valued G-contraction defined on a complete metric space endowed with a graph, which is a map preserving the graph and satisfying a contraction condition only between points related by an edge. He proved some generalizations of the Banach contraction principle to single-valued G-contractions. In particular, he generalized many contractions results in partially ordered sets, see [16, 17, 18, 19].

In [4], Dinevari and Frigon generalized Jachymski's fixed point results to multi-valued maps by introducing the notions of multi-valued G-contraction and weak G-contraction on a complete metric space endowed with a graph. Other generalizations of Jachymski's results to multi-valued maps were obtained in [15].

In 1982, Gheorgiu [10] presented a fixed point result for general single-valued contractions in complete gauge spaces. In [2], Chiş and Precup extended this result and they presented a continuation principle for such contractions. Another approach to obtain fixed point results was developed in [7] for single-valued contractions and in [8] for multi-valued contractions on complete gauge spaces, (see also [9] for a survey of results on that subject).

In this paper, we consider a complete gauge space X endowed with a directed graph G. We introduce the notions of multi-valued G-contraction and G-Lipschitz multi-valued map in the sense of Gheorgiu on X. Then, we establish a fixed point result for such multi-valued maps. This result generalizes fixed point results for single-valued and multi-valued contractions on complete metric spaces endowed with a graph obtained in [13] and [4] respectively. It is worthwhile to notice that our fixed point result is new even in the particular case where the map is single-valued and defined on X.

In this paper, we are also interested to apply our fixed point result to infinite iterated function systems.

An iterated function system (IFS) is a finite set of self-maps $\{T_i : i = 1, \dots, n\}$ defined on a complete metric space (M, d). Using the Banach contraction principle, Hutchinson [12] proved that if each T_i is a contraction, then there exists a unique nonempty compact set $K \subset M$, called the attractor of the IFS, such that

$$K = \bigcup_{i=1}^{n} T_i(K).$$

This result was popularized by Barnsley [1] as the main method of constructing fractals.

Geometric graph-directed constructions are generalizations of iterated function systems. Mauldin and Williams [14] were the firsts who introduced the notion of graph-directed constructions in $\mathbb{R}^m$ governed by a finite directed graph H and similarity maps $T_{i,j}$ which are labeled by the edges of the graph. They established that each geometric graph-directed construction has a unique attractor. Graph-directed constructions have been studied and generalized by many authors, see for example [3, 6, 11] and the references therein.

Recently, Dinevari and Frigon [5] applied their fixed point results for multi-valued G-contractions established in [4] to obtain more information on the attractor K of a graph-directed iterated function system governed by a finite directed graph and a finite family of contractions $\{T_{i,j}\}$ defined on complete metric spaces and labeled by the edges of the graph. To this aim, they defined a complete metric space, a suitable directed graph G on this space, and an appropriate multi-valued G-contraction. Using the fixed points of this G-contraction, they studied certain subsets of the attractor K and the relations between these sub-attractors.

In this paper, we consider a directed graph $H = (V(H), E(H))$ such that $V(H)$ the set of vertices and $E(H)$ the set of edges are countably infinite sets. We study infinite graph-directed iterated function systems over the graph H (H-IIFS). Such an H-IIFS contains a family of contractions $\{T_{i,j}\}_{(i,j)\in E(H)}$ on complete metric spaces. We give conditions insuring the existence of a unique attractor to this H-IIFS. Our result relies on a generalization of Gheorgiu's fixed point theorem on gauge spaces due to Chiş and Precup [2].

Then, under an extra assumption on the H-IIFS, we apply our fixed point result for multi-valued contractions on complete gauge spaces endowed with graphs to obtain more information on the attractor of this H-IIFS. Those results are obtained in Section 6. In order to prove those results, taking into account the H-IIFS, we construct a suitable complete gauge space on which we define an appropriate directed graph G in Section 4. In Section 5, we define a multi-valued map on this gauge space and we show that it is a G-contraction.

2. Main results

In this section, we introduce the notions of infinite MW-graph H and infinite graph iterated function system over the graph H. We give conditions insuring the existence of a unique attractor to an infinite graph iterated function system over the graph H.

Definition 2.1. A directed graph $H = (V(H), E(H))$ is called an *infinite MW-directed graph* if

(i) $V(H)$ is countable;
(ii) H has no parallel edges;
(iii) $1 \leq \text{outdeg}(i) < \infty$ for every $i \in V(H)$, where $\text{outdeg}(i)$ is the number of outward directed edges emanating from vertex i.

Definition 2.2. Let $H = (V(H), E(H))$ be an infinite MW-directed graph. An *infinite graph iterated function system over the graph H* (H-IIFS) is a family of nonempty complete metric spaces, $\{M_i : i \in V(H)\}$, and, for each $(i,j) \in E(H)$, a single-valued contraction $T_{i,j} : M_j \to M_i$ with constant of contraction $\lambda_{i,j}$. An H-IIFS is denoted by $\{T_{i,j}\}_H$.

An attractor of an H-IIFS is defined as follows.

Definition 2.3. Let $\{T_{i,j}\}_H$ be an H-IIFS. An *attractor* K of this H-IIFS is a family of nonempty compact sets $K = (K_i)_{i\in V(H)}$ such that $K_i \subset M_i$ and

$$K_i = \bigcup_{(i,j)\in E(H)} T_{i,j}(K_j) \quad \forall i \in V(H).$$

In order to establish the existence of an attractor to some H-IIFS, we will use the following generalization of Gheorghiu's fixed point result due to Chiş and Precup [2] that we recall for sake of completeness.

Theorem 2.4 ([2])**.** *Let $(X, \{q_s\}_{s\in S})$ be a complete gauge space, and $f : X \to X$ a single-valued map. Assume that*

(i) *there exist a function $\psi : S \to S$ and $k = (k_s)_{s\in S}$ such that $k_s \geq 0$ for all $s \in S$,*

$$q_s\big(f(x), f(y)\big) \leq k_s q_{\psi(s)}(x,y) \quad \forall s \in S,\ \forall x,y \in X, \tag{2.1}$$

and

$$\sum_{n=1}^{\infty} k_s k_{\psi(s)} \cdots k_{\psi^{n-1}(s)} q_{\psi^n(s)}(x,y) < \infty \quad \forall s \in S,\ \forall x,y \in X,$$

where ψ^n is the n-th iteration of ψ;

(ii) *for every $x_0 \in X$, if $\{f^n(x_0)\}$ converges to some $x \in X$, then $x = f(x)$.*

Then f has a unique fixed point.

We need to introduce some notations. In what follows, H is an infinite MW-directed graph and $\{T_{i,j}\}_H$ is an H-IIFS.

Let

$$\Gamma_0 = \big\{ I = \{i_1, \dots, i_n\} \subset V(H) : n \in \mathbb{N} \big\}. \tag{2.2}$$

We denote

$$k_I = \max\big\{ \lambda_{i,j} : (i,j) \in E(H) \text{ and } i \in I \big\} \quad \forall I \in \Gamma_0,$$

and we define the map $\varphi : \Gamma_0 \to \Gamma_0$ by

$$\varphi(I) = I \cup \big\{ j \in V(H) : \exists i \in I \text{ such that } (i,j) \in E(H) \big\}. \tag{2.3}$$

We consider the space

$$\mathcal{Y} = \big\{ Y = (Y_i)_{i \in V(H)} : \emptyset \neq Y_i \subset M_i \text{ is compact} \big\}. \tag{2.4}$$

For every $I \in \Gamma_0$ and $Y, \hat{Y} \in \mathcal{Y}$, let

$$p_I(Y, \hat{Y}) = \max\big\{ D_i(Y_i, \hat{Y}_i) : i \in I \big\}, \tag{2.5}$$

where D_i is the Hausdorff metric on M_i. It is easy to see that $(\mathcal{Y}, \{p_I\}_{I \in \Gamma_0})$ is a complete gauge space.

We are ready to establish the existence of an attractor of the H-IIFS.

Theorem 2.5. *Let $\{T_{i,j}\}_H$ be an H-IIFS. Assume that*

$$\sum_{n=1}^{\infty} k_I k_{\varphi(I)} \cdots k_{\varphi^{n-1}(I)} p_{\varphi^n(I)}(Y, \hat{Y}) < \infty \quad \forall I \in \Gamma_0,\ \forall Y, \hat{Y} \in \mathcal{Y}. \tag{2.6}$$

Then $\{T_{i,j}\}_H$ has a unique attractor K.

Proof. Let us define $f : \mathcal{Y} \to \mathcal{Y}$ by

$$f_i(Y) = \bigcup_{(i,j) \in E(H)} T_{i,j}(Y_j).$$

Using the fact that every $T_{i,j}$ is a contraction in the classical sense, we prove that

$$p_I(f(Y), f(\hat{Y})) \le k_I p_{\varphi(I)}(Y, \hat{Y}) \quad \forall I \in \Gamma_0,\ \forall Y, \hat{Y} \in \mathcal{Y}.$$

Indeed,

$$\begin{aligned}
p_I(f(Y), f(\hat{Y})) &= \max\big\{D_i(f_i(Y), f_i(\hat{Y})) : i \in I\big\} \\
&= \max\left\{D_i\Big(\bigcup_{(i,j)\in E(H)} T_{i,j}(Y_j), \bigcup_{(i,j)\in E(H)} T_{i,j}(\hat{Y}_j)\Big) : i \in I\right\} \\
&\le \max\left\{\max_{(i,j)\in E(H)} D_i(T_{i,j}(Y_j), T_{i,j}(\hat{Y}_j)) : i \in I\right\} \\
&\le \max\left\{\max_{(i,j)\in E(H)} \lambda_{i,j} D_j(Y_j, \hat{Y}_j) : i \in I\right\} \\
&\le k_I \max\big\{D_i(Y_i, \hat{Y}_i) : i \in \varphi(I)\big\} \\
&= k_I p_{\varphi(I)}(Y, \hat{Y}).
\end{aligned}$$

We claim that (ii) of Theorem 2.4 is satisfied. Indeed, let us assume that $Y^0 \in \mathcal{Y}$ is such that $\{f^n(Y^0)\}$ converges to some $Y \in \mathcal{Y}$. If $Y \neq f(Y)$, there exists $i \in V(H)$ such that

$$D_i(Y_i, f(Y)_i) = r > 0.$$

Let $N \in \mathbb{N}$ be such that

$$p_{\varphi(\{i\})}\big(f^n(Y^0), Y\big) < \frac{r}{2} \quad \forall n \ge N.$$

So,

$$\begin{aligned}
r = p_{\{i\}}(Y, f(Y)) &\le p_{\{i\}}\big(Y, f^{N+1}(Y^0)\big) + p_{\{i\}}\big(f^{N+1}(Y^0), f(Y)\big) \\
&\le p_{\varphi(\{i\})}\big(Y, f^{N+1}(Y^0)\big) + k_{\{i\}} p_{\varphi(\{i\})}\big(f^N(Y^0), Y\big) < r.
\end{aligned}$$

Contradiction.

It follows from Theorem 2.4 that f has a unique fixed point $K \in \mathcal{Y}$, and hence, K is an attractor of $\{T_{i,j}\}_H$. □

Remark 2.6. Observe that (2.6) is satisfied if:

$$\sup\{\lambda_{i,j} : (i,j) \in E(H)\} < 1 \quad \text{and} \quad \sup\{\mathrm{diam}(M_i) : i \in V(H)\} < \infty. \tag{2.7}$$

So, every H-IIFS satisfying (2.7) has a unique attractor.

Example 2.7. Let $H = (V(H), E(H))$ (see Figure 2.1) be given by

$$V(H) = \mathbb{Z} \quad \text{and} \quad E(H) = \{(n, n+1), (n, n+2) : n \in \mathbb{Z}\}.$$

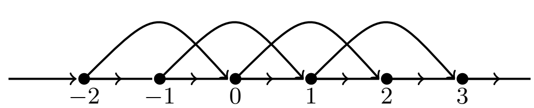

FIGURE 2.1. The MW-directed graph H of Example 2.7.

For $n \in \mathbb{Z}$, let $M_n = [n, n+1]$ and $T_{n,n+1} : M_{n+1} \to M_n$, $T_{n,n+2} : X_{n+2} \to X_n$ contractions with constants of contraction $\lambda_{n,n+1} < 1$ and $\lambda_{n,n+2} < 1$ respectively. We define

$$\lambda_n = \max\{\lambda_{n,n+1}, \lambda_{n,n+2}\}.$$

We assume that $n \mapsto \lambda_n$ is nonincreasing.

It follows from Theorem 2.5 that the H-IIFS, $\{T_{i,j}\}_H$, has a unique attractor K. Indeed, one has

$$\begin{aligned}
&\Gamma_0 = \{I \subset \mathbb{Z} : 0 < \operatorname{card}(I) < \infty\},\\
&\mathcal{Y} = \{Y = (Y_n)_{n\in\mathbb{Z}} : \emptyset \neq Y_n \subset [n, n+1] \text{ closed } \forall n \in \mathbb{Z}\},\\
&p_I(Y, \widehat{Y}) = \max\{D(Y_i, \widehat{Y}_i) : i \in I\} \quad \forall Y, \widehat{Y} \in \mathcal{Y}, \forall I \in \Gamma_0,\\
&\varphi : \Gamma_0 \to \Gamma_0 \quad \text{given by} \quad \varphi(I) = I \cup \{i+1, i+2 : i \in I\}.
\end{aligned}$$

Observe that

$$k_I = \max\{\lambda_{i,j} : (i,j) \in E(H) \text{ and } i \in I\} = \lambda_{i_0}, \quad \text{where} \quad i_0 = \min I,$$

and $k_I = k_{\varphi(I)}$ for every $I \in \Gamma_0$. Therefore,

$$\begin{aligned}
\sum_{n=1}^{\infty} k_I k_{\varphi(I)} \cdots k_{\varphi^{n-1}}(I) p_{\varphi^n(I)}(Y, \widehat{Y}) &\leq \sum_{n=1}^{\infty} \lambda_{i_0}^n p_{\varphi^n(I)}(Y, \widehat{Y})\\
&\leq \sum_{n=1}^{\infty} \lambda_{i_0}^n < \infty \quad \forall Y, \widehat{Y} \in \mathcal{Y}.
\end{aligned}$$

Hence, $\{T_{i,j}\}_H$ satisfies the assumptions of Theorem 2.5.

3. Multi-valued contraction on gauge spaces endowed with a graph

In this section, we consider $\big(X, \{q_s\}_{s\in S}\big)$ a complete gauge space endowed with a directed graph $G = (V(G), E(G))$ such that the set of vertices $V(G) = X$ and the set of edges $E(G)$ has no parallel edges and it contains the diagonal. We generalize Theorem 2.4 to multi-valued map $F : X \to X$ satisfying a condition analogous to (2.1) only for $x, y \in X$ related by an edge $(x, y) \in E(G)$.

Definition 3.1. Let $F : X \to X$ be a multi-valued map with nonempty values. We say that F is a *G-Lipschitz* map in the sense of Gheorghiu with map $\psi : S \to S$ and constant $\lambda = (\lambda_s)_{s\in S}$ such that $\lambda_s \geq 0$ for all $s \in S$, if, for every $(x, y) \in E(G)$ and every $u \in F(x)$, there exists $v \in F(y)$ such that $(u, v) \in E(G)$ and

$$q_s(u, v) \leq \lambda_s q_{\psi(s)}(x, y) \quad \forall s \in S. \tag{3.1}$$

The map F is called a *G-contraction* if it is a G-Lipschitz map with $\lambda_s < 1$ for every $s \in S$.

We consider suitable trajectories in X.

Definition 3.2. Let $F : X \to X$ be a multi-valued mapping and $x_0 \in X$. We say that a sequence $\{x_n\}$ is a *G-Picard trajectory from* x_0, if $x_n \in F(x_{n-1})$ and $(x_{n-1}, x_n) \in E(G)$ for all $n \in \mathbb{N}$. The set of all such G-Picard trajectories from x_0 is denoted by $T(F, G, x_0)$.

Here is our main fixed point result for multi-valued contractions in the sense of Gheorgiu on the gauge space X endowed with a directed graph G.

Theorem 3.3. *Let $F : X \to X$ be a multi-valued G-Lipschitz map with constant $\lambda = (\lambda_s)_{s\in S}$ and map $\psi : S \to S$. Assume that there exists $(x_0, x_1) \in E(G)$ such that $x_1 \in F(x_0)$ and*

$$\sum_{n=1}^{\infty} \lambda_s \lambda_{\psi(s)} \cdots \lambda_{\psi^{(n-1)}(s)} q_{\psi^n(s)}(x_0, x_1) < \infty \quad \forall s \in S. \tag{3.2}$$

Then, there exists a G-Picard trajectory from x_0 converging to some $\hat{x} \in X$. In addition, assume that one of the following conditions holds:

(i) *F is G-*Picard continuous from *x_0, i.e. the limit of any convergent G-Picard trajectory $\{x_n\} \in T(F, G, x_0)$ is a fixed point of F;*
(ii) *F has closed values and, for every $\{x_n\}$ in $T(F, G, x_0)$ converging to some $x \in X$, there exists a subsequence $\{x_{n_k}\}$ such that $(x_{n_k}, x) \in E(G)$ for all $k \in \mathbb{N}$.*

Then, $\hat{x}$ is a fixed point of F. Moreover, every converging G-Picard trajectory from x_0 converges to a fixed point of F.

Proof. Let x_0 and $x_1 \in F(x_0)$ be given by assumption. Since F is a G-Lipschitz map, one can choose a sequence $\{x_n\}$ such that $x_{n+1} \in F(x_n)$, $(x_n, x_{n+1}) \in E(G)$ and

$$q_s(x_n, x_{n+1}) \leq \lambda_s q_{\psi(s)}(x_{n-1}, x_n) \leq \ \ldots \ \leq \lambda_s \lambda_{\psi(s)} \ldots \lambda_{\psi^{n-1}(s)} q_{\psi^n(s)}(x_0, x_1),$$

for every $s \in S$ and $n \in \mathbb{N}$. Moreover, for every $m \in \mathbb{N}$,

$$q_s(x_n, x_{n+m}) \leq \sum_{i=n}^{n+m-1} q_s(x_i, x_{i+1}) \leq \sum_{i=n}^{n+m-1} \lambda_s \lambda_{\psi(s)} \ldots \lambda_{\psi^{i-1}(s)} q_{\psi^i(s)}(x_0, x_1).$$

Therefore, $\{x_n\}$ is a Cauchy sequence and hence converges to some $\hat{x} \in X$.

If the condition (i) is satisfied, then clearly $\hat{x}$ is a fixed point of F.

On the other hand, if the condition (ii) is satisfied, then there exists a subsequence $\{x_{n_k}\}$ such that $(x_{n_k}, \hat{x}) \in E(G)$ for every $k \in \mathbb{N}$. Since F is a G-Lipschitz map, for each $k \in \mathbb{N}$, there exists $y_{n_k+1} \in F(\hat{x})$ such that $(x_{n_k+1}, y_{n_k+1}) \in E(G)$ and

$$q_s(x_{n_k+1}, y_{n_k+1}) \leq \lambda_s q_{\psi(s)}(x_{n_k}, \hat{x}) \quad \forall s \in S.$$

Therefore, for every $s \in S$,

$$q_s(y_{n_k+1}, \hat{x}) \leq q_s(y_{n_k+1}, x_{n_k+1}) + q_s(x_{n_k+1}, \hat{x}) \leq \lambda_s q_{\psi(s)}(x_{n_k}, \hat{x}) + q_s(x_{n_k+1}, \hat{x}).$$

Consequently, $y_{n_k+1} \to \hat{x}$, and hence $\hat{x} \in F(\hat{x})$ since F has closed values. □

Remark 3.4. We could have formulated a more general result by considering two families of gauges as it is done in [2, 10]. We preferred not to do so for sake a simplicity.

In the particular case where X is a metric space, the previous result generalizes a fixed point result for multi-valued contraction obtained in [4]. If, in addition F is single-valued, the fixed point result for G-contraction due to Jachymski [13] is generalized by the following result.

Corollary 3.5. *Let $f : X \to X$ be a single-valued map such that there exist $\psi : S \to S$ and $\lambda = (\lambda_s)_{s\in S}$ such that $\lambda_s \geq 0$ for all $s \in S$, and for every $(x, y) \in E(G)$*

$$(f(x), f(y)) \in E(G) \quad \text{and} \quad q_s(f(x), f(y)) \leq \lambda_s q_{\psi(s)}(x, y) \quad \forall s \in S. \tag{3.3}$$

Assume that there exists $x_0 \in X$ such that $(x_0, f(x_0)) \in E(G)$ and

$$\sum_{n=1}^{\infty} \lambda_s \lambda_{\psi(s)} \cdots \lambda_{\psi^{(n-1)}(s)} q_{\psi^n(s)}(x_0, f(x_0)) < \infty \quad \forall s \in S. \tag{3.4}$$

Then, the sequence $\{f^n(x_0)\}$ converges to some $\hat{x} \in X$. In addition, assume that one of the following conditions holds:

(i) $f\big(f^n(x_0)\big) \to f(\hat{x})$;
(ii) *there exists a subsequence $\{f^{n_k}(x_0)\}$ such that $(f^{n_k}(x_0), \hat{x}) \in E(G)$ for all $k \in \mathbb{N}$.*

Then, $\hat{x}$ is a fixed point of f.

It is worthwhile to point out that in Theorem 3.3, we did not assume the continuity of the G-Lipschitz map F. The following lemma could be useful to deduce that the limit of a convergent G-Picard trajectory is a fixed point of F.

Lemma 3.6. *Let $F : X \to X$ be a multi-valued G-Lipschitz map with constant $\lambda = (\lambda_s)_{s\in S}$ and map $\psi : S \to S$. Assume that there exists $x_0 \in X$ and a G-Picard trajectory $\{x_n\}$ from x_0 converging to some $\hat{x} \in X$. In addition, assume that there exists $\hat{u} \in F(\hat{x})$ such that, for every $s \in S$, the following conditions hold:*

(i) *there exists a subsequence $\{x_{n_k}\}$ such that there exists $\{\hat{x}_{n_k}\}$ a sequence in X satisfying*

$$(\hat{x}, \hat{x}_{n_k}) \in E(G)\ \forall k \in \mathbb{N}, \quad \text{and} \quad q_{\psi(s)}(x_{n_k}, \hat{x}_{n_k}) \to 0;$$

(ii) *for every $k \in \mathbb{N}$, one can choose $u_{n_k} \in F(\hat{x}_{n_k})$ such that*

$$(\hat{u}, u_{n_k}) \in E(G) \quad \text{and} \quad q_s(\hat{u}, u_{n_k}) \leq \lambda_s q_{\psi(s)}(\hat{x}, \hat{x}_{n_k}),$$

satisfying

$$q_s(u_{n_k}, x_{n_k+1}) \to 0 \quad \text{as } k \to \infty.$$

Then, $\hat{x} = \hat{u} \in F(\hat{x})$.

Proof. Let us suppose that $\hat{x} \neq \hat{u}$. Then, there exists $s \in S$ such that

$$q_s(\hat{u}, \hat{x}) = r > 0.$$

Observe that

$$\begin{aligned} q_s(\hat{u},\hat{x}) &\le q_s(\hat{u},u_{n_k}) + q_s(u_{n_k},x_{n_k+1}) + q_s(x_{n_k+1},\hat{x}) \\ &\le \lambda_s q_{\psi(s)}(\hat{x},\hat{x}_{n_k}) + q_s(u_{n_k},x_{n_k+1}) + q_s(x_{n_k+1},\hat{x}) \\ &\le \lambda_s q_{\psi(s)}(\hat{x},x_{n_k}) + \lambda_s q_{\psi(s)}(x_{n_k},\hat{x}_{n_k}) + q_s(u_{n_k},x_{n_k+1}) + q_s(x_{n_k+1},\hat{x}) \\ &\to 0. \end{aligned}$$

Contradiction. So, $\hat{x} = \hat{u} \in F(\hat{x})$. □

4. A suitable gauge space endowed with a directed graph

In order to get more information on the attractor to the H-IIFS, we will apply our main fixed point result for multi-valued G-contraction. In this section, we will define a suitable complete gauge space.

First, we need to introduce some notations. For a graph $H = (V(H), E(H))$, we denote an N-*directed path* in H from i_0 to i_N by $[i_n]_{n=0}^N$, and we denote the set of vertices from which there is a directed path in H reaching $i \in H$ by

$$[i]_{\leftarrow} = \{j \in V(H) : \text{ there is a directed path from } j \text{ to } i \text{ in } H\}. \tag{4.1}$$

We say that a subgraph $C = (V(C), E(C))$ of H is *connected* if for every $i, j \in V(C)$ there exists a directed path from i to j in C. A *connected component* of H is a maximal connected subgraph of H. A subgraph $C = (V(C), E(C))$ of H is *weakly connected* if the undirected graph induced by C is connected. Let C and $\widehat{C}$ be two connected components of H. We write

$$C \preceq \widehat{C} \quad \Longleftrightarrow \quad \text{there is a directed path from } C \text{ to } \widehat{C}.$$

Also, we write $C \prec \widehat{C}$ if $C \preceq \widehat{C}$ and $C \neq \widehat{C}$. We say that C and $\widehat{C}$ are *incomparable* if $C \npreceq \widehat{C}$ and $\widehat{C} \npreceq C$.

Let H be an infinite MW-directed graph and $\{T_{i,j}\}_H$ an H-IIFS with M_i a complete metric space for every $i \in V(H)$. We denote the set of all connected components of H by

$$C(H) = \{C : C \text{ is a connected component of } H\}. \tag{4.2}$$

In what follows, we will make the following assumption:

(H) H is an infinite MW-directed graph and $\{T_{i,j}\}_H$ is an H-IIFS such that
- (H1) H is weakly connected and
$$V(H) = \bigcup_{C \in C(H)} V(C);$$
- (H2) for every $i, j \in V(H)$, the length of directed paths from i to j is bounded, i.e.
$$\sup\big\{N : \exists [i_n]_{n=0}^N \text{ from } i = i_0 \text{ to } j = i_N \text{ containing no cycle}\big\} < \infty;$$
- (H3) the metric spaces M_i are bounded and
$$R = \sup\{diam(M_i) : i \in V(H)\} < \infty.$$

It follows from Definition 2.1 that $C(H)$ is countable. Let

$$\Gamma = \big\{ I \subset V(H) : 0 < \text{card}(I) < \infty, \text{ and } V(C) \subset I \ \forall C \in C(H) \text{ such that } V(C) \cap I \neq \emptyset \big\}. \tag{4.3}$$

We define the map $\phi : \Gamma \to \Gamma$ by

$$\phi(I) = I \cup \big\{ k \in V(H) : \text{ there exist } (i,j) \in E(H) \text{ and } C \in C(H) \text{ such that } i \in I \text{ and } j,k \in V(C) \big\}. \tag{4.4}$$

We are ready to define our suitable gauge space.

(X) Let $\mathcal{X}$ be the space of elements $X = (X_i)_{i \in V(H)}$ satisfying the following properties:

(X1) X_i is a compact subset of M_i for every $i \in V(H)$;

(X2) there exists $i \in V(H)$ such that $X_i \neq \emptyset$;

(X3) if $X_i \neq \emptyset$ for some $i \in V(C)$ and $C \in C(H)$, then $X_j \neq \emptyset$ for all $j \in V(C)$.

Taking into account the graph H, we endow $\mathcal{X}$ with a directed graph defined as follows.

(G) Let $G = (V(G), E(G))$ be the directed graph such that $V(G) = \mathcal{X}$ and, for $X, Y \in \mathcal{X}$, $(X,Y) \in E(G)$ if and only if, for every $i \in V(H)$, one of the following properties holds:

(Ga) $X_i = Y_i = \emptyset$, or $X_i \neq \emptyset$ and $Y_i \neq \emptyset$;

(Gb) $X_i = \emptyset$, $Y_i \neq \emptyset$ and, for $C \in C(H)$ such that $i \in V(C)$, there exist $k \in V(C)$ and $j \in V(H) \backslash V(C)$ such that $(k,j) \in E(H)$ and $X_j \neq \emptyset$.

We endow $\mathcal{X}$ with the family of gauges $\{d_I\}_{I \in \Gamma}$, where

$$d_I(X,Y) = \max \big\{ \overline{D}_i(X_i, Y_i) : i \in I \big\}, \tag{4.5}$$

with

$$\overline{D}_i(X_i, Y_i) = \begin{cases} D_i(X_i, Y_i), & \text{if } X_i \neq \emptyset, Y_i \neq \emptyset, \\ 0, & \text{if } X_i = \emptyset = Y_i, \\ R_i, & \text{otherwise,} \end{cases} \tag{4.6}$$

where D_i the Hausdorff metric in M_i and

(R) the family of constants $(R_i)_{i \in V(H)}$ is such that

(R1) for every $i \in V(H)$, $R_i > R$;

(R2) for every $C \in C(H)$, $R_i = R_j$ for all $i, j \in V(C)$;

(R3) for every $i, j \in V(H)$, if $R_i < R_j$, then $j \notin [i]_{\leftarrow}$;

(R4) for every $I \in \Gamma$, one has $R_i < R_j$ for every $i \in I$ and $j \in \phi(I) \backslash I$.

It is clear that $(\mathcal{X}, \{d_I\}_{I \in \Gamma})$ is a complete gauge space.

Now, we show that we can easily find $(R_i)_{i \in V(H)}$ satisfying (R).

Lemma 4.1. *Let H be an infinite MW-directed graph and $\{T_{i,j}\}_H$ an H-IIFS satisfying* (H). *Then, there exists $\{V_\mu : \mu \in L\}$ a family of non empty disjoint subsets with $L \subset \mathbb{Z}$ countable such that*

(1) *$V(H)=\bigcup_{\mu\in L} V_\mu$;*
(2) *for every $C\in C(H)$, if $V(C)\cap V_\mu\neq\emptyset$ for some $\mu\in L$, one has $V(C)\subset V_\mu$;*
(3) *for every $C,\widehat{C}\in C(H)$ such that $C\prec\widehat{C}$, $V(C)\subset V_\mu$ and $V(\widehat{C})\subset V_\nu$, one has $\mu<\nu$;*
(4) *if $\mu<\nu$ in L, then $j\notin[i]_\leftarrow$ for all $i\in V_\mu$ and $j\in V_\nu$.*

Moreover, for every strictly increasing map $\sigma: L\to\,]1,\infty[$, the family of constants $(R_i)_{i\in V(H)}$ defined by

$$R_i=\sigma(\mu)R \quad \text{if } i\in V_\mu,$$

satisfies (R).

Proof. Let $\mathcal{S}_0\subset C(H)$ be such that $\{C : C\in\mathcal{S}_0\}$ is a maximal set of incomparable connected components of H. We denote

$$\mathcal{S}_0^+=\{C\in C(H):\exists\widehat{C}\in\mathcal{S}_0 \text{ such that } \widehat{C}\prec C\};$$
$$\mathcal{S}_0^-=\{C\in C(H):\exists\widehat{C}\in\mathcal{S}_0 \text{ such that } C\prec\widehat{C}\}.$$

It follows from (H1) that $C(H)=\mathcal{S}_0\cup\mathcal{S}_0^+\cup\mathcal{S}_0^-$. We denote

$$\mathcal{S}_1=\{C\in\mathcal{S}_0^+ :\nexists\widehat{C}\in\mathcal{S}_0^+ \text{ such that } \widehat{C}\prec C\},$$

and we define inductively for each $n\in\mathbb{N}$,

$$\mathcal{S}_{n+1}=\left\{C\in\mathcal{S}_0^+\setminus\bigcup_{k=1}^n\mathcal{S}_k :\nexists\widehat{C}\in\mathcal{S}_0^+\setminus\bigcup_{k=1}^n\mathcal{S}_k \text{ such that } \widehat{C}\prec C\right\}.$$

Similarly, we denote

$$\mathcal{S}_{-1}=\{C\in\mathcal{S}_0^- :\nexists\widehat{C}\in\mathcal{S}_0^- \text{ such that } C\prec\widehat{C}\},$$

and we define inductively for each $n\in\mathbb{N}$,

$$\mathcal{S}_{-(n+1)}=\left\{C\in\mathcal{S}_0^-\setminus\bigcup_{k=1}^n\mathcal{S}_{-k} :\nexists\widehat{C}\in\mathcal{S}_0^-\setminus\bigcup_{k=1}^n\mathcal{S}_{-k} \text{ such that } C\prec\widehat{C}\right\}.$$

Let $L=\{\mu\in\mathbb{Z}:\mathcal{S}_\mu\neq\emptyset\}$ endowed with the natural order. We define

$$V_\mu=\bigcup_{C\in\mathcal{S}_\mu}V(C)\quad\forall\mu\in L.$$

Therefore, by (H),

$$V(H)=\bigcup_{\mu\in L}V_\mu.$$

By construction, (2), (3) and (4) are satisfied.

Let $\sigma: L\to\,]1,\infty[$ be a strictly increasing map, and the family of constants $(R_i)_{i\in V(H)}$ defined by

$$R_i=\sigma(\mu)R \quad\text{for}\quad i\in V_\mu.$$

The property (R) follows directly from (1)–(4) and the fact that $\sigma(L)\subset\,]1,\infty[$. □

5. A suitable G-contraction

We consider H an infinite MW-directed graph and $\{T_{i,j}\}_H$ an H-IIFS satisfying the condition (H). In this section, we will define an appropriate multi-valued G-contraction on $\mathcal{X}$, where $\mathcal{X}$ is the space endowed with the family of gauges $\{d_I\}_{I\in\Gamma}$ and endowed with the directed graph G defined in the previous section. This G-contraction will be used to get more information on the attractor of this infinite H-IIFS.

Let $X \in \mathcal{X}$. If $j \in V(H)$ is such that $X_j \neq \emptyset$, then $T_{i,j}(X_j) \neq \emptyset$ for all i such that $(i,j) \in E(H)$. So, it is important to distinguish all those edges. To this aim, we introduce the following notation. For $C \in C(H)$,

$$E_C(X) = \{(k,j) \in E(H) : k \in V(C), j \notin V(C), X_j \neq \emptyset\}. \tag{5.1}$$

Let us notice that the cardinality of $E_C(X)$ is finite since $\mathrm{outdeg}(i)$ is finite for every $i \in V(H)$.

For $C \in C(H)$ and $i, k \in V(C)$, we define $T_{i\to k} : M_k \to M_i$ by

$$T_{i\to k}(x) = \left\{T_{i_0,i_1} \circ \cdots \circ T_{i_{N-1},i_N}(x) : [i_n]_{n=0}^N \in \{i \xrightarrow{C} k\}\right\}, \tag{5.2}$$

where

$$\{i \xrightarrow{C} k\} = \{[i_n]_{n=0}^N : [i_n]_{n=0}^N \text{ is an } N\text{-directed path in } C \text{ from } i = i_0 \text{ to } k = i_N \text{ containing no cycle}\}. \tag{5.3}$$

For $i \in V(C)$ with $C \in C(H)$, we define the following subsets of M_i:

$$O_i(X,P) = \begin{cases} \emptyset, & \text{if } P = \emptyset, \\ \displaystyle\bigcup_{(k,j)\in P} T_{i\to k} \circ T_{k,j}(X_j), & \text{if } \emptyset \neq P \subset E_C(X); \end{cases} \tag{5.4}$$

and

$$W_i(X) = \begin{cases} \emptyset, & \text{if } X_i = \emptyset, \\ \displaystyle\bigcup_{(i,j)\in E(C)} T_{i,j}(X_j), & \text{if } X_i \neq \emptyset, \end{cases} \tag{5.5}$$

where $E(C) = \{(k,j) \in E(H) : k, j \in V(C)\}$.

We have all the ingredients to introduce a suitable multi-valued map. We define $F : \mathcal{X} \to \mathcal{X}$ by

$$F(X) = \left\{U = (U_i)_{i\in V(H)} \in \mathcal{X} : U_i \in F_i(X)\ \forall i \in V(H)\right\}, \tag{5.6}$$

where, for $i \in V(C)$ for some $C \in C(H)$, $F_i(X)$ is defined as follows:

$$F_i(X) = \begin{cases} \emptyset, & \text{if } X_i = \emptyset \text{ and } E_C(X) = \emptyset, \\ \left\{O_i(X,P) : \emptyset \neq P \subset E_C(X)\right\}, & \text{if } X_i = \emptyset \text{ and } E_C(X) \neq \emptyset, \\ \left\{W_i(X) \cup O_i(X,P) : P \subset E_C(X)\right\}, & \text{if } X_i \neq \emptyset. \end{cases} \tag{5.7}$$

It is easy to see that F is well defined and has finite, and hence closed values.

We show that F is a multi-valued G-contraction.

Proposition 5.1. *Let H be an infinite MW-directed graph and $\{T_{i,j}\}_H$ an H-IIFS satisfying* (H). *Let $(R_i)_{i\in V(H)}$ be a family of constants satisfying* (R). *Then, the multi-valued map defined as above, $F:\mathcal{X}\to\mathcal{X}$ is a G-contraction.*

Proof. We show that F is a G-contraction with constant of contraction $\lambda=(\lambda_I)_{I\in\Gamma}$, where

$$\lambda_I=\max\Bigg\{\max\{\lambda_{i,j}:i\in I,(i,j)\in E(H)\},\max\Big\{\frac{R}{R_i}:i\in I\Big\},\\ \max\Big\{\frac{R_i}{R_j}:i\in I,j\in\phi(I)\backslash I\Big\}\Bigg\},\quad (5.8)$$

where ϕ is defined in (4.4).

For $i,k\in V(C)$ for some $C\in C(H)$, we denote

$$\lambda_{i\to k}=\max\big\{\lambda_{i_0,i_1}\cdots\lambda_{i_{N-1},i_N}:[i_n]_{n=0}^N\in\{i\xrightarrow{C}k\}\big\},\quad (5.9)$$

where $\{i\xrightarrow{C}k\}$ is given in (5.3). Observe that $\lambda_{i\to k}\le\lambda_I$ for all $I\in\Gamma$ such that $i\in I$.

Let $X,Y\in\mathcal{X}$ be such that $(X,Y)\in E(G)$ and $U\in F(X)$. We look for $\widetilde{U}\in F(Y)$ such that $(U,\widetilde{U})\in E(G)$ and $d_I(U,\widetilde{U})\le\lambda_I d_{\phi(I)}(X,Y)$ for every $I\in\Gamma$.

Step 1: For $\mathbf{I}\subset\mathbf{\Gamma}$, different cases of U_i for $\mathbf{i}\in\mathbf{I}$:

Let $C\in C(H)$ be such that $i\in V(C)\subset I$.

Case 1: $U_i=\emptyset$ and $\widetilde{U}_i\neq\emptyset$ for every $\widetilde{U}\in F(Y)$.

In this case, $X_i=E_C(X)=\emptyset$ and $Y_i\cup E_C(Y)\neq\emptyset$ by (5.7).

If $Y_i\neq\emptyset$, since $(X,Y)\in E(G)$, by condition (Gb), there exist $k\in V(C)$ and $j\in V(H)\backslash V(C)$ such that $(k,j)\in E(H)$ and $X_j\neq\emptyset$. So, $(k,j)\in E_C(X)$. This contradicts the fact that $E_C(X)=\emptyset$.

If $E_C(Y)\neq\emptyset$, by (5.1), there exist $k\in V(C)$ and $j\in V(\widehat{C})$ such that $(k,j)\in E(H)$, $Y_j\neq\emptyset$ and $\widehat{C}\neq C$. One has $j\in\phi(I)\backslash I$ and $R_i<R_j$. Since $E_C(X)=\emptyset$, one has $X_j=\emptyset$. By condition (Gb), there exist $m\in V(\widehat{C})$, $l\in V(H)\backslash V(\widehat{C})$ such that $(m,l)\in E(H)$ and $X_l\neq\emptyset$. So, $E_{\widehat{C}}(X)\neq\emptyset$ and $U_j\neq\emptyset$ by (5.7). So, we obtain

$$U_i=\emptyset,\ \widetilde{U}_i\neq\emptyset\quad\text{and}\quad U_j\neq\emptyset\text{ for some }(k,j)\in E_C(Y)\quad (5.10)$$

$$\text{with }k\in V(C)\text{ and }j\in\phi(I)\backslash I.\quad (5.11)$$

Moreover, by (4.5), (4.6) and (5.8),

$$\overline{D}_i(U_i,\widetilde{U}_i)=R_i=\frac{R_i}{R_j}\overline{D}_j(X_j,Y_j)\le\lambda_I d_{\phi(I)}(X,Y)\quad\forall\widetilde{U}\in F(Y).\quad (5.12)$$

Case 2: $U_i\neq\emptyset$ and $\widetilde{U}_i=\emptyset$ for every $\widetilde{U}\in F(Y)$.

In this case, $X_i\cup E_C(X)\neq\emptyset$ and $Y_i\cup E_C(Y)=\emptyset$ by (5.7). Since $(X,Y)\in E(G)$, we deduce that $X_i=Y_i=\emptyset$ and hence $E_C(X)\neq\emptyset$. Let $(k,j)\in E_C(X)$. One has

$X_j \neq \emptyset$ and $Y_j = \emptyset$, since $(k,j) \notin E_C(Y)$. This contradicts $(X,Y) \in E(G)$ (see condition (Ga)). Thus,

$$U_i \neq \emptyset \text{ and } \widetilde{U}_i = \emptyset \text{ for every } \widetilde{U} \in F(Y) \text{ is impossible.} \tag{5.13}$$

Case 3: $U_i \neq \emptyset$ and $\widetilde{U}_i \neq \emptyset$ for every $\widetilde{U} \in F(Y)$

In this case, $X_i \cup E_C(X) \neq \emptyset$ and $Y_i \cup E_C(Y) \neq \emptyset$ by (5.7).

If $X_i \neq \emptyset$, by condition (Ga), $Y_i \neq \emptyset$. So $W_i(X) \neq \emptyset$, $W_i(Y) \neq \emptyset$, and by (4.5), (5.5), and (5.8),

$$\begin{aligned} D_i(W_i(X), W_i(Y)) &= D_i\left(\bigcup_{(i,j)\in E(C)} T_{i,j}(X_j), \bigcup_{(i,j)\in E(C)} T_{i,j}(Y_j)\right) \\ &\leq \max_{(i,j)\in E(C)} D_i\big(T_{i,j}(X_j), T_{i,j}(Y_j)\big) \\ &\leq \max_{(i,j)\in E(C)} \lambda_{i,j} D_j(X_j, Y_j) \\ &\leq \lambda_I \max_{(i,j)\in E(C)} D_j(X_j, Y_j) \\ &\leq \lambda_I d_{\phi(I)}(X, Y). \end{aligned} \tag{5.14}$$

If $X_i = \emptyset$ and $Y_i \neq \emptyset$, then, for every $\widetilde{U}_i \in F_i(Y_i)$, one has by (4.6) and (5.8),

$$D_i(U_i, \widetilde{U}_i) \leq R = \frac{R}{R_i}\overline{D}_i(X_i, Y_i) \leq \lambda_I d_{\phi(I)}(X, Y). \tag{5.15}$$

If $E_C(X) \neq \emptyset$, for $\emptyset \neq P \subset E_C(X)$ such that $P \subset E_C(Y)$, for every $(k,j) \in P$, one has $j \in \phi(I)$, and, by (4.5), (5.2), (5.4), (5.8) and (5.9),

$$\begin{aligned} D_i(O_i(X,P), O_i(Y,P)) &= D_i\left(\bigcup_{(k,j)\in P} T_{i\to k} \circ T_{k,j}(X_j), \bigcup_{(k,j)\in P} T_{i\to k} \circ T_{k,j}(Y_j)\right) \\ &\leq \max_{(k,j)\in P} \lambda_{i\to k} D_k\big(T_{k,j}(X_j), T_{k,j}(Y_j)\big) \\ &\leq \max_{(k,j)\in P} \lambda_{i\to k}\lambda_{k,j} D_j(X_j, Y_j) \\ &\leq \lambda_I \max_{(k,j)\in P} D_j(X_j, Y_j) \\ &\leq \lambda_I d_{\phi(I)}(X, Y). \end{aligned} \tag{5.16}$$

If $P \subset E_C(X)$ and $P \not\subset E_C(Y)$, then there exists $(k,j) \in P$ such that $X_j \neq \emptyset$ and $Y_j = \emptyset$ which is impossible since $(X,Y) \in E(G)$.

Combining (5.7), (5.14), (5.15) and (5.16), we choose $\widetilde{U}_i \in F_i(Y)$ such that

$$\widetilde{U}_i = \begin{cases} W_i(Y), & \text{if } U_i = W_i(X), \\ O_i(Y,P), & \text{if } Y_i = \emptyset, \text{ and } U_i = O_i(X,P) \\ & \quad \text{for } \emptyset \neq P \subset E_C(X) \cap E_C(Y), \\ W_i(Y) \cup O_i(Y,P), & \text{if } Y_i \neq \emptyset, \text{ and} \\ & \quad U_i \in \{O_i(X,P), W_i(X) \cup O_i(X,P)\} \\ & \quad \text{for } \emptyset \neq P \subset E_C(X) \cap E_C(Y); \end{cases} \tag{5.17}$$

and we get

$$\overline{D}_i(U_i, \widetilde{U}_i) \leq \lambda_I d_{\phi(I)}(X,Y). \tag{5.18}$$

Step 2: Choice of an appropriate $\widetilde{\mathbf{U}} \in \mathbf{F}(\mathbf{Y})$:

Finally, we choose $\widetilde{U} = \left(\widetilde{U}_i\right)_{i \in V(H)} \in F(Y)$ as follows:

$$\widetilde{U}_i = \begin{cases} \emptyset, & \text{if } i \in V(C),\ U_i = \emptyset,\ Y_i \cup E_C(Y) = \emptyset, \\ \text{some } \widetilde{U}_i \in F_i(Y), & \text{if } i \in V(C),\ U_i = \emptyset,\ Y_i \cup E_C(Y) \neq \emptyset, \\ \widetilde{U}_i \text{ given by (5.17)}, & \text{if } i \in V(C),\ U_i \neq \emptyset,\ Y_i \cup E_C(Y) \neq \emptyset. \end{cases} \tag{5.19}$$

It follows from (5.10) and (5.17) that

$$(U, \widetilde{U}) \in E(G).$$

Finally, from (5.12) and (5.18), we deduce that

$$d_I(U, \widetilde{U}) \leq \lambda_I d_{\phi(I)}(X,Y) \quad \forall I \in \Gamma.$$

Therefore, F is a G-contraction. □

Remark 5.2. From the proof of the previous proposition, we already know that for $(X,Y) \in E(G)$ and $U \in F(X)$, the choice of $\widetilde{U} \in F(Y)$ such that $(U, \widetilde{U}) \in E(G)$ and $d_I(U, \widetilde{U}) \leq \lambda_I d_{\phi(I)}(X,Y)$ for all $I \in \Gamma$ is not necessarily unique. Moreover, if for some $C \in C(H)$, one has $E_C(X) \neq \emptyset$, then, from the previous proof, we deduce that $E_C(X) \subset E_C(Y)$. So, for

$$\emptyset \neq P \subsetneq \widetilde{P}, \quad \text{with } P \subset E_C(X), \widetilde{P} \subset E_C(Y), \tag{5.20}$$

there exists $(k,j) \in \widetilde{P} \backslash P$ with $X_j = \emptyset$ and $Y_j \neq \emptyset$. So, $j \in \phi(I) \backslash I$. By (4.5), (4.6) and (5.8),

$$\overline{D}_i(O_i(X,P), O_i(Y,\widetilde{P})) \leq R_i = \frac{R_i}{R_j} \overline{D}_j(X_j, Y_j) \leq \lambda_I d_{\phi(I)}(X,Y) \quad \forall i \in I.$$

Therefore, for $i \in V(C) \subset I$, $\widetilde{U}_i$ can be chosen as follows

$$\widetilde{U}_i = \begin{cases} W_i(Y), & \text{if } U_i = W_i(X), \\ O_i(Y, \widetilde{P}), & \text{if } Y_i = \emptyset \text{ and } U_i = O_i(X, P) \\ & \quad \text{with } \widetilde{P} \text{ as in (5.20)}, \\ W_i(Y) \cup O_i(Y, \widetilde{P}), & \text{if } Y_i \neq \emptyset, \text{ and} \\ & \quad U_i \in \{O_i(X, P), W_i(X) \cup O_i(X, P)\} \\ & \quad \text{with } \widetilde{P} \text{ as in (5.20)}. \end{cases}$$

6. Some properties of the attractor of an infinite H-IIFS

For $H = (V(H), E(H))$ an infinite MW-directed graph, and $\{T_{i,j}\}_H$ an infinite graph-directed iterated function system over the graph H. Theorem 2.5 gave conditions insuring the existence of K an attractor of this H-IIFS. We want to get more information on K by taking into account the connected components of H. To this aim, we will consider $F : \mathcal{X} \to \mathcal{X}$ the G-contraction defined on the gauge space $\mathcal{X}$ endowed with the graph G introduced in sections 4 and 5.

Theorem 6.1. *Let $H = (V(H), E(H))$ be an infinite MW-directed graph and $\{T_{i,j}\}_H$ an H-IIFS satisfying* (H). *Let $(R_i)_{i\in V(H)}$ be a family of constants satisfying* (R). *Assume that $X^0 \in \mathcal{X}$ and $X^1 \in F(X^0)$ are such that*

$$\sum_{n=1}^{\infty} \lambda_I \lambda_{\phi(I)} \cdots \lambda_{\phi^{n-1}(I)} d_{\phi^n(I)}(X^0, X^1) < \infty \quad \forall I \in \Gamma, \tag{6.1}$$

where λ_I is defined in (5.8). *Then, there exists $K(X^0) \in \mathcal{X}$ such that*

1. *$K_i(X^0) \neq \emptyset$ for every $i \in V(H)$ such that $X_i^0 \neq \emptyset$;*
2. *$K_i(X^0) \neq \emptyset$ if and only if $i \in [j]_{\leftarrow}$, for some $j \in V(H)$ such that $X_j^0 \neq \emptyset$;*
3. *$K(X^0)$ is a fixed point of the multi-valued map F;*
4. *if $\{T_{i,j}\}_H$ has an attractor K, then $K(X^0) \subset K$.*

Proof. Let $F : \mathcal{X} \to \mathcal{X}$ be the multi-valued map defined in (5.6) and (5.7). We know that F is a G-contraction by Proposition 5.1. Also, if $\{T_{i,j}\}_H$ has an attractor K, the definition of F implies that fixed points of F are included in K.

Let $X^0 \in \mathcal{X}$ and $X^1 \in F(X^0)$ be such that (6.1) is satisfied. We want to show that there exists $K(X^0)$ a fixed point of F satisfying the required properties.

For $n \in \mathbb{N}$, we choose inductively

$$X^{n+1} \in F(X^n) \quad \text{the biggest element of } F(X^n), \tag{6.2}$$

that is $X^{n+1} = (X_i^{n+1})_{i\in V(H)} \in F(X^n)$ is chosen as follows. For $i \in V(C)$ for some $C \in C(H)$,

$$X_i^{n+1} = \begin{cases} \emptyset, & \text{if } X_i^n = E_C(X^n) = \emptyset; \\ O_i\big(X^n, E_C(X^n)\big), & \text{if } X_i^n = \emptyset,\ E_C(X^n) \neq \emptyset; \\ W_i(X^n) \cup O_i\big(X^n, E_C(X^n)\big), & \text{if } X_i^n \neq \emptyset; \end{cases} \tag{6.3}$$

where E_C, O_i and W_i are defined in (5.1), (5.4) and (5.5) respectively.

Arguing as in the proof of Proposition 5.1 and by Remark 5.2, one has that $(X^{n-1}, X^n) \in E(G)$ and

$$d_I(X^n, X^{n+1}) \leq \lambda_I d_{\phi(I)}(X^{n-1}, X^n) \quad \forall I \in \Gamma.$$

By the proof of Theorem 3.3, the sequence $\{X^n\}$ is a G-Picard trajectory converging to some $K(X^0) \in \mathcal{X}$.

Observe that for every $i \in V(H)$ such that $X_i^0 \neq \emptyset$, one has $X_i^n \neq \emptyset$ for every $n \in \mathbb{N}$. Therefore, $K(X^0)$ satisfies (1).

By construction, for $i \in V(C)$ for $C \in C(H)$, if there is a directed path $[i_n]_{n=0}^N$ in H from $i = i_0$ to $j = i_N$ such that $X_j^0 \neq \emptyset$, then $X_i^n \neq \emptyset$ for every $n > N$. Therefore, $K(X^0)_i \neq \emptyset$. On the other hand, if $i \notin [j]_{\leftarrow}$, for all $j \in V(H)$ such that $X_j^0 \neq \emptyset$, then $X_i^n = \emptyset$ for every $n \in \mathbb{N}$, and hence $K(X^0)_i = \emptyset$. So, $K(X^0)$ satisfies (2).

To conclude, we have to show that $K(X^0)$ is a fixed point of F. This will imply that $K(X^0) \subset K$ if the attractor K of $\{T_{i,j}\}_H$ exists.

Let us denote

$$V(X^0) = \{i \in V(H) : i \in [j]_{\leftarrow} \text{ for some } j \in V(H) \text{ such that } X_j^0 \neq \emptyset\}. \tag{6.4}$$

It follows from (2) that

$$\begin{aligned} &\text{if } i \in V(X^0), \quad K(X^0)_i \neq \emptyset, \\ &\text{if } i \notin V(X^0), \quad K(X^0)_i = E_C(K(X^0)) = \emptyset. \end{aligned} \tag{6.5}$$

Let $\widehat{U} = (\widehat{U})_{i \in V(H)} \in \mathcal{X}$ be defined by

$$\widehat{U}_i = \begin{cases} \emptyset, & \text{if } i \in V(H) \backslash V(X^0), \\ W_i(K(X^0)) \cup O_i\big(K(X^0), E_C(K(X^0))\big), & \text{if } i \in V(X^0) \cap V(C) \\ & \quad \text{for } C \in C(H). \end{cases} \tag{6.6}$$

So, by (6.5) and the definition of F (see (5.7)),

$$\widehat{U} \in F(K(X^0)). \tag{6.7}$$

We claim that $K(X^0) = \widehat{U}$.

Let $\hat{I} \in \Gamma$. For every $C \in C(H)$ such that $V(C) \subset \hat{I}$, we denote

$$N_C = \begin{cases} \sup\big\{ \inf\{n : X_j^n \neq \emptyset\} : (k,j) \in E_C(K(X^0))\big\}, & \text{if } E_C(K(X^0)) \neq \emptyset, \\ 0, & \text{otherwise.} \end{cases}$$

From the fact that $\operatorname{outdeg}(k) < \infty$ for every $k \in V(C)$ and by (H), we deduce that $N_C < \infty$. Let

$$N = \max\big\{N_C : V(C) \subset \hat{I}\big\}. \tag{6.8}$$

So,

$$E_C(K(X^0)) = E_C(X^n) \quad \forall V(C) \subset \hat{I},\ \forall n > N. \tag{6.9}$$

For $n > N$, let us define $\widehat{X}^n = (\widehat{X}_i^n)_{i\in V(H)}$, $\widehat{U}^n = (\widehat{U}_i^n)_{i\in V(H)} \in \mathcal{X}$ by

$$\widehat{X}_i^n = \begin{cases} X_i^n, & \text{if } i \in \phi(\hat{I}),\\ K(X^0)_i, & \text{otherwise;}\end{cases}$$

and

$$\widehat{U}_i^n = \begin{cases} \emptyset, & \text{if } i \in V(H)\backslash V(X^0),\\ W_i(\widehat{X}^n) \cup O_i\big(\widehat{X}^n, E_C(\widehat{X}^n)\big), & \text{if } i \in V(X^0)\cap V(C) \text{ for } C \in C(H).\end{cases}$$

It follows from (6.9) and the definitions of $E(G)$ and F (see (5.6)) that

$$(K(X^0), \widehat{X}^n) \in E(G), \quad (\widehat{U}, \widehat{U}^n) \in E(G) \quad \text{and} \quad \widehat{U}^n \in F(\widehat{X}^n). \tag{6.10}$$

Arguing as in the proof of Proposition 5.1, we can show that

$$d_{\hat{I}}(\widehat{U}^n, \widehat{U}) \le \lambda_{\hat{I}} d_{\phi(\hat{I})}(\widehat{X}^n, K(X^0)). \tag{6.11}$$

Observe that, for every $n > N$,

$$\widehat{X}_i^n = X_i^n \ \ \forall i \in \phi(\hat{I}) \quad \text{and} \quad \widehat{U}_i^n = X_i^{n+1} \ \ \forall i \in \hat{I}. \tag{6.12}$$

So,

$$d_{\phi(\hat{I})}(\widehat{X}^{N+k}, X^{N+k}) \to 0 \quad \text{and} \quad d_{\hat{I}}(\widehat{U}^{N+k}, X^{N+k+1}) \to 0 \quad \text{as } k \to \infty. \tag{6.13}$$

Combining (6.7), (6.10), (6.11), and (6.13), it follows from Lemma 3.6 that

$$K(X^0) = \widehat{U} \in F(K(X^0)).$$

□

Theorem 6.2. *Let $H = (V(H), E(H))$ be an infinite MW-directed graph and $\{T_{i,j}\}_H$ an H-IIFS satisfying* (H). *Let $(R_i)_{i\in V(H)}$ be a family of constants satisfying* (R). *Assume that, for $X^0, Y^0 \in \mathcal{X}$, (6.1) is satisfied with (X^0, X^1) and (Y^0, Y^1), where X^1 and Y^1 are the biggest elements of $F(X^0)$ and $F(Y^0)$ respectively. Then the following statements hold:*

1. *If X^0, Y^0 are such that $\{i \in V(H) : X_i^0 \neq \emptyset\} = \{i \in V(H) : Y_i^0 \neq \emptyset\}$ and $X_i^0 \subset Y_i^0$ for every $i \in V(H)$, then $K(X^0) = K(Y^0)$.*
2. *If X^0, Y^0 are such that $\{i \in V(H) : X_i^0 \neq \emptyset\} \subset \{i \in V(H) : Y_i^0 \neq \emptyset\}$, then $K(X^0)_i \subset K(Y^0)_i$ for every $i \in V(H)$.*
3. *If there is $N \in \mathbb{N}$ such that $\{i \in V(H) : X_i^0 \neq \emptyset\} \subset \{[j]_{\leftarrow}^N : Y_j^0 \neq \emptyset\}$, then $K(X^0)_i \subset K(Y^0)_i$ for every $i \in V(H)$, where $[j]_{\leftarrow}^N = \{k \in V(H) :$ there is a directed path $[i_n]_{n=0}^{N_k}$ in H from $k = i_0$ to $j = i_{N_k}$ with $N_k \le N\}$.*

Proof. (1) Let $\{X^n\}$ and $\{Y^n\}$ be the G-Picard trajectories defined inductively by (6.2) and such that $X^n \to K(X^0)$ and $Y^n \to K(Y^0)$. Observe that $(X^n, Y^n) \in E(G)$ for every $n \in \{0\} \cup \mathbb{N}$. Arguing as in the proof of Proposition 5.1, we deduce that

$$d_I(X^n, Y^n) \le \lambda_I d_{\phi(I)}(X^{n-1}, Y^{n-1}) \quad \forall n \in \mathbb{N},\ \forall I \in \Gamma.$$

Therefore, $\{X^n\}$ and $\{Y^n\}$ have the same limit; that is $K(X^0) = K(Y^0)$.

(2) Let $Z^0 = (Z_i^0)_{i\in V(H)} \in \mathcal{X}$ be defined by $Z_i^0 = X_i^0 \cup Y_i^0$. Let Z^1 be the biggest element of $F(Z^0)$. One can check that

$$\overline{D}_i(Z_i^0, Z_i^1) \leq \overline{D}_i(X_i^0, X_i^1) + \overline{D}_i(Y_i^0, Y_i^1) \quad \forall i \in V(H),$$

and hence

$$d_I(Z^0, Z^1) \leq d_I(X^0, X^1) + d_I(Y^0, Y^1) \quad \forall I \in \Gamma.$$

Thus, (Z^0, Z^1) satisfies (6.1). So, Y^0 and Z^0 verify the assumptions of (1). Therefore,

$$K(Y^0) = K(Z^0).$$

Let $\{X^n\}$ and $\{Z^n\}$ be the G-Picard trajectories defined inductively by (6.2) and such that $X^n \to K(X^0)$ and $Z^n \to K(Z^0)$. Since $X_i^0 \subset Z_i^0$, one has $X_i^n \subset Z_i^n$ for every $i \in V(H)$ and every $n \in \mathbb{N}$. Thus,

$$K(X^0)_i \subset K(Z^0)_i = K(Y^0)_i \quad \forall i \in V(H).$$

(3) Let $\{X^n\}$ and $\{Y^n\}$ be the G-Picard trajectories defined inductively by (6.2) and such that $X^n \to K(X^0)$ and $Y^n \to K(Y^0)$. The assumption implies that

$$\{i \in V(H) : X_i^0 \neq \emptyset\} \subset \{i \in V(H) : Y_i^N \neq \emptyset\}.$$

From the proof of Proposition 5.1,

$$d_I(Y^N, Y^{N+1}) \leq \lambda_I \cdots \lambda_{\phi^{N-1}(I)} d_{\phi^N(I)}(Y^0, Y^1) \quad \forall I \in \Gamma.$$

Therefore, (Y^N, Y^{N+1}) satisfies (6.1). It follows from (2) that

$$K(X^0)_i \subset K(Y^N)_i \quad \forall i \in V(H).$$

Since

$$K(Y^N) = \lim_{k\to\infty} Y^{N+k} = \lim_{n\to\infty} Y^n = K(Y^0),$$

one has

$$K(X^0)_i \subset K(Y^0)_i \quad \forall i \in V(H).$$

□

Example 6.3. Let $H = (V(H), E(H))$ be given by $V(H) = \mathbb{Z} \times \{0, 1\}$ and

$$E(H) = \Big\{\big((0,0),(1,1)\big), \big((0,1),(1,0)\big)\Big\} \cup \Big\{\big((i,a),(i+1,a)\big), \big((3i,a),(3i-2,a)\big) : i \in \mathbb{Z}, a = 0, 1\Big\}.$$

For $a = 0, 1$, and $i \in \mathbb{Z}$, let $M_{(i,a)} = [i, i+1] \times [a, a+1]$ be endowed with the norm $\|(x,y)\| = \max\{|x|, |y|\}$. For $(i,j) = \big((i_1,a),(j_1,b)\big) \in E(H)$, let $T_{i,j} : M_j \to M_i$ be a contraction with constant of contraction $\lambda_{i,j} < 1$. We assume that

$$k_n := \frac{1+e^n}{1+e^{n+1}} \geq \max\big\{\lambda_{i,j} : (i,j) \in E(H), i = (i_1, a) \text{ for } a \in \{0,1\} \text{ and } i_1 \in \{3n-1, 3n-2, 3n\}\big\}. \quad (6.14)$$

We observe that $n \mapsto k_n$ is nonincreasing. Arguing as in Example 2.7, it can be shown that Theorem 2.5 implies that this H-IIFS, $\{T_{i,j}\}_H$, has a unique attractor K.

Moreover, for this H-IIFS, one has for $n \in \mathbb{Z}$ and $a = 0, 1$, the connected component of H, $C_n^a = \big(V(C_n^a), E(C_n^a)\big)$, given by

$$V(C_n^a) = \{(3n-2, a), (3n-1, a), (3n, a)\},$$
$$E(C_n^a) = \Big\{\big((3n-2, a), (3n-1, a)\big), \big((3n-1, a), (3n, a)\big), \big((3n, a), (3n-2, a)\big)\Big\}.$$

So, as shown in Figure 6.1, the set of all connected components of H is

$$C(H) = \big\{C_n^a : n \in \mathbb{Z}, a = 0, 1\big\}.$$

Observe that

$$C_m^a \preceq C_n^b \quad \iff \quad (a = b \text{ and } m \leq n) \ \text{ or } \ (a \neq b \text{ and } m \leq 0 < n).$$

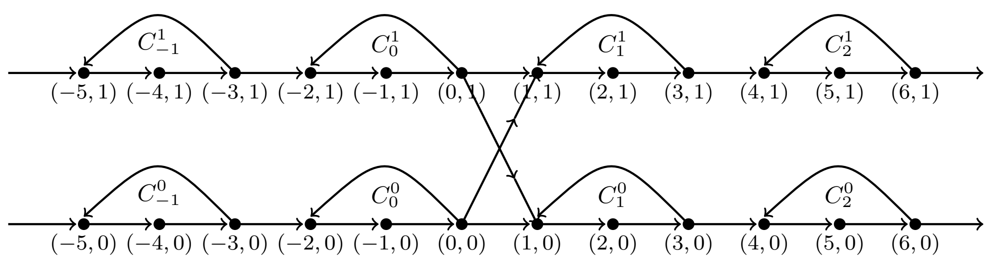

FIGURE 6.1. The set of connected components $C(H)$.

Let Γ and $\phi : \Gamma \to \Gamma$ be given by

$$\Gamma = \{I \subset \mathbb{Z} \times \{0,1\} : 0 < \text{card}(I) < \infty, \text{ and } V(C_n^a) \subset I \ \ \forall V(C_n^a) \cap I \neq \emptyset\},$$
$$\begin{aligned} \phi(I) = I &\cup \{(i+1, a), (i+2, a), (i+3, a) : (i, a) \in I\} \\ &\cup \{(1,1), (2,1), (3,1) : \text{if } (0,0) \in I\} \\ &\cup \{(1,0), (2,0), (3,0) : \text{if } (0,1) \in I\}. \end{aligned}$$

Also, let

$$\begin{aligned} \mathcal{X} = \Big\{ X = \big(X_{(i,a)}\big)_{(i,a) \in V(H)} &: X_{(i,a)} \subset M_{(i,a)} \text{ closed } \forall (i,a) \in V(H), \\ &\text{if } X_{(i,a)} \neq \emptyset \text{ for } (i,a) \in C_n^a, \text{ then } X_{(j,a)} \neq \emptyset \ \forall (j,a) \in C_n^a, \\ &\text{card}\{(i,a) : X_{(i,a)} \neq \emptyset\} \neq 0 \Big\}. \end{aligned}$$

We fix $R = 1$ and $(R_{(i,a)})_{(i,a) \in V(H)}$ given by

$$R_{(i,a)} = 1 + e^n \quad \text{for } (i,a) \in C_n^a.$$

This permits to define $\{d_I\}_{I \in \Gamma}$ by

$$d_I(X, \widehat{X}) = \max \Big\{ \overline{D}_{(i,a)}\big(X_{(i,a)}, \widehat{X}_{(i,a)}\big) : (i,a) \in I \Big\},$$

where

$$\overline{D}_{(i,a)}\big(X_{(i,a)}, \widehat{X}_{(i,a)}\big) = \begin{cases} D\big(X_{(i,a)}, \widehat{X}_{(i,a)}\big), & \text{if } X_{(i,a)} \neq \emptyset, \widehat{X}_{(i,a)} \neq \emptyset, \\ 0, & \text{if } X_{(i,a)} = \emptyset, \widehat{X}_{(i,a)} = \emptyset, \\ R_{(i,a)}, & \text{otherwise.} \end{cases}$$

Observe that

$$\begin{aligned} \lambda_I = \max\Bigg\{ & \max\Big\{\lambda_{(i,a),(j,b)} : \big((i,a),(j,b)\big) \in E(H)\Big\}, \max\Big\{\frac{1}{R_{(i,a)}} : (i,a) \in I\Big\}, \\ & \max\Big\{\frac{R_{(i,a)}}{R_{(j,b)}} : (i,a) \in I, (j,b) \in \phi(I)\backslash I\Big\}\Bigg\} \\ \leq k_{n_0}, & \end{aligned}$$

where k_n is defined in (6.14) and

$$n_0 = \min\{n : I \cap C_n^0 \neq \emptyset \text{ or } I \cap C_n^1 \neq \emptyset\}.$$

Also $\lambda_I = \lambda_{\phi(I)}$ for every $I \in \Gamma$. Therefore,

$$\sum_{n=1}^{\infty} \lambda_I \lambda_{\phi(I)} \cdots \lambda_{\phi^{n-1}}(I) d_{\phi^n(I)}(X, \widehat{X}) \leq \sum_{n=1}^{\infty} k_{n_0}^n d_{\phi^n(I)}(X, \widehat{X}) \quad \forall X, \widehat{X} \in \mathcal{X}.$$

This sum is finite in particular for every $X = X^0 \in \mathcal{X}$ and every $\widehat{X} = X^1 \in F(X^0)$ such that $\sup\{i : X^0_{(i,a)} \neq \emptyset\} \neq \sup\{i : X^0_{(i,a)} = \emptyset\}$, where $F : \mathcal{X} \to \mathcal{X}$ is defined in (5.6). Therefore, this H-IIFS, $\{T_{i,j}\}_H$, satisfies all the assumptions of Theorems 6.1 and 6.2. In particular, for such $X^0 \in \mathcal{X}$, there exists a subattractor $K(X^0) \subset K$ satisfying all the properties stated in those theorems.

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