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ON A NEW ITERATIVE METHOD FOR SOLVING EQUILIBRIUM PROBLEMS AND FIXED POINT PROBLEMS

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Abstract. A new iterative algorithm is proposed for finding a common solution of an equilibrium problem and a fixed point problem. Then, a strong convergence theorem is proved. As a consequence, they can be obtained some strong convergence theorems for an equilibrium problem and a split common fixed point problem. The obtained theorems can be applied to solve an equilibrium problem and a split common null point problem. The results presented in this paper extend and improve some corresponding ones in the literature. Finally, a numerical example is given to show the validity of the algorithm.
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1. Introduction

Let H be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$ and let C be a nonempty closed convex subset of H. A mapping T of C into itself is called nonexpansive if $\|Tx-Ty\| \le \|x-y\|$ for all $x, y \in C$. We use $Fix(T)$ to denote the set of fixed points T, i.e., $Fix(T) = \{x \in C : Tx = x\}$. Also, a contraction on C is a self-mapping f of C such that $\|f(x)-f(y)\| \le \kappa\|x-y\|$ for all $x, y \in C$ and some constant $\kappa \in [0,1)$. In this case f is said to be a κ-contraction.

Consider an equilibrium problem (EP) which is to find a point $u \in C$ satisfying the property:

$$\phi(u,v) \ge 0 \quad \text{for all } v \in C, \tag{1.1}$$

where $\phi : C \times C \to \mathbb{R}$ is a bifunction of C. We use $EP(\phi)$ to denote the set of solutions of EP (1.1), that is, $EP(\phi) = \{u \in C : (1.1) \text{ holds}\}$.

The EP (1.1) includes, as special cases, numerous problems in physics, optimization and economics. Some authors (e.g., [22, 24, 28]) have proposed some useful methods for solving the EP (1.1). Let C and Q be nonempty closed convex subsets of real Hilbert spaces H_1 and H_2, respectively. The split feasibility problem (SFP) is formulated as finding a point x satisfying the property

$$x \in C \text{ such that } Ax \in Q,$$

where $A : H_1 \to H_2$ is a (nonzero) bounded linear operator. Recently, the SFP has been widely studied by many authors (see [8, 9, 6, 10, 7, 21, 19, 20, 23, 25, 3, 27, 30, 31, 34, 33, 35]), due to its application in signal processing [4].

In 2002, Byrne [3] introduced the so-called CQ algorithm which starts an arbitrary initial guess $x_0 \in H_1$ and generates a sequence $\{x_n\}$ via the iteration process

$$x_{n+1} = P_C(I - \gamma A^*(I - P_Q)A)x_n, \tag{1.2}$$

where $0 < \gamma < \frac{2}{\rho(A^*A)}$, P_C and P_Q are the projections onto C and Q, respectively, and $\rho(A^*A)$ is the spectral radius of the operator A^*A, with A^* the adjoint of A. It is known that the CQ algorithm converges weakly to a solution of the SFP if such a solution exists. In the case where both C and Q are the sets of fixed points of some nonlinear operators, the SFP is known as the split common fixed point problem (SCFP). More specifically, the SCFP is to find a point x with the property

$$x \in Fix(S) \text{ and } Ax \in Fix(T), \tag{1.3}$$

where $S : H_1 \to H_1$ and $T : H_2 \to H_2$ are nonlinear mappings. We denote by Γ the solution set of the SCFP, that is,

$$\Gamma := \{x \in H_1 : x \in Fix(S) \text{ and } Ax \in Fix(T)\}. \tag{1.4}$$

In 2009, Censor and Segal [5] proposed and proved the convergence of the following algorithm in the setting of the finite-dimensional spaces when S and T are directed operators:

$$x_{n+1} = S(x_n - \gamma A^*(I - T)Ax_n). \tag{1.5}$$

Note that the class of directed operators includes the metric projections. Therefore, the results of Censor and Segal recover Byrne's CQ algorithm.

In 2010, Moudafi [17] studied the following algorithm:

$$\begin{cases} u_n = x_n - \gamma A^*(I - T)Ax_n, \\ x_{n+1} = (1 - \alpha_n)u_n + \alpha_n S u_n, \end{cases}$$

where $\{\alpha_n\}$ is a real sequence to solve the SCFP for demicontractive operators and obtained the weak convergence. It is known that demicontractive operators include the directed operators. Hence, Moudafi's algorithm is an extension of the algorithm (1.5).

In 2012, Zhao and He [37] introduced the following viscosity approximation method for solving the SCFP:

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)((1 - \lambda_n)x_n + \lambda_n U x_n),$$

where $f : H_1 \to H_1$ is a contraction, $U = S(I - \gamma A^*(I - T)A)$, S and T are two quasi-nonexpansive operators, $\alpha_n \in (0, 1)$ and $\lambda_n \in (0, \frac{1}{2})$.

Recently, Thong [26] proposed the following generalized algorithm:

$$x_{n+1} = (1-\alpha_n)f(x_n) + \alpha_n U x_n, \tag{1.6}$$

where $f : H \to H$ is a contraction, $U : H \to H$ is an α-strongly quasi-nonexpansive operator and $\alpha_n \in (0,1)$. He proved the sequence generated by (1.6) converges strongly to an element of $Fix(U)$. Then, he obtained some strong convergence theorems for solving the SCFP which extended the corresponding results announced by many others.

In this paper, motivated by the above results, we propose a new iterative algorithm to solve an equilibrium problem and a fixed point problem in Hilbert spaces. Then, we obtain some strong convergence theorems for equilibrium problems and the split common fixed point problems.

2. Preliminaries

Let H be a real Hilbert space. We use $\rightharpoonup$ and $\to$ to denote the weak and strong convergence in H, respectively. The following identity holds:

$$\begin{aligned}\|\alpha x + \beta y + \gamma z\|^2 =& \alpha\|x\|^2 + \beta\|y\|^2 + \gamma\|z\|^2 \\ &- \alpha\beta\|x-y\|^2 - \beta\gamma\|z-y\|^2 - \alpha\gamma\|z-x\|^2,\end{aligned}$$

for all $x, y, z \in H$ and $\alpha, \beta, \gamma \in [0,1]$ such that $\alpha + \beta + \gamma = 1$.

Let C be a nonempty closed convex subset of H. Then, for any $x \in H$, there exists a unique nearest point in C, denoted by $P_C(x)$, such that

$$\|x - P_C(x)\| \leq \|x - y\| \quad \text{for all } y \in C.$$

P_C is called the metric projection of H onto C. It is known that P_C is nonexpansive. Further, for $x \in H$ and $z \in C$,

$$z = P_C(x) \;\Leftrightarrow\; \langle x - z, z - y\rangle \geq 0 \quad \text{for all } y \in C.$$

Definition 2.1. A mapping $T : H \to H$ is called firmly nonexpansive if for any $x, y \in H$,

$$\|Tx - Ty\|^2 \leq \langle Tx - Ty, x - y\rangle.$$

We recal that for a real Hilbert space H and all $x, y \in H$,

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle. \tag{2.1}$$

Lemma 2.2. [2] *Let C be a nonempty closed convex subset of H and $\phi : C \times C \to \mathbb{R}$ be a bifunction satisfying the following conditions:*

- (A_1) *$\phi(x,x) = 0$ for all $x \in C$;*
- (A_2) *ϕ is monotone, i.e., $\phi(x,y) + \phi(y,x) \leq 0$, for all $x, y \in C$;*
- (A_3) *for each $x, y, z \in C$, $\lim_{t\downarrow 0} \phi(tz + (1-t)x, y) \leq \phi(x,y)$;*
- (A_4) *for each $x \in C, y \mapsto \phi(x,y)$ is convex and weakly lower semicontinuous.*

Let $r > 0$ and $x \in H$. Then, there exists $z \in C$ such that

$$\phi(z,y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \quad \text{for all } y \in C.$$

Lemma 2.3. [11] *Assume $\phi : C \times C \to \mathbb{R}$ satisfies the conditions (A_1)-(A_4). For $r > 0$, define a mapping $Q_r : H \to C$ by*

$$Q_r x := \{z \in C : \phi(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \quad \forall y \in C\} \tag{2.2}$$

for all $x \in H$. Then, the following hold:

(i) Q_r *is single-valued;*
(ii) Q_r *is firmly nonexpansive;*
(iii) $Fix(Q_r) = EP(\phi)$*;*
(iv) $EP(\phi)$ *is closed and convex.*

Definition 2.4. Assume $T : H \to H$ is a mapping. Then, $I - T$ is said to be demiclosed at zero if for any $\{x_n\}$ in H, the following implication holds:

$$x_n \rightharpoonup x \text{ and } (I - T)x_n \to 0 \Rightarrow \; x \in Fix(T).$$

Definition 2.5. Let $T : H \to H$ be a mapping with $Fix(T) \neq \emptyset$. Then

(i) $T : H \to H$ is called directed if

$$\langle z - Tx, x - Tx\rangle \leq 0, \quad \forall z \in Fix(T),\; x \in H$$

or equivalently

$$\|Tx - z\|^2 \leq \|x - z\|^2 - \|x - Tx\|^2, \quad \forall z \in Fix(T),\; x \in H.$$

[Firmly nonexpansive mappings are directed.]

(ii) $T : H \to H$ is called $\alpha-$strongly quasi-nonexpansive with $\alpha > 0$ if

$$\|Tx - z\|^2 \leq \|x - z\|^2 - \alpha\|x - Tx\|^2,\; \forall z \in Fix(T),\; x \in H, \tag{2.3}$$

or equivalently

$$\langle Tx - x, x - z\rangle \leq -\frac{1 + \alpha}{2}\|x - Tx\|^2, \quad \forall z \in Fix(T),\; x \in H. \tag{2.4}$$

(iii) $T : H \to H$ is called quasi-nonexpansive if

$$\|Tx - z\| \leq \|x - z\|, \quad \forall z \in Fix(T),\; x \in H. \tag{2.5}$$

(iv) $T : H \to H$ is called β-demicontractive with $0 \leq \beta < 1$ if

$$\|Tx - z\|^2 \leq \|x - z\|^2 + \beta\|x - Tx\|^2, \quad \forall z \in Fix(T),\; x \in H. \tag{2.6}$$

Lemma 2.6. [32] *Let $T : H \to H$ be an α_1-strongly quasi-nonexpansive mapping and $S : H \to H$ be an α_2-strongly quasi-nonexpansive mapping with $Fix(T) \cap Fix(S) \neq \emptyset$. Then, the mapping TS is $\frac{\alpha_1\alpha_2}{\alpha_1+\alpha_2}$-strongly quasi-nonexpansive and $Fix(TS) = Fix(T) \cap Fix(S)$.*

Lemma 2.7. [26] *Let $T : H \to H$ be a β-demicontractivebe mapping and $S : H \to H$ be an α-strongly quasi-nonexpansive mapping with $\beta < \alpha$. Then, mapping TS is $\frac{\alpha\beta}{\alpha-\beta}$-demicontractive and $Fix(TS) = Fix(T) \cap Fix(S)$.*

Lemma 2.8. [26] *Let $T : H \to H$ be a β-demicontractive mapping with $Fix(T) \neq \emptyset$ and set $T_\lambda = (1 - \lambda)I + \lambda T$, called the λ-relaxation of T, with $\lambda \in (0, 1 - \beta)$. Then we have*

(i) $Fix(T) = Fix(T_\lambda)$.
(ii) $\|T_\lambda x - z\|^2 \le \|x - z\|^2 - \frac{1}{\lambda}(1 - \beta - \lambda)\|x - T_\lambda x\|^2,\ \forall z \in Fix(T),\ x \in H$.
(iii) $Fix(T)$ *is a closed convex subset of* H.

Lemma 2.9. [26] *Let* $A : H_1 \to H_2$ *be a linear bounded operator with* $L = \|A^*A\|$ *and* $T : H_2 \to H_2$ *be a* $\beta-$*demicontractive mapping. For a positive real number* γ, *define the mapping* $V : H_1 \to H_1$ *by* $V := I + \gamma A^*(T - I)A$. *Then*

(i) for all $x \in H_1$ *and* $z \in A^{-1}(Fix(T))$,

$$\|Vx - z\|^2 \le \|x - z\|^2 - \frac{1}{\gamma L}(1 - \beta - \gamma L)\|x - Vx\|^2.$$

(ii) for all $x \in H_1$ *and* $z \in A^{-1}(Fix(T))$,

$$\|Vx - z\|^2 \le \|x - z\|^2 - \gamma(1 - \beta - \gamma L)\|Ax - T(Ax)\|^2.$$

(iii) $x \in Fix(V)$ *if* $Ax \in Fix(T)$ *provided that* $\gamma \in (0, \frac{1-\beta}{L})$.

Lemma 2.10. [1] *Assume* $\{a_n\}$ *is a sequence of nonnegative real numbers such that*

$$a_{n+1} \le (1 - \gamma_n)a_n + \gamma_n v_n + \mu_n,$$

where $\{\gamma_n\}$ *is a sequence in* $[0, 1]$, $\{\mu_n\}$ *a sequence of nonnegative real numbers, and* $\{v_n\}$ *a sequence in* $\mathbb{R}$ *such that* $\sum_{n=1}^{\infty} \gamma_n = \infty, \limsup_{n\to\infty} v_n \le 0$ *and* $\sum_{n=1}^{\infty} \mu_n < \infty$. *Then* $\lim_{n\to\infty} a_n = 0$.

3. Main result

In this section, we introduce an iterative algorithm for finding a common element of the set of solutions of the equilibrium problem (1.1) and the fixed point set of a nonexpansive mapping. We will prove strong convergence of the algorithm. As a consequence, we also obtain two strong convergence theorems for equilibrium problems and the split common fixed point problems.

Theorem 3.1. *Let* H *be a real Hilbert space,* C *a nonempty closed convex subset of* H, $T : C \to C$ *an* α-*strongly quasi-nonexpansive mapping such that* $I - T$ *is demiclosed at zero,* $\phi : H \times H \to \mathbb{R}$ *a bifunction satisfying the conditions* (A_1)-(A_4) *of Lemma 2.2, and* $f : C \to C$ *a* κ-*contraction for some* $\kappa \in [0, 1)$. *Set* $F := EP(\phi) \cap Fix(T)$ *and assume* $F \ne \emptyset$. *Suppose* $\{\alpha_n\}$ *and* $\{r_n\}$ *are real sequences satisfying the following conditions:*

(B_1) $\{\alpha_n\} \subset (0, 1)$, $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(B_2) $\{r_n\} \subset (a, \infty)$ *for some* $a > 0$.

Let $\{x_n\}$ *be a sequence generated by the two-layer iteration process*

$$\begin{cases} u_n = Q_{r_n} x_n, & \text{(3.1a)} \\ x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)Tu_n, \quad n \ge 1, & \text{(3.1b)} \end{cases}$$

where the initial guess $x_0 \in C$ *is arbitrary and* Q_r *is defined by* (2.2) *for* $r > 0$. *Then, the sequences* $\{x_n\}$ *and* $\{u_n\}$ *converge strongly to* $q \in F$, *where* $q = P_F f(q)$, *which uniquely solves the following variational inequality (VI):*

$$\langle (I - f)q, q - x\rangle \le 0, \quad \text{for all } x \in F. \tag{3.2}$$

Proof. Since $P_F f$ is a contraction on F, it has a unique fixed point $q \in F$; equivalently, q is the unique solution of VI (3.2).

We first claim that $\{x_n\}$ and $\{u_n\}$ are bounded. To see this, taking $p \in F$ and noticing $u_n = Q_{r_n} x_n$ and $Q_{r_n} p = p$, we get $\|u_n - p\| \le \|x_n - p\|$, and from (3.1b),

$$\begin{aligned}\|x_{n+1} - p\| \le& \|\alpha_n(f(x_n) - p) + (1-\alpha_n)(Tu_n - p)\| \\ \le& \alpha_n(\|f(x_n) - f(p)\| + \|f(p) - p\|) + (1-\alpha_n)\|u_n - p\| \\ \le& (1 - \alpha_n(1-\kappa))\|x_n - p\| + \alpha_n\|f(p) - p\| \\ \le& \max\{\|x_n - p\|, \|f(p) - p\|/(1-\kappa)\}.\end{aligned}$$

By induction, we obtain

$$\|x_n - p\| \le \max\left\{\|x_0 - p\|, \frac{\|f(p) - p\|}{1-\kappa}\right\} \tag{3.3}$$

for all $n \ge 0$. Hence $\{x_n\}$ is bounded, so are $\{u_n\}$, $\{f(x_n)\}$ and $\{Tu_n\}$.

By (3.1b) and (2.1), we derive that

$$\begin{aligned}\|x_{n+1} - q\|^2 &= \|\alpha_n(f(x_n) - q) + (1-\alpha_n)(Tu_n - q)\|^2 \\ &= \|(\alpha_n(f(x_n) - f(q)) + (1-\alpha_n)(Tu_n - q)) + \alpha_n(f(q) - q)\|^2 \\ &\le \|\alpha_n(f(x_n) - f(q)) + (1-\alpha_n)(Tu_n - q)\|^2 \\ &\quad + 2\alpha_n\langle f(q) - q, x_{n+1} - q\rangle.\end{aligned}$$

By convexity of $\|\cdot\|^2$, κ-contraction of f, and α-strong quasi-nonexpansivity of T, we further derive that

$$\begin{aligned}\|x_{n+1} - q\|^2 &\le \alpha_n\|f(x_n) - f(q)\|^2 + (1-\alpha_n)\|Tu_n - q\|^2 \\ &\quad + 2\alpha_n\langle f(q) - q, x_{n+1} - q\rangle \\ &\le \alpha_n\kappa^2\|x_n - q\|^2 + (1-\alpha_n)(\|u_n - q\|^2 - \alpha\|Tu_n - u_n\|^2) \\ &\quad + 2\alpha_n\langle f(q) - q, x_{n+1} - q\rangle. \end{aligned} \tag{3.4}$$

Since Q_r is firmly nonexpansive for each $r > 0$ and $u_n = Q_{r_n} x_n$, we get (cf. Definition 2.4(i))

$$\|u_n - q\|^2 \le \|x_n - q\|^2 - \|u_n - x_n\|^2. \tag{3.5}$$

Substituting (3.5) into (3.4) we obtain

$$\begin{aligned}\|x_{n+1} - q\|^2 &\le (1 - \alpha_n(1-\kappa^2))\|x_n - q\|^2 + 2\alpha_n\langle f(q) - q, x_{n+1} - q\rangle \\ &\quad - (1-\alpha_n)(\|u_n - x_n\|^2 + \alpha\|Tu_n - u_n\|^2).\end{aligned} \tag{3.6}$$

Put $\tilde{\alpha}_n := \alpha_n(1-\kappa^2)$ and $\tilde{\beta}_n := \beta_n/(1-\kappa^2)$, where

$$\beta_n := 2\langle f(q) - q, x_{n+1} - q\rangle - \frac{1-\alpha_n}{\alpha_n}(\|u_n - x_n\|^2 + \alpha\|Tu_n - u_n\|^2). \tag{3.7}$$

Then we can rewrite (3.6) as

$$\|x_{n+1} - q\|^2 \le (1 - \tilde{\alpha}_n)\|x_n - q\|^2 + \tilde{\alpha}_n\tilde{\beta}_n. \tag{3.8}$$

In order to prove $x_n \to q$ in norm via Lemma 2.10, we must verify two conditions:

(i) $\sum_{n=1}^{\infty} \tilde{\alpha}_n = \infty$,

(ii) $\limsup_{n\to\infty} \tilde{\beta}_n \le 0$, equivalently, $\limsup_{n\to\infty} \beta_n \le 0$.

Condition (i) is guaranteed by Assumption (B_1). To verify (ii), we first observe that $\{\beta_n\}$ is bounded from above (by $2\|f(q) - q\|(M + \|q\|)$ with $M \ge \sup\{\|x_n\| : n \ge 0\}$). Thus, $\limsup_{n\to\infty} \beta_n$ exists. Take a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\lim_{k\to\infty} \beta_{n_k} = \limsup_{n\to\infty} \beta_n$. We may also assume that $x_{n_k+1} \rightharpoonup p$. It then turns out from the definition of β_{n_k} that

$$\lim_{k\to\infty} \frac{1-\alpha_{n_k}}{\alpha_{n_k}} (\|u_{n_k} - x_{n_k}\|^2 + \alpha\|Tu_{n_k} - u_{n_k}\|^2) \tag{3.9}$$

exists (which actually equals to $2\langle f(q) - q, p - q\rangle - \lim_{k\to\infty} \beta_{n_k}$). Since $\alpha_{n_k} \to 0$, it turns out from (3.9) that

$$\lim_{k\to\infty} (\|u_{n_k} - x_{n_k}\|^2 + \alpha\|Tu_{n_k} - u_{n_k}\|^2) = 0. \tag{3.10}$$

Using (3.1b), we find that $\|x_{n+1} - Tu_n\| = \alpha_n\|f(x_n) - Tu_n\| \to 0$. Hence, by (3.10),

$$\|x_{n_k+1} - x_{n_k}\| \le \|x_{n_k+1} - Tu_{n_k}\| + \|Tu_{n_k} - u_{n_k}\| + \|u_{n_k} - x_{n_k}\| \to 0.$$

This asserts that $x_{n_k} \rightharpoonup p$ and then $u_{n_k} \rightharpoonup p$ as well.

Since $I - T$ is demiclosed, we derive from the fact $\|Tu_{n_k} - u_{n_k}\| \to 0$ that $Tp = p$, namely, $p \in Fix(T)$. We now claim that $p \in EP(\phi)$ so that $p \in F = Fix(T) \cap EP(\phi)$. Since $u_{n_k} = Q_{r_{n_k}} x_{n_k}$, we have that for all $v \in C$

$$\phi(u_{n_k}, v) + \frac{1}{r_{n_k}} \langle v - u_{n_k}, u_{n_k} - x_{n_k}\rangle \ge 0.$$

Using the monotonicity condition (A_2), we get

$$\frac{1}{r_{n_k}} \langle v - u_{n_k}, u_{n_k} - x_{n_k}\rangle \ge \phi(v, u_{n_k}). \tag{3.11}$$

Since $r_{n_k} \ge a > 0$ for all k by (B_2), and since $\lim_{k\to\infty} \|x_{n_k} - u_{n_k}\| = 0$, it follows from (3.11) that

$$\limsup_{k\to\infty} \phi(v, u_{n_k}) \le 0. \tag{3.12}$$

By the weak lower semicontinuity of $\phi(v, \cdot)$ and the fact $u_{n_k} \rightharpoonup p$, we immediately get

$$\phi(v, p) \le 0. \tag{3.13}$$

Next, for each $v \in C$ and $t \in (0, 1)$, setting $v_t := tv + (1 - t)p$ and using properties (A_1) and (A_4) we get

$$0 = \phi(v_t, v_t) = \phi(v_t, tv + (1 - t)p) \le t\phi(v_t, v) + (1 - t)\phi(v_t, p) \le t\phi(v_t, v).$$

Consequently, $\phi(v_t, v) \ge 0$, which together with the monotonicity (A_2) implies that $\phi(v, v_t) \le 0$. This implies upon letting $t \to 0$, by the lower semicontinuity property (A_4), that $\phi(v, p) \le 0$ for each $v \in C$. Hence, $p \in EP(\phi)$.

Now we are ready to verify condition (ii). As a matter of fact, we have

$$\begin{aligned}\limsup_{n\to\infty}\beta_n &= \lim_{k\to\infty}\beta_{n_k}\\ &= \lim_{k\to\infty}\Big\{2\langle f(q)-q, x_{n_k+1}-q\rangle\\ &\quad -\frac{1-\alpha_{n_k}}{\alpha_{n_k}}(\|u_{n_k}-x_{n_k}\|^2+\alpha\|Tu_{n_k}-u_{n_k}\|^2)\Big\}\\ &\le \lim_{k\to\infty} 2\langle f(q)-q, x_{n_k+1}-q\rangle\\ &= 2\langle f(q)-q, p-q\rangle \le 0 \quad \text{(due to VI (3.2) and } p\in F).\end{aligned}$$

Finally, conditions (i)-(ii) make Lemma 2.10 applicable to the relation (3.8) and we obtain that $\|x_n-q\|^2\to 0$, i.e., $x_n\to q$ in norm as $n\to\infty$. □

Remark 3.2. The proof of Theorem 3.1 given here provided a novel way (see also [29]) in proving strong convergence of iterative algorithms for fixed-point and optimization problems compared with the way of Maingé [14]. Set $a_n := \|x_n-q\|^2$ for $n\ge 0$. Maingé's method always distinguishes two cases. The first case assumes that the sequence $\{a_n\}_{n=0}^\infty$ is eventually nonincreasing at infinity, which means that, for some integer $n_0\ge 0$, the sequence $\{a_n\}_{n\ge n_0}$ is nonincreasing; consequently, $\lim_{n\to\infty} a_n$ exists. The second case is the opposite to the first case; that is, for any integer $n_0\ge 0$, the sequence $\{a_n\}_{n\ge n_0}$ is not nonincreasing. Thus there exists a subsequence $\{n_j\}$ such that $a_{n_j} < a_{n_j+1}$ for all integer $j\ge 0$. Define $\tau(n) := \max\{j\le n : a_j < a_{j+1}\}$ for $n\ge n_0$. Then $\tau(n)\to\infty$ as $n\to\infty$ and $\max\{a_{\tau(n)}, a_n\}\le a_{\tau(n)+1}$.

Maingé used his method to prove the strong convergence [14, Theorem 3.1] of a projected subgradient method for a constrained nonsmooth convex optimization problem. His method has been followed by many researchers, including Thong [26]. However, it is not necessary to always use Maingé's technique (i.e., [14, Lemma 3.1]) by distinguishing two cases (as outlined above). The proof of Theorem 3.1 we gave here demonstrates an alternative way for proving strong convergence of iterative algorithms by deliberately estimating $\|x_{n+1}-q\|^2$ in terms of $\|x_n-q\|^2$.

Corollary 3.3. *Let all the assumptions of Theorem 3.1 hold except that the mapping $T: C\to C$ is now a β-demicontractive mapping for some $\beta\in(0,1)$. Let $\{x_n\}$ be generated by the iteration algorithm:*

$$\begin{cases} u_n = Q_{r_n}x_n,\\ x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)T_\lambda u_n,\end{cases} \tag{3.14}$$

for $n\ge 0$, where the starting point $x_0\in C$ is arbitrary and $T_\lambda = (1-\lambda)I+\lambda T$ is the λ-relaxation of T. Assume $\lambda\in(0,1-\beta)$. Then, the sequences $\{x_n\}$ and $\{u_n\}$ converge strongly to $q\in F$, where $q = P_F f(q)$.

Proof. From Lemma 2.8, T_λ is α-strongly quasi-nonexpansive with $\alpha := \frac{1}{\lambda}(1-\beta-\lambda) > 0$ for $\lambda\in(0,1-\beta)$. On the other hand, $Fix(T) = Fix(T_\lambda)$ and $\lambda(I-T) = I-T_\lambda$. Thus $I-T_\lambda$ is also demiclosed at zero. Consequently, applying Theorem 3.1 to the mapping T_λ yields the strong convergence of $\{x_n\}$ and $\{u_n\}$ defined by the algorithm (3.14). □

Remark 3.4. Corollary 3.3 is a generalization of [26, Corollary 3.2].

Remark 3.5. Since every quasi-nonexpansive mapping is β-demicontractive mapping, Corollary 3.3 remains true when T is quasi-nonexpansive mapping on H. So, Corollary 3.3 is a generalization of [16, Theorem 3.1].

The next result is regarding an iteration method for SCFP (1.3).

Theorem 3.6. *Let H_1 and H_2 be real Hilbert spaces, $S : H_1 \to H_1$ an $\hat{\alpha}_1$-strongly quasi-nonexpansive mapping and $T : H_2 \to H_2$ a β-demicontractive mapping. Suppose that $I - S$ and $I - T$ are demiclosed at zero. Let $A : H_1 \to H_2$ be a bounded linear operator, $\phi : H_1 \times H_1 \to \mathbb{R}$ a bifunction satisfying the conditions (A_1)–(A_4) of Lemma 2.2, f a κ-contraction on H_1 for some $\kappa \in [0, 1)$. Set $F := EP(\phi) \cap \Gamma$ and assume $F \neq \emptyset$, where Γ is the solution set (1.4) of SCFP (1.3). Suppose $\{\alpha_n\}$ and $\{r_n\}$ are real sequences satisfying the following conditions:*

(B_1) $\{\alpha_n\} \subset (0, 1)$, $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(B_2) $\{r_n\} \subset (a, \infty)$ for some $a > 0$.

Let $\{x_n\} \subset H_1$ be a sequence generated by the iteration process

$$\begin{cases} \phi(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad y \in H_1, \\ x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)S(I + \gamma A^*(T - I)A)u_n, \quad n \geq 0, \end{cases} \tag{3.15}$$

where $x_0 \in H_1$ and $\gamma \in (0, \frac{1-\beta}{L})$ with $L = \|A^*A\|$. Then, the sequences $\{x_n\}$ and $\{u_n\}$ converge strongly to $q \in F$, where $q = P_F f(q)$.

Proof. Set $V := I + \gamma A^*(T - I)A$ for a fixed $\gamma \in (0, \frac{1-\beta}{L})$. Then the algorithm (3.15) is reduced to the algorithm (3.1) associating with the mapping SV. By Theorem 3.1, it is suffices to show that $F = EP(\phi) \cap Fix(SV)$ and $I - SV$ is demiclosed at zero. From Lemma 2.9, V is $\hat{\alpha}_2$-strongly quasi-nonexpansive mapping with $\hat{\alpha}_2 := \frac{1}{\gamma L}(1 - \beta - \gamma L)$. Hence SV is α-strongly quasi-nonexpansive and $Fix(S) \cap Fix(V) = Fix(SV)$ by Lemma 2.6 where $\alpha = \frac{\hat{\alpha}_1\hat{\alpha}_2}{\hat{\alpha}_1+\hat{\alpha}_2}$. It follows from Lemma 2.9 that

$$\begin{aligned} \Gamma &= \{x \in H_1 : x \in Fix(S) \text{ and } Ax \in Fix(T)\} \\ &= \{x \in H_1 : x \in Fix(S) \text{ and } x \in Fix(V)\} \\ &= Fix(S) \cap Fix(V) = Fix(SV). \end{aligned}$$

Now, we show that $I - SV$ is demiclosed at zero. Let $\{x_n\} \subset H_1$ be a sequence such that $x_n \rightharpoonup x$ and $x_n - SVx_n \to 0$. To prove $x \in Fix(SV)$, take $z \in F$ and use the fact that S and V are $\hat{\alpha}_1$- and $\hat{\alpha}_2$-strongly quasi-nonexpansive, respectively, to derive that

$$\begin{aligned} \|SVx_n - z\|^2 \leq &\|Vx_n - z\|^2 - \hat{\alpha}_1\|Vx_n - SVx_n\|^2 \\ \leq &\|x_n - z\|^2 - \hat{\alpha}_2\|x_n - Vx_n\| - \hat{\alpha}_1\|Vx_n - SVx_n\|^2. \end{aligned}$$

It turns out that, for a certain appropriate constant $M > 0$

$$\begin{aligned} &\hat{\alpha}_2\|x_n - Vx_n\| + \hat{\alpha}_1\|Vx_n - SVx_n\|^2 \leq \|x_n - z\|^2 - \|SVx_n - z\|^2 \\ &= (\|x_n - z\| + \|SVx_n - z\|)(\|x_n - z\| - \|SVx_n - z\|) \\ &\leq M\|x_n - SVx_n\| \to 0. \end{aligned}$$

Therefore, $\|x_n - Vx_n\| \to 0$ and $\|Vx_n - SVx_n\| \to 0$. This implies $Vx_n \rightharpoonup x$. By the demiclosedness of $I - S$, we get $x \in Fix(S)$. Moreover, from Lemma 2.9(ii), we obtain

$$\begin{aligned}\|SVx_n - z\|^2 &\le \|Vx_n - z\|^2 \\ &\le \|x_n - z\|^2 - \gamma(1-\beta-\gamma L)\|Ax_n - TAx_n\|^2.\end{aligned}$$

It follows that

$$\begin{aligned}\gamma(1-\beta-\gamma L)\|Ax_n - TAx_n\|^2 &\le \|x_n - z\|^2 - \|SVx_n - z\|^2 \\ &\le M\|x_n - SVx_n\| \to 0.\end{aligned}$$

Since $Ax_n \rightharpoonup Ax$, the demiclosedness of $I - T$ implies $Ax \in Fix(T)$. Hence, $x \in Fix(S) \cap Fix(V) = Fix(SV)$ and $I - SV$ is demiclosed at zero. □

Corollary 3.7. *Let all the assumptions of Theorem 3.6 hold except that the mapping $S : H_1 \to H_1$ is a μ-demicontractive mapping for some $\mu \in (0,1)$ [instead of $\hat{\alpha}_1$-strongly quasi-nonexpansive mapping]. Let $\{x_n\}$ be generated by the iteration process*

$$\begin{cases} \phi(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \ge 0, \quad y \in H_1, \\ x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n) S_\lambda (I + \gamma A^*(T - I)A)u_n, \quad n \ge 0, \end{cases} \tag{3.16}$$

where $x_0 \in H$ and $\lambda \in (0, 1-\mu)$. Then, the sequences $\{x_n\}$ and $\{u_n\}$ converge strongly to $q \in F$, where $q = P_F f(q)$.

Proof. From Lemma 2.8, $V = I + \gamma A^*(T - I)A)$ is $\hat{\alpha}_2$-strongly quasi-nonexpansive with $\hat{\alpha}_2 = \frac{1}{\gamma L}(1 - \beta - \gamma L)$ and from Lemma 2.8, S_λ is ν-strongly quasi-nonexpansive with $\nu = \frac{1}{\lambda}(1 - \mu - \lambda)$. The remaining of the proof follows from that of Theorem 3.6. □

Remark 3.8. Corollary 3.7 remains true when T and S are quasi-nonexpansive mappings, similar to Remark 3.5.

Theorem 3.9. *Let H_1 and H_2 be real Hilbert spaces, $S : H_1 \to H_1$ a β-demicontractive mapping, $T : H_2 \to H_2$ a μ-demicontractive mapping, $A : H_1 \to H_2$ a bounded linear operator with $L = \|A^*A\|$, $\phi : H_1 \times H_1 \to \mathbb{R}$ a bifunction satisfying the conditions (A_1)-(A_4) of Lemma 2.2, f a κ-contraction on H_1 for some $\kappa \in [0,1)$, $F := EP(\phi) \cap \Gamma \ne \emptyset$. Suppose $\{\alpha_n\}$ and $\{r_n\}$ are real sequences satisfying the following conditions:*

- (B_1) *$\{\alpha_n\} \subset (0,1)$, $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$;*
- (B_2) *$\{r_n\} \subset (a, \infty)$ for some $a > 0$.*

Let $\{x_n\} \subset H_1$ be a sequence generated by the iterative process

$$\begin{cases} \phi(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \ge 0, \quad y \in H_1, \\ x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n) W_\lambda u_n, \quad n \ge 1, \end{cases} \tag{3.17}$$

where $x_0 \in H_1$, $W = S(I + \gamma A^*(T - I)A)$, and $W_\lambda = I + \lambda(W - I)$ is the λ-relaxation of W. Assume $I - W$ is demiclosed at zero, $\beta < \alpha$, where $\alpha = \frac{1}{\gamma L}(I - \mu - \gamma L)$, $\lambda \in (0, 1 - \frac{\alpha\beta}{\alpha+\beta})$ and $\gamma \in (0, \frac{1-\mu}{L})$. Then, the sequences $\{x_n\}$ and $\{u_n\}$ converge strongly to $q \in F$, where $q = P_F f(q)$.

Proof. Set $V := I + \gamma A^*(T - I)A)$ for a fixed $\gamma \in (0, \frac{1-\mu}{L})$. From Lemma 2.9, V is α-strongly quasi-nonexpansive mapping. It turns out that SV is $\frac{\alpha\beta}{\alpha+\beta}$-demicontractive and $Fix(S) \cap Fix(V) = Fix(SV)$ from Lemma 2.7. Now, we show that $\Gamma = Fix(S) \cap Fix(V) = Fix(SV)$. In fact, it follows from Lemma 2.9 that

$$\begin{aligned}\Gamma &= \{x \in H_1 : x \in Fix(S) \text{ and } Ax \in Fix(T)\}\\ &= \{x \in H_1 : x \in Fix(S) \text{ and } x \in Fix(V)\}\\ &= F(S) \cap Fix(V) = Fix(SV).\end{aligned}$$

□

Corollary 3.10. *In Theorem 3.9, suppose S is quasi-nonexpansive (instead of being β-demicontractive), and $I - S$ and $I - T$ (instead of $I - W$) are demiclosed at zero; suppose also $\lambda \in (0, 1)$. Then, the sequences $\{x_n\}$ and $\{u_n\}$ generated by the algorithm* (3.17) *converge strongly to $q \in F$, where $q = P_F f(q)$.*

Proof. According to the proof of Theorem 3.9, it is suffices to show that $I - SV$ is demiclosed at zero, which can be proved similarly by repeating the proof of Theorem 3.6. □

Remark 3.11. Corollary 3.10 is a generalization of [26, Corollary 3.8], [37, Corollary 3.2] and [18, Theorem 2.1].

4. Application

Let H_1 and H_2 be real Hilbert spaces, $B_1 : H_1 \to 2^{H_1}$ and $B_2 : H_2 \to 2^{H_2}$ be set-valued operators, and $A : H_1 \to H_2$ be a (nonzero) bounded linear operator. The split common null point problem (SCNP) is the problem of finding a point $x \in H_1$ with the property

$$0 \in B_1(x) \text{ and } 0 \in B_2(Ax). \tag{4.1}$$

Recently, Byrne et al. [3] and Kazmi et al. [13] proposed a strongly convergent algorithm for finding a solution of SCNP (4.1) when B_1 and B_2 are maximal monotone. Recall that $B : H \to 2^H$ is said to be monotone if

$$\langle x - y, u - v\rangle \geq 0, \quad x, y \in D(B),\, u \in Bx,\, v \in By,$$

where $D(B) := \{x \in H : Bx \neq \emptyset\}$ is the (effective) domain of B.

A monotone operator is said to be maximal if its graph is not properly contained in the graph of any other monotone operator. For a maximal monotone operator $B : H \to 2^H$ and $\lambda > 0$, we can define the following single-valued operator (referred to as resolvent):

$$J_\lambda^B := (I + \lambda B)^{-1} : H \to H.$$

It is known J_λ^B is firmly nonexpansive and $0 \in B(x)$ if and only if $x \in Fix(J_\lambda^B)$. Therefore, the problem (4.1) is equivalent to the problem of finding a point $x \in H_1$ satisfying the property

$$x \in Fix(J_\lambda^{B_1}) \text{ and } Ax \in Fix(J_\lambda^{B_2}),$$

where $\lambda > 0$, that is, the SCNP is reduced to the split common fixed point problem (SCFP).

Theorem 4.1. *Let $B_1 : H_1 \to 2^{H_1}$ and $B_2 : H_2 \to 2^{H_2}$ be set-valued maximal monotone operators, and $A : H_1 \to H_2$ be a bounded linear operator. Set Γ to be the solution set of SCNP (4.1) and $F = EP(\phi) \cap \Gamma$. Assume $F \neq \emptyset$. Let $f : H_1 \to H_1$ be a κ-contraction for some $\kappa \in [0, 1)$ and $\gamma \in (0, \frac{1}{L})$ with $L = \|A^*A\|$. Suppose $\{\alpha_n\}$and $\{r_n\}$ are real sequences satisfying the following conditions:*

(B_1) $\{\alpha_n\} \subset (0, 1)$, $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(B_2) $\{r_n\} \subset (a, \infty)$ *for some* $a > 0$.

Let $\{x_n\} \subset H_1$ be a sequence generated by the iterative algorithm

$$\begin{cases} \phi(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \text{for all } y \in H_1, \\ x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) J_\lambda^{B_1}(I - \gamma A^*(I - J_\lambda^{B_2})A)u_n, \end{cases} \tag{4.2}$$

for $n \geq 0$, where the starting point $x_0 \in H_1$ is arbitrary. Then, the sequences $\{x_n\}$ and $\{u_n\}$ converge strongly to $q \in F$, where $q = P_F f(q)$.

Proof. The resolvents $J_\lambda^{B_1}$ and $J_\lambda^{B_2}$ are firmly nonexpansive operators, and $I - J_\lambda^{B_1}$ and $I - J_\lambda^{B_2}$ are demiclosed at zero. Also, $J_\lambda^{B_1}$ is 1-strongly quasi-nonexpansive and $J_\lambda^{B_2}$ is 0-demicontractive. Consequently, the strong convergence of the algorithm (4.2) immediately follows from Theorem 3.6. □

Remark 4.2. Theorem 4.1 is a generalization of [4, Theorem 4.5] when we take $\phi = 0$ and the contraction $f(x) \equiv u$ to be constant.

5. Numerical Test

In this section, we give a numerical example to illustrate the convergence of the algorithm (3.15) in Theorem 3.6.

Example 5.1. Let $H_1 = H_2 = \mathbb{R}^2$ and define

$$\phi((x, y)^t, (u, v)^t) := -6(x^2 + y^2) + xu + yv + 5(u^2 + v^2). \tag{5.1}$$

It is easy to verify that ϕ satisfies the conditions (A_1)-(A_4). First, we deduce a formula for $Q_r((x, y)^t)$. For any $(u, v)^t \in \mathbb{R}^2$ and $r > 0$,

$$\phi((x, y)^t, (u, v)^t) + \frac{1}{r}\langle (u - z, v - w)^t, (z - x, w - y)^t\rangle \geq 0, \tag{5.2}$$

if and only if

$$\begin{aligned} &5r(u^2 + v^2) + ((r + 1)z - x)u + ((r + 1)w - y)v + xz + yw \\ &- (6r + 1)(z^2 + w^2) \geq 0. \end{aligned} \tag{5.3}$$

Set

$$G(u) = 5ru^2 + ((r + 1)z - x)u + xz - (6r + 1)z^2,$$

and

$$J(v) = 5rv^2 + ((r + 1)w - y)v + yw - (6r + 1)w^2.$$

Therefore, from (5.3), it is clear that (5.2) holds if and only if

$$G(u) + J(v) \geq 0 \text{ for all } u, v \in \mathbb{R}. \tag{5.4}$$

Also, we know $G(u)$ is a quadratic function of u with coefficients

$$a := 5r,\ b := (r+1)z - x,\ c := xz - (6r+1)z^2.$$

The discriminant of $G(u)$, $\Delta = b^2 - 4ac$, is given by $\Delta_G = [(11r+1)z - x]^2$. Similarly, $J(v)$ is a quadratic function of v and its discriminant is $\Delta_J = [(11r+1)w - y]^2$. Hence $\Delta_G \geq 0$ and $\Delta_J \geq 0$. Next, we will show (5.4) is true if and only if

$$G(u) \geq 0 \text{ for all } u \in \mathbb{R} \text{ and } J(v) \geq 0 \text{ for all } v \in \mathbb{R}. \tag{5.5}$$

Obviously, (5.5) concludes (5.4). Inversely, suppose (5.4) holds and there exists $u_0 \in \mathbb{R}$ such that $G(u_0) < 0$. From (5.4), we get $J(v) > 0$ for all $v \in \mathbb{R}$. So, the discriminant of $J(v)$ should be negative. This is a contradiction with $\Delta_J \geq 0$. Hence, $G(u) \geq 0$ for all $u \in \mathbb{R}$. By the same argument as above, we can prove $J(v) \geq 0$ for all $v \in \mathbb{R}$. This shows (5.4) derives (5.5).

Now, from (5.5), we have $\Delta_G \leq 0$. That is, $[(11r+1)z - x]^2 \leq 0$. So, $z = \frac{x}{11r+1}$. Similarly, we get $w = \frac{y}{11r+1}$. By Lemma 2.3, Q_r is single-valued for $r > 0$. Hence $Q_r((x,y)^t) = (\frac{x}{11r+1}, \frac{y}{11r+1})^t$. Thus, from Lemma 2.3, we get $EP(\phi) = \{(0,0)^t\}$. Let $\alpha_n = \frac{1}{n}$ and $r_n = 1$, for all $n \in \mathbb{N}$, $T((x,y)^t) = (0,y)^t$, $f((x,y)^t) = \frac{1}{2}(x,y)^t$ and $S((x,y)^t) = \frac{1}{4}(x,y)^t$, $A = \begin{bmatrix} 1 & 3 \\ -2 & 1 \end{bmatrix}$, $\gamma = \frac{1}{5}$ and $x_n = (a_n, b_n)^t$. Hence $F = \Gamma \cap EP(\phi) = (0,0)^t$. Then, from Theorem 3.6, the sequences $\{x_n\}$ and $\{u_n\}$, generated iteratively by

$$\begin{cases} u_n = (\frac{1}{12}a_n, \frac{1}{12}b_n)^t, \\ x_{n+1} = \left(\frac{(292n-12)a_n + (n-1)b_n}{560n}, \frac{(881n-41)b_n + 2(n-1)a_n}{1680n}\right)^t, \end{cases} \tag{5.6}$$

converge strongly to $(0,0)^t \in F$, where $(0,0)^t = P_F(f)((0,0)^t)$.

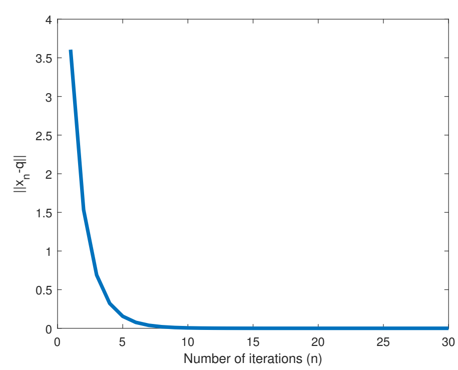

FIGURE 1. The convergence of $\{x_n\}$ with initial value $x_0 = (-3, 2)^t$

TABLE 1. The values of the sequences $\{a_n\}$, $\{b_n\}$ and $\{\|x_n - q\|\}$

Numerical results for $x_1 = (-3, 2)^t$			
n	a_n	b_n	$\|x_n - q\|$
1	-3	2	3.6056
2	-1.1625	1	1.5334
3	-0.46203	0.51151	0.68929
⋮	⋮	⋮	⋮
14	$-1.7602{-}5$	0.00038738	0.00038778
15	$-6.5285e^{-6}$	0.00020245	0.00020255
16	$-2.323e^{-6}$	0.00010583	0.00010585
⋮	⋮	⋮	⋮
28	$5.7821e^{-10}$	$4.4568e^{-8}$	$4.4572e^{-8}$
29	$3.1275e^{-10}$	$2.3333e^{-8}$	$2.3335e^{-8}$
30	$11.6789e^{-10}$	$1.2217e^{-8}$	$1.2218e^{-8}$

Table 1 indicates the values of the sequences $\{a_n\}$, $\{b_n\}$ and $\|x_n - q\|$ for algorithm (5.6), where $x_1 = (-3, 2)^t$ and $n = 30$.

Figure 1 shows the behavior of $\|x_n - q\|$ that corresponds to Table 1 and also the convergence to $0 \in F$ of the sequence $\{x_n\}$.

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