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EXISTENCE AND ASYMPTOTICAL BEHAVIOR OF GROUND STATE SOLUTIONS FOR FRACTIONAL SCHRÖDINGER-KIRCHHOFF TYPE EQUATIONS

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Abstract. In this paper, we study the following Schrödinger-Kirchhoff type equations involving the fractional p-Laplacian $M([u]^p_{s,p})(-\Delta)^s_p u + (1+\lambda g(x))u^{p-1} = H(x)u^{q-1}$, $u > 0$, $x \in \mathbb{R}^N$, where $s \in (0,1)$, $2 \le p < \infty$, $ps < N$ and $(-\Delta)^s_p$ is the fractional p-Laplacian operator. $M(t) = a + bt^k$, where $a, k > 0$ and $b \ge 0$ are constants. $\lambda > 0$ is a real parameter. $p(k+1) < q < p^*_s$, where $p^*_s = \frac{Np}{N-ps}$ is the fractional Sobolev critical exponent. Under some appropriate assumptions on $g(x)$ and $H(x)$, we obtain the existence of positive ground state solutions and discuss their asymptotical behavior via the method used by Bartsch and Wang [Multiple positive solutions for a nonlinear Schrödinger equation. Z. Angew. Math. Phys. 51 (2000) 366-384].
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1. Introduction and main results

In this paper, we consider the following equation:

$$M\left(\iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}} \mathrm{d}x\mathrm{d}y\right)(-\Delta)^s_p u + (1+\lambda g(x))u^{p-1} = H(x)u^{q-1}, \quad (1.1)$$

$u > 0$, $x \in \mathbb{R}^N$, where $s \in (0,1)$, $2 \le p < \infty$ and $ps < N$. Here $(-\Delta)^s_p$ is the fractional p-Laplacian operator which (up to normalization factors) may be defined along a function $\varphi \in \mathrm{C}^\infty_0(\mathbb{R}^N)$ as

$$(-\Delta)^s_p \varphi(x) = 2 \lim_{\epsilon \to 0^+} \int_{\mathbb{R}^N \setminus B_\epsilon(x)} \frac{|\varphi(x)-\varphi(y)|^{p-2}(\varphi(x)-\varphi(y))}{|x-y|^{N+ps}} \mathrm{d}y, \quad x \in \mathbb{R}^N,$$

where $B_\epsilon(x) := \{y \in \mathbb{R}^N : |x-y| < \epsilon\}$. In relation to the fractional p-Laplacian operator, we recommend readers to read [14, 19, 12] and the references therein.

When $p=2$ and $M \equiv 1$, Eq. (1.1) transform into the fraction Laplacian equation

$$(-\Delta)^s u + V(x)u = h(x,u), \qquad x \in \mathbb{R}^N,$$

which can been seen as the fractional form of the following classical stationary Schrödinger equation

$$-\Delta u + V(x)u = h(x,u), \qquad x \in \mathbb{R}^N.$$

In recent years, a great interest has devoted to studying the existence of solutions via variational methods for the Kirchhoff equation:

$$u_{tt} - \left(a + b\int_\Omega |\nabla u|^2 \,\mathrm{d}x\right)\Delta u = h(x,u).$$

It comes from the well-known D'Alembert wave equation

$$\rho\frac{\partial^2 u}{\partial t^2} - \left(\frac{p_0}{h} + \frac{E}{2L}\int_0^L \left|\frac{\partial u}{\partial x}\right|^2 \mathrm{d}x\right)\frac{\partial^2 u}{\partial x^2} = h(x,u)$$

for free vibrations of elastic strings, see[15]. Here, u denotes the displacement of a string, ρ is the mass density, p_0 is the initial tension, h is the area of cross-section, E is the Young modulus of the material and L is the length of the string. For this, Kirchhoff's model takes into account the changes in length of the string produced by transverse vibrations. For more details on the Kirchhoff equation, we recommend readers to read [1, 9, 15] and the references therein. More recently, in [11], Fiscella and Valdinoci provide a detailed discussion about the physical meaning underlying the fractional Kirchhoff problems and their applications. Indeed, in Appendix, they construct a stationary Kirchhoff variational problem which models, as a special significant case, the nonlocal aspect of the tension arising from nonlocal measurements of the fractional length of the string.

Nonlocal fractional problems have been appearing in the literature in many different contexts, both in the pure mathematical research and in concrete real-world application. Indeed, fractional and nonlocal operators appear in many diverse fields such as optimization, finance, phase transitions, stratified materials, anomalous diffusion, crystal dislocation, soft thin films, semipermeable membranes, flame propagation, conversion laws, ultra-relativistic of quantum mechanics, quasi-geotrophic flows, multiple scattering, minimal surface, materials science, and water waves, see [22, 10] for more details.

Indeed, for the Schrödinger-Kirchhoff type equations involving the fractional p-Laplacian, many people have made great contributions, see e.g. [7, 26, 24, 23, 16, 31, 30, 28, 27, 25, 3, 17, 18, 20, 2] and the references therein. For exmple, in [24], Pucci and Xiang obtain two solutions for nonhomogeneous fractional p-Laplacian equations of Schrödinger-Kirchhoff type

$$M\left(\iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\mathrm{d}x\mathrm{d}y\right)(-\Delta)_p^s u + V(x)\,|u|^{p-2}\,u = f(x,u) + g(x), \ \text{ in } \mathbb{R}^N$$

by assuming the potential function $V(x)$ satisfies following conditions:
(V_1) $V(x) \in C(\mathbb{R}^N)$ satisfies $\inf_{x\in\mathbb{R}^N} V(x) \geq V_0$, where $V_0 > 0$ is a constant;
(V_2) there exists $h > 0$ such that $\lim_{|y|\to\infty} \mathrm{meas}(\{x \in B_h(y) : V(x) \leq c\}) = 0$ for any $c > 0$, where $\mathrm{meas}(A)$ denotes the Lebesgue measure of A on $\mathbb{R}^N$.
In addition, in [31], Zhang et al. get the existence of infinitely many solutions for fractional p-Laplacian Schrödinger-Kirchhoff type equations with sign-changing potential

$$\left(a + b\iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\mathrm{d}x\mathrm{d}y\right)^{p-1} (-\Delta)^s_p u + V(x)\,|u|^{p-2}\,u = f(x,u), \ \text{ in } \mathbb{R}^N.$$

They assume the potential function $V(x)$ satisfies following conditions:
$(V_1^{'})$ $V(x) \in C(\mathbb{R}^N,\mathbb{R})$ and $\inf_{x\in\mathbb{R}^N} V(x) > -\infty$;
$(V_2^{'})$ There exists $d > 0$ such that $\lim_{|y|\to+\infty} \mathrm{meas}(\{x \in B_d(y) : V(x) \leq M\}) = 0$ for any $M > 0$.
However, to our knowledge, there is no study for elliptic equations of Schödinger-Kirchhoff type involving the fractional p-Laplacian and steep well potential $V(x) = 1 + \lambda g(x)$ yet.

In this paper, we are interested in the existence of positive ground state solutions of problem (1.1) and the asymptotical behavior of the solutions as $\lambda \to +\infty$. To obtain these results, the potential in (1.1) needs some restricted conditions to overcome the loss of compactness caused by $\mathbb{R}^N$. Specifically, for $V_\lambda(x) = 1 + \lambda g(x)$, we can have some compactness by letting the parameter λ large enough, while any limit is not needed to be posed on the potentials. The method above is first proposed by Bartsch and Wang[4, 5]. In relation to the steep well potentail, we recommend readers to read [21, 29] and the references therein.

Throughout this paper, we make the following assumptions:
(M) $M(t) = a + b\,t^k$ with $a, k > 0$ and $b \geq 0$;
(A_1) $2 \leq p < p(k+1) < q < p_s^*$;
(A_2) $g(x)$, $H(x)$ are two nonnegative, locally Hölder continuous, and bounded potential functions. Moreover, $0 \leq g(x) \leq g_\infty$ and $0 < H(x) \leq H_\infty$;
(A_3) There exists $M_0 > 0$ such that the set $A = \{x \in \mathbb{R}^N : g(x) \leq M_0\}$ is nonempty and $\mathrm{meas}(A) < \infty$;
(A_4) $\Omega := \mathrm{int}\{g^{-1}(0)\}$ is nonempty, bounded, and has smooth boundary, $\overline{\Omega} = g^{-1}(0)$.

Before stating our main results, we give the variational setting for (1.1). The fractional Sobolev space $W^{s,p}(\mathbb{R}^N)$ defined by

$$W^{s,p}(\mathbb{R}^N) = \{u \in L^p(\mathbb{R}^N) : [u]_{s,p} < \infty\},$$

where

$$[u]^p_{s,p} = \iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}},$$

and $W^{s,p}(\mathbb{R}^N)$ is equipped with the norm

$$\|u\|_{W^{s,p}(\mathbb{R}^N)} = \left([u]^p_{s,p} + \|u\|^p_p\right)^{\frac{1}{p}}.$$

It is clear that $W^{s,p}(\mathbb{R}^N)$ is a uniformly convex Banach space (see Appendix in [24]). For more basic properties of fractional Sobolev space, we refer the readers to [10].

The nature energy functional associated with (1.1) is given by

$$I_\lambda(u) = \frac{1}{p}\left(\mathcal{M}([u]_{s,p}^p) + \int_{\mathbb{R}^N} V_\lambda(x)\,|u|^p\right) - \frac{1}{q}\int_{\mathbb{R}^N} H(x)\,|u|^q\,, \tag{1.2}$$

where $\mathcal{M}(t) = at + \frac{b}{k+1}\,t^{k+1}$ and $V_\lambda(x) = 1 + \lambda g(x)$. It is well known that $I_\lambda(u)$ is well defined. Furthermore, $I_\lambda(u) \in \mathrm{C}^1(W^{s,p}, \mathbb{R})$ and for any $v \in \mathrm{C}_0^\infty(\mathbb{R}^N)$, there holds

$$\begin{aligned}\langle I_\lambda'(u), v\rangle = M([u]_{s,p}^p)\iint_{\mathbb{R}^{2N}} &\frac{|u(x)-u(y)|^{p-2}\,(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+ps}}\\ + \int_{\mathbb{R}^N} V_\lambda(x)\,|u|^{p-1}\,v - \int_{\mathbb{R}^N} &H(x)\,|u|^{q-1}\,v, \qquad \forall\, u \in W^{s,p}(\mathbb{R}^N).\end{aligned} \tag{1.3}$$

Define

$$N_\lambda := \{u \in W^{s,p}(\mathbb{R}^N)\setminus\{0\} \mid \langle I_\lambda'(u), u\rangle = 0\}$$

and

$$c_\lambda := \inf_{u\in N_\lambda} I_\lambda(u).$$

We call that u_λ is ground state solution of (1.1) if $u_\lambda \in W^{s,p}(\mathbb{R}^N)$ is a critical point of $I_\lambda(u)$ such that c_λ is achieved.

The following Dirichlet problem is a kind of 'limit' problem for (1.1):

$$\begin{cases} M\left([u]_{s,p}^p\right)(-\Delta)_p^s u + u^{p-1} = H(x)u^{q-1} & \text{in } \Omega,\\ u > 0 & \text{in } \Omega,\\ u = 0 & \text{on } \partial\Omega.\end{cases} \tag{1.4}$$

Similarly, we define a functional Φ_Ω on $W^{s,p}(\Omega)$ by

$$\Phi_\Omega(u) = \frac{1}{p}\left(\mathcal{M}([u]_{s,p}^p) + \int_\Omega |u|^p\right) - \frac{1}{q}\int_\Omega H(x)\,|u|^q\,, \tag{1.5}$$

and for any $v \in \mathrm{C}_0^\infty(\Omega)$, there holds

$$\begin{aligned}\langle \Phi_\Omega'(u), v\rangle = M([u]_{s,p}^p)\iint_{\Omega\times\Omega} &\frac{|u(x)-u(y)|^{p-2}\,(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+ps}}\\ + \int_\Omega |u|^{p-1}\,v - \int_\Omega &H(x)\,|u|^{q-1}\,v, \qquad \forall\, u \in W^{s,p}(\Omega).\end{aligned} \tag{1.6}$$

Moreover, we define the Nehari manifold N_Ω by

$$N_\Omega := \{u \in W^{s,p}(\Omega)\setminus\{0\} \mid \langle \Phi_\Omega'(u), u\rangle = 0\}$$

and c_Ω by

$$c_\Omega := \inf_{u\in N_\Omega} \Phi_\Omega(u).$$

We call that u is ground state solution of (1.4) if $u \in W^{s,p}(\Omega)$ is a critical point of $\Phi_\Omega(u)$ such that c_Ω is achieved.

Now we state the first result of this paper.

Theorem 1.1. *Assume that (M), $(A_1) - (A_4)$ hold. Then, for $\lambda > 0$ sufficiently large, c_λ is achieved by a critical point u_λ of I_λ such that u_λ is a ground state solution of problem (1.1).*

In fact, in order to get the ground state solution, another aim of this paper is to show that the energy of the ground state solution of (1.1) is equal to the energy of the mountain pass type solution.To this end, we need to obtain that there exists a unique $t(u) > 0$ such that $t(u)t \in N_\lambda$, $I_\lambda(t(u)t) = \max_{t\geq 0} I_\lambda(tu)$ and $\lim_{t\to\infty} I_\lambda(tu) = -\infty$ for any $u \in W^{s,p} \setminus \{0\}$. Hence, we need to assume that $g(x)$ is bounded.It is easy to check that $V(x) = 1+g(x)$ is not satisfying the coercivity condition.Then it will cause the lack of compactness. In order to overcome the difficulty, we let $V(x) = 1 + \lambda g(x)$ and λ is sufficiently large, where λ is a real parameter. Thus, we obtain the existence of ground state solutions of problem (1.1) for every $\lambda > \Lambda^*$, where Λ^* is a constant as in Lemma 3.9. Because of the introduction of λ, a natural question is whether the ground solutions of problem (1.1) have asymptotical behavior when $\lambda \to +\infty$. Then, we give the second result of this paper.

Theorem 1.2. *Assume that (M), $(A_1) - (A_4)$ hold. Then, for any sequence $\lambda_n \to +\infty$, u_{λ_n} has a subsequence converging to u in $W^{s,p}(\mathbb{R}^N)$ such that u is a ground state solution of problem (1.4).*

Remark 1.1. *As mentioned above, for λ large enough, we can prove that the functional I_λ satisfies $(PS)_c$ condition when c lies in suitable range (see Lemma* 3.9 *below).*

Remark 1.2. *In this paper, we extend the results of Chen and Guo* [8] *to the Schrödinger-Kirchhoff type equations involving the fractional p-Laplacian. Moreover, comparing with* [24, 7, 31]*, we consider the steep potential well question. Our results are new. Therefore, the results of this paper can enrich and supplement the previous one in the literature.*

This paper is organized as follows. In Sect. 2, we give some basic properties of the space $W^{s,p}(\mathbb{R}^N)$ and Nehari manifold N_λ. In Sect. 3, using the Mountain Pass theorem and comparing some energy levels, we obtain the existence of ground state solutions of problem (1.1). In Sect. 4, we complete the proof of Theorem 1.2.

Notations. In this paper, we make use of the following notations:

- we use C and C_i to denote various positive constants in context;
- $W^{s,p}$ denotes $W^{s,p}(\mathbb{R}^N)$;
- $L^r(\mathbb{R}^N)$ denotes the Lebesgue space with norm $\|u\|_r = (\int_{\mathbb{R}^N} |u|^r \,\mathrm{d}x)^{\frac{1}{r}}$, where $1 \leq r < \infty$;
- the weak convergence is denoted by $\rightharpoonup$, and the strong convergence is denoted by $\to$;
- $G^{(i)}(t)$ denote the i-order derivative of $G(t)$;
- $\int_{\mathbb{R}^N} \clubsuit$ denotes $\int_{\mathbb{R}^N} \clubsuit \mathrm{d}x$.

2. PRELIMINARIES

In this section, we give some properties of the fractional Sobolev spaces $W^{s,p}$ and energy functional $I_\lambda(u)$.

Lemma 2.1. *(Theorem 6.7 and Corollary 7.2 of* [10]*) If $\nu \in [p, p_s^*]$, then the embedding $W^{s,p}(\mathbb{R}^N) \hookrightarrow L^\nu(\mathbb{R}^N)$ is continuous. In particular, there exists a constant $C_\nu > 0$ such that*

$$\|u\|_{L^\nu(\mathbb{R}^N)} \le C_\nu \|u\|_{W^{s,p}}, \quad \forall\, u \in W^{s,p}. \tag{2.1}$$

If $\nu \in [1, p_s^*)$, then the embedding $W^{s,p}(B_R) \hookrightarrow\hookrightarrow L^\nu(B_R)$ is compact, where $B_R(x) := \{y \in \mathbb{R}^N : |x-y| < R\}$ and we simply write B_R when $x = 0$.

Lemma 2.2. *For any $u \in N_\lambda \setminus \{0\}$, there exists $\sigma \in (0,1)$ which is independent of λ such that*

$$\|u\|_{W^{s,p}} > \sigma, \quad I_\lambda(u) \ge \frac{(q-p)\min\{a,1\}}{pq}\sigma^p. \tag{2.2}$$

Proof. By $u \in N_\lambda \setminus \{0\}$, we have $\langle I_\lambda^{'}(u), u\rangle = 0$. Then

$$M([u]_{s,p}^p)[u]_{s,p}^p + \int_{\mathbb{R}^N} V_\lambda(x)\,|u|^p = \int_{\mathbb{R}^N} H(x)\,|u|^q. \tag{2.3}$$

By (A_2) and (2.1), we have

$$\begin{aligned} H_\infty C_q^q \|u\|_{W^{s,p}}^q &\ge \int_{\mathbb{R}^N} H(x)\,|u|^q \\ &= M([u]_{s,p}^p)[u]_{s,p}^p + \int_{\mathbb{R}^N} V_\lambda(x)\,|u|^p \\ &\ge a[u]_{s,p}^p + \int_{\mathbb{R}^N} V_\lambda(x)\,|u|^p \\ &\ge \min\{a,1\}\,\|u\|_{W^{s,p}}^p. \end{aligned}$$

Then,

$$\|u\|_{W^{s,p}} \ge \left(\frac{\min\{a,1\}}{H_\infty C_q^q}\right)^{\frac{1}{q-p}} > \sigma > 0.$$

On the other hand, by (1.2) and (2.3), we have

$$\begin{aligned} I_\lambda(u) &= \frac{1}{p}\left(\mathcal{M}([u]_{s,p}^p) + \int_{\mathbb{R}^N} V_\lambda(x)\,|u|^p\right) - \frac{1}{q}\int_{\mathbb{R}^N} H(x)\,|u|^q \\ &= \frac{1}{p}\left(\mathcal{M}([u]_{s,p}^p) + \int_{\mathbb{R}^N} V_\lambda(x)\,|u|^p\right) - \frac{1}{q}\left(M([u]_{s,p}^p)[u]_{s,p}^p + \int_{\mathbb{R}^N} V_\lambda(x)\,|u|^p\right) \\ &= \left(\frac{a}{p} - \frac{a}{q}\right)[u]_{s,p}^p + \left(\frac{b}{p(k+1)} - \frac{b}{q}\right)[u]_{s,p}^{p(k+1)} + \left(\frac{1}{p} - \frac{1}{q}\right)\int_{\mathbb{R}^N} V_\lambda(x)\,|u|^p \\ &\ge \left(\frac{1}{p} - \frac{1}{q}\right)\min\{a,1\}\,\|u\|_{W^{s,p}}^p \\ &\ge \frac{(q-p)\min\{a,1\}}{pq}\sigma^p. \end{aligned}$$

□

Lemma 2.3. *For any $u \in W^{s,p} \setminus \{0\}$, there exists a unique $t(u) > 0$ such that*

$$t(u)u \in N_\lambda, \quad I_\lambda(t(u)u) = \max_{t\geq 0} I_\lambda(tu).$$

Proof. Let

$$G(t) = I_\lambda(tu), \quad t > 0.$$

Then,

$$G(t) = \frac{a}{p}[u]_{s,p}^p t^p + \frac{b}{p(k+1)}[u]_{s,p}^{p(k+1)} t^{p(k+1)} + \frac{t^p}{p}\int_{\mathbb{R}^N} V_\lambda(x)\,|u|^p - \frac{t^q}{q}\int_{\mathbb{R}^N} H(x)\,|u|^q\,,$$

$$G^{(1)}(t) = a[u]_{s,p}^p t^{p-1} + b[u]_{s,p}^{p(k+1)} t^{p(k+1)-1} + t^{p-1}\int_{\mathbb{R}^N} V_\lambda(x)\,|u|^p - t^{q-1}\int_{\mathbb{R}^N} H(x)\,|u|^q$$

$$\vdots$$

$$\begin{aligned}G^{(p)}(t) &= (p-1)!a[u]_{s,p} + \frac{(pk+p-1)!}{(pk)!} b[u]_{s,p}^{p(k+1)} t^{pk} + (p-1)!\int_{\mathbb{R}^N} V_\lambda(x)\,|u|^p \\ &\quad - \frac{(q-1)!}{(q-p)!}\left(\int_{\mathbb{R}^N} H(x)\,|u|^q\right) t^{q-p},\end{aligned}$$

$$\vdots$$

$$G^{(pk+p)}(t) = (pk+p-1)!b[u]_{s,p}^{p(k+1)} - \frac{(q-1)!}{(q-p(k+1))!}\left(\int_{\mathbb{R}^N} H(x)\,|u|^q\right) t^{q-p(k+1)},$$

$$G^{(p(k+1)+1)}(t) = -\frac{(q-1)!}{(q-p(k+1)-1)!}\left(\int_{\mathbb{R}^N} H(x)\,|u|^q\right) t^{q-p(k+1)-1} < 0.$$

So, there exists $t_i > 0$, such that

$$\begin{cases} t < t_i, & G^{(i)}(t) > 0, \\ t = t_i, & G^{(i)}(t) = 0, \quad (i = 1, 2, \ldots, pk+p). \\ t > t_i, & G^{(i)}(t) < 0, \end{cases}$$

Therefore, there exists a unique $t(u) > 0$, such that $t(u)u \in N_\lambda$ and $I_\lambda(t(u)u) = \max_{t\geq 0} I_\lambda(tu)$. □

3. Proof of Theorem 1.1.

Lemma 3.1. *(i) There exist $\eta > 0$, $0 < \rho < 1$ both independent of λ, such that $I_\lambda(u) \geq \eta$ for all $u \in W^{s,p}$ with $\|u\|_{W^{s,p}} = \rho$.*

(ii) For any $u \in W^{s,p} \setminus \{0\}$, $\lim_{t\to\infty} I_\lambda(tu) = -\infty$.

Proof. By (1.2) and (2.1), we have

$$\begin{aligned} I_\lambda(u) &= \frac{a}{p}[u]_{s,p}^p + \frac{b}{p(k+1)}[u]_{s,p}^{p(k+1)} + \frac{1}{p}\int_{\mathbb{R}^N} V_\lambda(x)\,|u|^p - \frac{1}{q}\int_{\mathbb{R}^N} H(x)\,|u|^q \\ &\geq 2\sqrt{\frac{ab}{p^2(k+1)}}[u]_{s,p}^{\frac{p(k+2)}{2}} + \frac{1}{p}\,\|u\|_p^{\frac{p(k+2)}{2}} - \frac{C_q^q H_\infty}{q}\,\|u\|_{W^{s,p}}^q \\ &\geq \frac{1}{2^{\frac{k}{2}}}\min\left\{2\sqrt{\frac{ab}{p^2(k+1)}}, \frac{1}{p}\right\}\|u\|_{W^{s,p}}^{\frac{p(k+2)}{2}} - \frac{C_q^q H_\infty}{q}\,\|u\|_{W^{s,p}}^q. \end{aligned}$$

Since $q > p(k+1) > \frac{p(k+2)}{2}$, then there exists $\|u\|_{W^{s,p}} = \rho > 0$ small enough such that $I_\lambda(u) \geq \eta$.

On the other hand, for any $u \in W^{s,p} \setminus \{0\}$, we have

$$\begin{aligned} I_\lambda(tu) &= \frac{a}{p}[tu]_{s,p}^p + \frac{b}{p(k+1)}[tu]_{s,p}^{p(k+1)} + \frac{1}{p}\int_{\mathbb{R}^N} V_\lambda(x)\,|tu|^p - \frac{1}{q}\int_{\mathbb{R}^N} H(x)\,|tu|^q \\ &\leq \frac{at^p}{p}\,[u]_{s,p}^p + \frac{bt^{p(k+1)}}{p(k+1)}\,[u]_{s,p}^{p(k+1)} + \frac{(1+\lambda g_\infty)t^p}{p}\,\|u\|_p^p - \frac{t^q}{q}\int_{\mathbb{R}^N} H(x)\,|u|^q. \end{aligned}$$

Since $q > p(k+1) > p$, then $I_\lambda(tu) \to -\infty$ as $t \to \infty$. □

Let

$$c_\lambda^* = \inf_{u\in W^{s,p}\setminus\{0\}} \max_{t\geq 0} I_\lambda(tu), \quad c_\lambda^{**} = \inf_{\gamma\in\Gamma}\sup_{t\in[0,1]} I_\lambda(\gamma(t)),$$

where

$$\Gamma = \{\gamma(t) \in \mathrm{C}([0,1],\mathbb{R}) : \gamma(0) = 0, I_\lambda(\gamma(1)) < 0\}.$$

Lemma 3.2. *For $\lambda > 0$, $c_\lambda = c_\lambda^* = c_\lambda^{**}$.*

Proof. We divided the proof into three steps.

Step 1 $c_\lambda^* = c_\lambda$.

By Lemma 2.3, we have

$$c_\lambda^* = \inf_{u\in W^{s,p}\setminus\{0\}} \max_{t\geq 0} I_\lambda(tu) = \inf_{u\in W^{s,p}\setminus\{0\}} I_\lambda(t(u)u) = \inf_{u\in N_\lambda} I_\lambda(u) = c_\lambda.$$

Step 2 $c_\lambda^* \geq c_\lambda^{**}$.

Form Lemma 3.1, for $u \in W^{s,p} \setminus \{0\}$, there exists t_0 large enough, such that $I_\lambda(t_0u) < 0$. Define $\gamma_0(t) = tt_0u$ for $t \in [0,1]$. Then $\gamma_0(t) \in \Gamma$, and thus,

$$c_\lambda^{**} = \inf_{\gamma\in\Gamma}\sup_{t\in[0,1]} I_\lambda(\gamma(t)) \leq \sup_{t\in[0,1]} I_\lambda(\gamma_0(t)) \leq \max_{t\geq 0} I_\lambda(tu).$$

Then, $c_\lambda^* \geq c_\lambda^{**}$.

Step 3 $c_\lambda^{**} \geq c_\lambda$.

The manifold N_λ separates $W^{s,p}$ into two component. By Lemma 3.1, the component containing the origin also contains a small ball around the origin. Moreover, $I_\lambda(u) \geq 0$ in this component, because

$$G^{(1)}(t) \geq 0, \quad \forall t \in [0, t_1(u)].$$

Thus, every $\gamma \in \Gamma$ has to cross N_λ. Therefore, $c_\lambda^{**} \geq c_\lambda$. □

Lemma 3.3. *For any $\lambda > 0$, we have $c_\lambda \leq c_\Omega$.*

Proof. For each $u \in N_\Omega$, we have $\langle \Phi_\Omega^{'}(u), u\rangle = 0$. Then

$$M([u]_{s,p}^p)\iint_{\Omega\times\Omega}\frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}+\int_\Omega |u|^p=\int_\Omega H(x)\,|u|^q\,.$$

By $V_\lambda(x)=1$ in Ω and $u=0$ in $\mathbb{R}^N\setminus\Omega$, the above equality can be written as

$$M([u]_{s,p}^p)\iint_{\mathbb{R}^{2N}}\frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}+\int_{\mathbb{R}^N} V_\lambda(x)\,|u|^p=\int_{\mathbb{R}^N} H(x)\,|u|^q\,.$$

Thus, $u \in N_\lambda$.
On the other hand, we can easy to see

$$\Phi_\Omega(u) = I_\lambda(u), \quad \forall u \in N_\Omega.$$

Therefore, $c_\lambda \leq c_\Omega$. □

Lemma 3.4. *For any $\lambda > 0$, there exists a constant $K > 0$ independent of λ such that $c_\lambda \leq K$.*

Proof. Let $v \in C_0^\infty(\Omega)\setminus\{0\}$. By Lemma 2.3, we have

$$\begin{aligned}
I_\lambda(tv) &= \frac{1}{p}\left(\mathcal{M}([tv]_{s,p}^p)+\int_{\mathbb{R}^N} V_\lambda(x)\,|tv|^p\right)-\frac{1}{q}\int_{\mathbb{R}^N} H(x)\,|tv|^q\\
&= \frac{1}{p}\left(\mathcal{M}([tv]_{s,p}^p)+\int_{\Omega} |tv|^p\right)-\frac{1}{q}\int_{\Omega} H(x)\,|tv|^q\\
&= \frac{at^p}{p}[v]_{s,p}^p+\frac{bt^{p(k+1)}}{p(k+1)}\,[v]_{s,p}^{p(k+1)}+\frac{t^p}{p}\int_\Omega |v|^p-\frac{t^q}{q}\int_\Omega H(x)\,|v|^q\\
&\leq K.
\end{aligned}$$

Then, by Lemma 3.1, there exists t_1 large enough such that $I_\lambda(t_1 v) < 0$.
Define $\gamma_1(t) = tt_1 v$ for $t \in [0,1]$. Then, $\gamma_1(t) \in \Gamma$ and thus

$$c_\lambda = c_\lambda^{**} \leq \max_{t\in[0,1]} I_\lambda(\gamma_1(t)) \leq \max_{t\geq 0} I_\lambda(tv) \leq K.$$

The proof is completed. □

Lemma 3.5. *Any of the $(PS)_c$ sequence $\{u_n\}$ for I_λ is bounded and*

$$\lim_{n\to\infty}\sup\|u_n\|_{W^{s,p}} \leq \left(\frac{pqc}{(q-p)\min\{a,1\}}\right)^{\frac{1}{p}}. \tag{3.1}$$

Proof. Suppose that $\{u_n\}$ is a $(PS)_c$ sequence of I_λ, we have

$$I_\lambda(u_n) \to c, \quad I_\lambda^{'}(u_n) \to 0.$$

Then,

$$\begin{aligned}
c+o(1)+o(1)\|u_n\|_{W^{s,p}} &\geq I_\lambda(u_n)-\frac{1}{q}\langle I_\lambda^{'}(u_n),u_n\rangle\\
&= \frac{a}{p}[u_n]_{s,p}^p+\frac{b}{p(k+1)}[u_n]_{s,p}^{p(k+1)}+\frac{1}{p}\int_{\mathbb{R}^N}V_\lambda(x)|u_n|^p-\frac{1}{q}\int_{\mathbb{R}^N}H(x)|u_n|^q\\
&\quad -\frac{a}{q}[u_n]_{s,p}^p-\frac{b}{q}[u_n]_{s,p}^{p(k+1)}-\frac{1}{q}\int_{\mathbb{R}^N}V_\lambda(x)|u_n|^p+\frac{1}{q}\int_{\mathbb{R}^N}H(x)|u_n|^q\\
&\geq \left(\frac{a}{p}-\frac{a}{q}\right)[u_n]_{s,p}^p+\left(\frac{b}{p(k+1)}-\frac{b}{q}\right)[u_n]_{s,p}^{p(k+1)}+\left(\frac{1}{p}-\frac{1}{q}\right)\int_{\mathbb{R}^N}V_\lambda(x)|u_n|^p\\
&\geq \left(\frac{1}{p}-\frac{1}{q}\right)\left(a[u_n]_{s,p}^p+\int_{\mathbb{R}^N}V_\lambda(x)|u_n|^p\right).
\end{aligned}$$

It follows that

$$\min\{a,1\}\|u_n\|_{W^{s,p}}^p\leq a[u_n]_{s,p}^p+\int_{\mathbb{R}^N}V_\lambda(x)|u_n|^p\leq\frac{pqc}{(q-p)}+o(1)+o(1)\|u_n\|_{W^{s,p}}. \tag{3.2}$$

Then, $\{u_n\}$ is bounded in $W^{s,p}$ and (3.1) holds. □

Lemma 3.6. *Let $K>0$ is the number given in Lemma* 3.4. *Then for any $\epsilon>0$, there exist $\Lambda_\epsilon, R_\epsilon>0$ such that if $\{u_n\}$ is a $(PS)_c$ sequence of I_λ with $\lambda>\Lambda_\epsilon$, $c\leq K$, then*

$$\lim_{n\to\infty}\sup\int_{\mathbb{R}^N\setminus B_{R_\epsilon}(0)}|u_n|^\nu\leq\epsilon,\quad \nu\in(p,p_s^*).$$

Proof. For all $R>0$, let

$$\begin{aligned}
A(R)&:=\{x\in\mathbb{R}^N\mid |x|\geq R, g(x)\geq M_0\},\\
B(R)&:=\{x\in\mathbb{R}^N\mid |x|\geq R, g(x)\leq M_0\}.
\end{aligned}$$

When n large enough, by (3.2) we have

$$\begin{aligned}
\int_{A(R)}|u_n|^p &\leq \frac{1}{\lambda M_0+1}\int_{A(R)}(\lambda g(x)+1)|u_n|^p\\
&\leq \frac{1}{\lambda M_0+1}\left[a[u_n]_{s,p}^p+\int_{\mathbb{R}^N}(\lambda g(x)+1)|u_n|^p\right]\\
&\leq \frac{1}{\lambda M_0+1}\left[\frac{pqc}{(q-p)}+o(1)+o(1)\|u_n\|_{W^{s,p}}\right]\\
&\leq \frac{1}{\lambda M_0+1}\left[\frac{pqK}{(q-p)}+1\right].
\end{aligned}\tag{3.3}$$

On the other hand, by the Hölder inequality and the boundedness of $\{u_n\}$ in $L^{p_s^*}(\mathbb{R}^N)$, we have

$$\begin{aligned}
\int_{B(R)}|u_n|^p &\leq \left(\int_{B(R)}|u_n|^{p_s^*}\right)^{\frac{p}{p_s^*}}\left(\int_{B(R)}1\right)^{\frac{p_s^*-p}{p_s^*}}\\
&\leq C(\mathrm{meas}(B(R)))^{\frac{p_s^*-p}{p_s^*}}.
\end{aligned}\tag{3.4}$$

By using the interpolation inequality, there exists $\sigma \in (0,1)$ such that $\frac{1}{\nu} = \frac{\sigma}{p} + \frac{1-\sigma}{p_s^*}$ and

$$\int_{\mathbb{R}^N \setminus B_R} |u_n|^\nu \leq \left(\int_{\mathbb{R}^N \setminus B_R} |u_n|^p \right)^{\frac{\nu\sigma}{p}} \left(\int_{\mathbb{R}^N \setminus B_R} |u_n|^{p_s^*} \right)^{\frac{\nu(1-\sigma)}{p_s^*}} \tag{3.5}$$

for any $\nu \in (p, p_s^*)$. Let λ and R large enough. For any $\epsilon > 0$, by (3.3), (3.4), (3.5), (A_3) and the boundedness of $\{u_n\}$ in $L^{p_s^*}(\mathbb{R}^N)$, we obtain

$$\lim_{n\to\infty} \sup \int_{\mathbb{R}^N \setminus B_{R_\epsilon}} |u_n|^\nu \leq \epsilon. \qquad \square$$

Corollary 3.1. *Let $L > 0$ be a constant, and let $\lambda_n \to \infty$ as $n \to \infty$. If*

$$a\,[u_n]_{s,p}^p + \int_{\mathbb{R}^N} V_{\lambda_n}(x)\,|u_n|^p \leq L$$

and $u_n \rightharpoonup 0$ in $W^{s,p}$, then

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} |u_n|^\nu = 0, \quad \nu \in (p, p_s^*).$$

Lemma 3.7. *There exists $\delta > 0$ such that any $(PS)_c$ sequence $\{u_n\}$ of I_λ with $\lambda > 0$, $c > 0$ satisfies*

$$\liminf_{n\to\infty} \int_{\mathbb{R}^N} |u_n|^q \geq \delta c.$$

Proof. From Lemma 3.5 and (A_2), we have

$$\begin{aligned}
c =& \liminf_{n\to\infty} \left(I_\lambda(u_n) - \frac{1}{p}\langle I_\lambda'(u_n), u_n\rangle \right) \\
=& \liminf_{n\to\infty} \left[\left(\frac{b}{p(k+1)} - \frac{b}{p} \right) [u_n]_{s,p}^{p(k+1)} + \left(\frac{1}{p} - \frac{1}{q} \right) \int_{\mathbb{R}^N} H(x)\,|u_n|^q \right] \\
\leq& \liminf_{n\to\infty} \left(\frac{1}{p} - \frac{1}{q} \right) \int_{\mathbb{R}^N} H(x)\,|u_n|^q \\
\leq& \liminf_{n\to\infty} \left(\frac{(q-p)H_\infty}{pq} \right) \int_{\mathbb{R}^N} |u_n|^q .
\end{aligned}$$

Then, there exists $\delta > 0$, such that

$$\liminf_{n\to\infty} \int_{\mathbb{R}^N} |u_n|^q \geq \frac{pqc}{(q-p)H_\infty} \geq \delta c. \qquad \square$$

Lemma 3.8. *For any $(PS)_c$ sequence $\{u_n\}$ of I_λ, there exists $u \in W^{s,p}$ such that $I_\lambda'(u) = 0$. Moreover if $u \neq 0$, then*

$$[u_n]_{s,p}^p \to [u]_{s,p}^p. \tag{3.6}$$

Proof. Assume that $\{u_n\}$ is a $(PS)_c$ sequence of I_λ. By lemma 3.5, we know that $\{u_n\}$ is doubded in $W^{s,p}$. Then there exists $u \in W^{s,p}$ such that $u_n \rightharpoonup u$ in $W^{s,p}$ and $[u_n]_{s,p}^p \to \beta$. If $u \equiv 0$, the proof is complete. If $u \neq 0$, then by Fatou lemma, we have

$$[u]_{s,p}^p \leq \beta.$$

Suppose that $[u]_{s,p}^p < \beta$. Then, by $I_\lambda^{'}(u_n) \to 0$, we have

$$\begin{aligned} & a \iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+ps}} \\ & + b\beta^k \iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+ps}} \\ & + \int_{\mathbb{R}^N} V_\lambda(x)|u|^{p-1} v - \int_{\mathbb{R}^N} H(x)|u|^{q-1} v = 0, \qquad \forall\, v \in W^{s,p}(\mathbb{R}^N). \end{aligned}$$

Taking $v = u$, then $\langle I_\lambda^{'}(u), u\rangle < 0$. From the proof of Lemma 2.3, we can easy to see that $\langle I_\lambda^{'}(tu), tu\rangle > 0$ for small $t > 0$. Hence, there exists a $t_0 \in (0,1)$ such that $\langle I_\lambda^{'}(t_0u), t_0u\rangle = 0$. Moreover, $I_\lambda(t_0u) = \max_{t>0} I_\lambda(tu)$. So, we have

$$\begin{aligned} c \leq & I_\lambda(t_0u) - \frac{1}{p(k+1)}\langle I_\lambda^{'}(t_0u), t_0u\rangle \\ = & \left(\frac{a}{p} - \frac{a}{p(k+1)}\right)[t_0u]_{s,p}^p + \left(\frac{1}{p} - \frac{1}{p(k+1)}\right)\int_{\mathbb{R}^N} V_\lambda(x)|t_0u|^p \\ & + \left(\frac{1}{p(k+1)} - \frac{1}{q}\right)\int_{\mathbb{R}^N} H(x)|t_0u|^q \\ < & \left(\frac{a}{p} - \frac{a}{p(k+1)}\right)[u]_{s,p}^p + \left(\frac{1}{p} - \frac{1}{p(k+1)}\right)\int_{\mathbb{R}^N} V_\lambda(x)|u|^p \\ & + \left(\frac{1}{p(k+1)} - \frac{1}{q}\right)\int_{\mathbb{R}^N} H(x)|u|^q \\ \leq & \left(\frac{a}{p} - \frac{a}{p(k+1)}\right)\liminf_{n\to\infty}[u_n]_{s,p}^p + \left(\frac{1}{p} - \frac{1}{p(k+1)}\right)\liminf_{n\to\infty}\int_{\mathbb{R}^N} V_\lambda(x)|u_n|^p \\ & + \left(\frac{1}{p(k+1)} - \frac{1}{q}\right)\liminf_{n\to\infty}\int_{\mathbb{R}^N} H(x)|u_n|^q \\ = & \liminf_{n\to\infty}[I_\lambda(u_n) - \frac{1}{p(k+1)}\langle I_\lambda^{'}(u_n), u_n\rangle] \\ \leq & c. \end{aligned}$$

Which is a contradiction. Then, $[u_n]_{s,p}^p \to [u]_{s,p}^p$ and $I_\lambda^{'}(u) = 0$. □

Lemma 3.9. *For any $\lambda > \Lambda^*$, I_λ satisfies $(PS)_c$ condition with $c \leq K$.*

Proof. Let $\{u_n\} \in W^{s,p}$ be any $(PS)_c$ sequence of I_λ, that is

$$I_\lambda(u_n) \to c, \quad I_\lambda^{'}(u_n) \to 0.$$

By Lemmas 3.5, $\{u_n\}$ is bounded. Then, up to a subsequence, we have

$$\begin{cases} u_n \rightharpoonup u \ \text{ in } \ W^{s,p}, \\ u_n \to u \ \text{ a.e. in } \ \mathbb{R}^N, \\ u_n \to u \ \text{ in } \ L_{loc}^{\nu}(\mathbb{R}^N), \ \ \nu \in [p, p_s^*). \end{cases} \tag{3.7}$$

Define $w_n = u_n - u$. Thus,

$$\begin{cases} w_n \rightharpoonup 0 \ \text{ in } \ W^{s,p}, \\ w_n \to 0 \ \text{ a.e. in } \ \mathbb{R}^N, \\ w_n \to 0 \ \text{ in } \ L^{\nu}_{loc}(\mathbb{R}^N), \ \ \nu \in [p, p_s^*). \end{cases} \tag{3.8}$$

By the Brézis-Lieb Lemma (see [6]), Lemma 3.8 and (3.8), for any $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^N)$, we have

$$\begin{aligned} &\lim_{n\to\infty} \langle I_\lambda^{'}(w_n), \varphi\rangle \\ =&M\left([u_n]^p_{s,p} - [u]^p_{s,p}\right) \iint_{\mathbb{R}^{2N}} \frac{|w_n(x)-w_n(y)|^{p-2}(w_n(x)-w_n(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+ps}} \\ &+ \int_{\mathbb{R}^N} V_\lambda(x)\,|w_n|^{p-1}\varphi - \int_{\mathbb{R}^N} H(x)\,|w_n|^{q-1}\varphi + o(1) \\ =&0 \end{aligned}$$

and

$$\begin{aligned} &\lim_{n\to\infty} I_\lambda(w_n) \\ =&\frac{a}{p}\,[u_n-u]^p_{s,p} + \frac{b}{p(k+1)}\,[u_n-u]^{p(k+1)}_{s,p} + \frac{1}{p}\int_{\mathbb{R}^N} V_\lambda(x)\,|u_n-u|^p \\ &- \frac{1}{q}\int_{\mathbb{R}^N} H(x)\,|u_n-u|^q + o(1) \\ =&\frac{a}{p}\,[u_n]^p_{s,p} - \frac{a}{p}\,[u]^p_{s,p} + \frac{b}{p(k+1)}\,[u_n]^{p(k+1)}_{s,p} - \frac{b}{p(k+1)}\,[u]^{p(k+1)}_{s,p} + \frac{1}{p}\int_{\mathbb{R}^N} V_\lambda(x)\,|u_n|^p \\ &- \frac{1}{p}\int_{\mathbb{R}^N} V_\lambda(x)\,|u|^p - \frac{1}{q}\int_{\mathbb{R}^N} H(x)\,|u_n|^q + \frac{1}{q}\int_{\mathbb{R}^N} H(x)\,|u|^q + o(1) \\ =&\lim_{n\to\infty} I_\lambda(u_n) - I_\lambda(u). \end{aligned}$$

Then we know that $\{w_n\}$ is also a (PS) sequence for I_λ and

$$I_\lambda(w_n) \to c - I_\lambda(u) := c_1, \quad I_\lambda^{'}(w_n) \to 0.$$

From Lemma 3.5, we have $c_1 \geq 0$. Hence, if $c_1 = 0$, by Lemma 3.5, we have $w_n \to 0$ in $W^{s,p}$, then $u_n \to u$ in $W^{s,p}$, the proof is completed. Now, we assume $c_1 > 0$. Then, by Lemma 3.7, we obtain

$$\liminf_{n\to\infty} \int_{\mathbb{R}^N} |w_n|^q \geq \delta c_1.$$

Next, if we choose $\epsilon = \frac{\delta c_1}{2}$ and $\Lambda^* = \Lambda_\epsilon$ and R_ϵ as in Lemma 3.6, then we have

$$\lim_{n\to\infty}\sup \int_{\mathbb{R}^N\setminus B_{R_\epsilon}} |w_n|^q \leq \epsilon = \frac{\delta c_1}{2}.$$

This implies $w_n \to w$ in $L^q(B_{R_\epsilon})$ such that $w \neq 0$, which contradicts (3.8). Therefore, $c_1 = 0$. Then $u_n \to u$, and the proof is completed. □

Proof of Theorem 1.1. From Lemma 3.1, I_λ satisfies the conditions of the Mountain Pass lemma. We can find a $(PS)_c$ sequence $\{u_n\} \in W^{s,p}$ for the functional I_λ. By

Lemma 3.9, for λ sufficiently large, there exists $u_\lambda \in W^{s,p}$ such that $I_\lambda(u_\lambda) = c_\lambda^{**}$. Then $I_\lambda(u_\lambda) = c_\lambda$ by Lemma 3.2. Therefore, c_λ is achieved by a critical point u_λ of I_λ and u_λ is a ground state solution of problem (1.1). The proof of Theorem 1.1 is completed. □

4. PROOF OF THEOREM 1.2.

In this section, we give a asymptotical behavior of ground state solutions for problem (1.1) when $\lambda_n \to +\infty$.

Proof of Theorem 1.2. By Theorem 1.1, suppose that $\lambda_n \to +\infty$ as $n \to \infty$ and $\{u_{\lambda_n}\} \subset N_{\lambda_n}$ is a sequence such that

$$u_{\lambda_n} > 0, \quad I_{\lambda_n}(u_{\lambda_n}) = c_{\lambda_n}, \quad I'_{\lambda_n}(u_{\lambda_n}) = 0.$$

By Lemmas 3.4 and 3.5, there exists $C > 0$, such that $\|u_{\lambda_n}\|_{W^{s,p}} \le C$. Thus, there exists $u \in W^{s,p}$ such that

$$\begin{cases} u_{\lambda_n} \rightharpoonup u \text{ in } W^{s,p}, \\ u_{\lambda_n} \to u \text{ a.e. in } \mathbb{R}^N, \\ u_{\lambda_n} \to u \text{ in } L^\nu_{loc}(\mathbb{R}^N), \ \ \nu \in [p, p_s^*). \end{cases} \tag{4.1}$$

We claim that $u|_{\Omega^c} = 0$, where $\Omega^c := \{x \mid x \in \mathbb{R}^N \setminus \Omega\}$. If not, we have $u|_{\Omega^c} \neq 0$. Then, there exists a compact subset $D \subset \Omega^c$ with $\operatorname{dist}\{D, \partial\Omega\} > 0$ such that $u|_D \neq 0$ and

$$\int_D |u_{\lambda_n}|^p \to \int_D |u|^p > 0.$$

Moreover, there exists $\epsilon_0 > 0$. such that $g(x) \ge \epsilon_0$ for any $x \in D$.

By the choice of $\{u_{\lambda_n}\}$, we have

$$\begin{aligned} 0 &= \langle I'_{\lambda_n}(u_{\lambda_n}), u_{\lambda_n} \rangle \\ &= M([u_{\lambda_n}]^p_{s,p})[u_{\lambda_n}]^p_{s,p} + \int_{\mathbb{R}^N} V_{\lambda_n}(x)\,|u_{\lambda_n}|^p - \int_{\mathbb{R}^N} H(x)\,|u_{\lambda_n}|^q . \end{aligned} \tag{4.2}$$

Then,

$$\begin{aligned} c_{\lambda_n} &= I_{\lambda_n}(u_{\lambda_n}) \\ &= \frac{1}{p}\left(\mathcal{M}([u_{\lambda_n}]^p_{s,p}) + \int_{\mathbb{R}^N} V_{\lambda_n}(x)\,|u_{\lambda_n}|^p\right) - \frac{1}{q}\int_{\mathbb{R}^N} H(x)\,|u_{\lambda_n}|^q \\ &= \frac{1}{p}\left(\mathcal{M}([u_{\lambda_n}]^p_{s,p}) + \int_{\mathbb{R}^N} V_{\lambda_n}(x)\,|u_{\lambda_n}|^p\right) \\ &\quad - \frac{1}{q}\left(M([u_{\lambda_n}]^p_{s,p})[u_{\lambda_n}]^p_{s,p} + \int_{\mathbb{R}^N} V_{\lambda_n}(x)\,|u_{\lambda_n}|^p\right) \\ &= \left(\frac{a}{p} - \frac{a}{q}\right)[u_{\lambda_n}]^p_{s,p} + \left(\frac{b}{p(k+1)} - \frac{b}{q}\right)[u_{\lambda_n}]^{p(k+1)}_{s,p} + \left(\frac{1}{p} - \frac{1}{q}\right)\int_{\mathbb{R}^N} V_{\lambda_n}(x)\,|u_{\lambda_n}|^p \\ &\ge \left(\frac{1}{p} - \frac{1}{q}\right)\int_D (\lambda_n\epsilon_0 + 1)\,|u_{\lambda_n}|^p \\ &\to +\infty \end{aligned}$$

as $n \to +\infty$, which contradicts Lemma 3.4. Then $u|_{\Omega^c} = 0$ and $u \in W^{s,p}(\Omega)$.
Now, we show $u \not\equiv 0$. If not, $u_{\lambda_n} \rightharpoonup 0$ in $W^{s,p}(\mathbb{R}^N)$. By the choice of $\{u_{\lambda_n}\}$, we have (4.2). From Corollary 3.1 and (A_2), we have

$$\int_{\mathbb{R}^N} H(x)\,|u_{\lambda_n}|^q \to 0$$

as $n \to +\infty$. Then

$$M([u_{\lambda_n}]_{s,p}^p)[u_{\lambda_n}]_{s,p}^p + \int_{\mathbb{R}^N} V_{\lambda_n}(x)\,|u_{\lambda_n}|^p \to 0$$

as $n \to +\infty$. This implies $\|u_{\lambda_n}\| \to 0$, which contradicts Lemma 2.2. Therefore, $u \not\equiv 0$.
Next, we prove $\Phi_\Omega'(u) = 0$. Consider a test function $\varphi \in C_0^\infty(\Omega)$.
By the $\langle I_{\lambda_n}'(u_{\lambda_n}), \varphi\rangle = 0$, we have

$$\begin{aligned} 0 = {} & M([u_{\lambda_n}]_{s,p}^p)\iint_{\mathbb{R}^{2N}} \frac{|u_{\lambda_n}(x) - u_{\lambda_n}(y)|^{p-2}\,(u_{\lambda_n}(x) - u_{\lambda_n}(y))(\varphi(x) - \varphi(y))}{|x-y|^{N+ps}} \\ & + \int_{\mathbb{R}^N} V_{\lambda_n}(x)\,|u_{\lambda_n}|^{p-1}\,\varphi - \int_{\mathbb{R}^N} H(x)\,|u_{\lambda_n}|^{q-1}\,\varphi. \end{aligned}$$

It follows from $u_{\lambda_n} \rightharpoonup u$ in $W^{s,p}$ that

$$\begin{aligned} & M([u_{\lambda_n}]_{s,p}^p)\iint_{\mathbb{R}^{2N}} \frac{|u_{\lambda_n}(x) - u_{\lambda_n}(y)|^{p-2}\,(u_{\lambda_n}(x) - u_{\lambda_n}(y))(\varphi(x) - \varphi(y))}{|x-y|^{N+ps}} \mathrm{d}y \to \\ & \qquad M([u]_{s,p}^p)\iint_{\Omega\times\Omega} \frac{|u(x) - u(y)|^{p-2}\,(u(x) - u(y))(\varphi(x) - \varphi(y))}{|x-y|^{N+ps}}, \end{aligned}$$

$$\int_{\mathbb{R}^N} V_{\lambda_n}(x)\,|u_{\lambda_n}|^{p-1}\,\varphi \to \int_\Omega |u|^{p-1}\,\varphi,$$

and

$$\int_{\mathbb{R}^N} H(x)\,|u_{\lambda_n}|^{q-1}\,\varphi \to \int_\Omega H(x)\,|u|^{q-1}\,\varphi.$$

Thus, we obtain

$$\begin{aligned} 0 = {} & M([u]_{s,p}^p)\iint_{\Omega\times\Omega} \frac{|u(x) - u(y)|^{p-2}\,(u(x) - u(y))(\varphi(x) - \varphi(y))}{|x-y|^{N+ps}} \\ & + \int_\Omega |u|^{p-1}\,\varphi - \int_\Omega H(x)\,|u|^{q-1}\,\varphi, \qquad \forall \varphi \in C_0^\infty(\Omega), \end{aligned}$$

which means $\Phi_\Omega'(u) = 0$. By using the strong maximum principle, we have $u > 0$ in Ω.

Next, we prove $\Phi_\Omega(u) = c_\Omega$. From the above discussion, we know $u \in N_\Omega$. Then, from Lemma 3.3, we have

$$\begin{aligned}
c_\Omega \leq{}& \Phi_\Omega(u) \\
={}&\Phi_\Omega(u) - \frac{1}{p(k+1)}\langle \Phi_\Omega^{'}(u), u\rangle \\
={}&\left(\frac{a}{p} - \frac{a}{p(k+1)}\right)[u]_{s,p}^p + \left(\frac{1}{p} - \frac{1}{p(k+1)}\right)\int_\Omega |u|^p + \left(\frac{1}{p(k+1)} - \frac{1}{q}\right)\int_\Omega H(x)\,|u|^q \\
\leq{}& \left(\frac{a}{p} - \frac{a}{p(k+1)}\right)\liminf_{n\to\infty}[u_{\lambda_n}]_{s,p}^p + \left(\frac{1}{p} - \frac{1}{p(k+1)}\right)\liminf_{n\to\infty}\int_\Omega |u_{\lambda_n}|^p \\
&+ \left(\frac{1}{p(k+1)} - \frac{1}{q}\right)\liminf_{n\to\infty}\int_\Omega H(x)\,|u_{\lambda_n}|^q \\
\leq{}& \left(\frac{a}{p} - \frac{a}{p(k+1)}\right)\liminf_{n\to\infty}[u_{\lambda_n}]_{s,p}^p + \left(\frac{1}{p} - \frac{1}{p(k+1)}\right)\liminf_{n\to\infty}\int_{\mathbb{R}^N} V_{\lambda_n}(x)\,|u_{\lambda_n}|^p \\
&+ \left(\frac{1}{p(k+1)} - \frac{1}{q}\right)\liminf_{n\to\infty}\int_{\mathbb{R}^N} H(x)\,|u_{\lambda_n}|^q \\
={}&\liminf_{n\to\infty}\left[I_{\lambda_n}(u_{\lambda_n}) - \frac{1}{p(k+1)}\langle I_{\lambda_n}^{'}(u_{\lambda_n}), u_{\lambda_n}\rangle\right] \\
={}&\liminf_{n\to\infty} c_{\lambda_n} \\
\leq{}& c_\Omega,
\end{aligned}$$

which implies that

$$\lim_{n\to\infty} I_{\lambda_n}(u_{\lambda_n}) = \lim_{n\to\infty} c_{\lambda_n} = c_\Omega = \Phi_\Omega(u). \tag{4.3}$$

Finally, we prove $u_{\lambda_n} \to u$ in $W^{s,p}(\mathbb{R}^N)$. Let us now recall the following equations:

$$\begin{aligned}
I_{\lambda_n}(u_{\lambda_n} - u) ={}&\frac{a}{p}\,[u_{\lambda_n} - u]_{s,p}^p + \frac{b}{p(k+1)}\,[u_{\lambda_n} - u]_{s,p}^{p(k+1)} \\
&+ \frac{1}{p}\int_{\mathbb{R}^N} V_{\lambda_n}(x)\,|u_{\lambda_n} - u|^p - \frac{1}{q}\int_{\mathbb{R}^N} H(x)\,|u_{\lambda_n} - u|^q\,,
\end{aligned} \tag{4.4}$$

$$\begin{aligned}
I_{\lambda_n}(u_{\lambda_n}) ={}&\frac{a}{p}\,[u_{\lambda_n}]_{s,p}^p + \frac{b}{p(k+1)}\,[u_{\lambda_n}]_{s,p}^{p(k+1)} + \frac{1}{p}\int_{\mathbb{R}^N} V_{\lambda_n}(x)\,|u_{\lambda_n}|^p \\
&- \frac{1}{q}\int_{\mathbb{R}^N} H(x)\,|u_{\lambda_n}|^q\,,
\end{aligned} \tag{4.5}$$

$$\Phi_\Omega(u) = \frac{a}{p}[u]_{s,p}^p + \frac{b}{p(k+1)}[u]_{s,p}^{p(k+1)} + \frac{1}{p}\int_\Omega |u|^p - \frac{1}{q}\int_\Omega H(x)\,|u|^q\,, \tag{4.6}$$

$$\begin{aligned}
\langle I_{\lambda_n}^{'}(u_{\lambda_n} - u), u_{\lambda_n} - u\rangle ={}&a\,[u_{\lambda_n} - u]_{s,p}^p + b\,[u_{\lambda_n} - u]_{s,p}^{p(k+1)} \\
&+ \int_{\mathbb{R}^N} V_{\lambda_n}(x)\,|u_{\lambda_n} - u|^p - \int_{\mathbb{R}^N} H(x)\,|u_{\lambda_n} - u|^q\,,
\end{aligned} \tag{4.7}$$

$$
\begin{aligned}
\langle I_{\lambda_n}^{'}(u_{\lambda_n}), u_{\lambda_n}\rangle =& a\,[u_{\lambda_n}]_{s,p}^{p} + b\,[u_{\lambda_n}]_{s,p}^{p(k+1)} + \int_{\mathbb{R}^N} V_{\lambda_n}(x)\,|u_{\lambda_n}|^p \\
&- \int_{\mathbb{R}^N} H(x)\,|u_{\lambda_n}|^q\,,
\end{aligned} \tag{4.8}
$$

and

$$
\langle \Phi_{\Omega}^{'}(u), u\rangle = a\,[u]_{s,p}^{p} + b\,[u]_{s,p}^{p(k+1)} + \int_{\Omega} |u|^p - \int_{\Omega} H(x)\,|u|^q\,. \tag{4.9}
$$

By the Brézis-Lieb Lemma (see [6])and Lemma 3.8, we obtain

$$
[u_{\lambda_n} - u]_{s,p}^{p} = [u_{\lambda_n}]_{s,p}^{p} - [u]_{s,p}^{p} + o(1), \tag{4.10}
$$

$$
\begin{aligned}
\int_{\mathbb{R}^N} V_{\lambda_n}(x)\,|u_{\lambda_n} - u|^p &= \int_{\mathbb{R}^N\setminus\Omega} V_{\lambda_n}(x)\,|u_{\lambda_n} - u|^p + \int_{\Omega} V_{\lambda_n}(x)\,|u_{\lambda_n} - u|^p \\
&= \int_{\mathbb{R}^N\setminus\Omega} V_{\lambda_n}(x)\,|u_{\lambda_n}|^p + \int_{\Omega} |u_{\lambda_n} - u|^p + o(1) \\
&= \int_{\mathbb{R}^N\setminus\Omega} V_{\lambda_n}(x)\,|u_{\lambda_n}|^p + \int_{\Omega} |u_{\lambda_n}|^p - \int_{\Omega} |u|^p + o(1) \\
&= \int_{\mathbb{R}^N} V_{\lambda_n}(x)\,|u_{\lambda_n}|^p - \int_{\Omega} |u|^p + o(1),
\end{aligned} \tag{4.11}
$$

and

$$
\int_{\mathbb{R}^N} H(x)\,|u_{\lambda_n} - u|^q = \int_{\mathbb{R}^N} H(x)\,|u_{\lambda_n}|^q - \int_{\Omega} H(x)\,|u|^q + o(1). \tag{4.12}
$$

Hence, according to (4.4)-(4.12), we have

$$
I_{\lambda_n}(u_{\lambda_n} - u) = I_{\lambda_n}(u_{\lambda_n}) - \Phi_{\Omega}(u) + o(1),
$$

$$
\langle I_{\lambda_n}^{'}(u_{\lambda_n} - u), u_{\lambda_n} - u\rangle = \langle I_{\lambda_n}^{'}(u_{\lambda_n}), u_{\lambda_n}\rangle - \langle \Phi_{\Omega}^{'}(u), u\rangle + o(1).
$$

Thus, by (4.3), we have

$$
I_{\lambda_n}(u_{\lambda_n} - u) = o(1).
$$

From $I_{\lambda_n}^{'}(u_{\lambda_n}) = 0$ and $\Phi_{\Omega}^{'}(u) = 0$, we obtain

$$
\langle I_{\lambda_n}^{'}(u_{\lambda_n} - u), u_{\lambda_n} - u\rangle = o(1).
$$

Then,

$$\begin{aligned}
o(1) &= I_{\lambda_n}(u_{\lambda_n}-u) - \frac{1}{q}\langle I'_{\lambda_n}(u_{\lambda_n}-u), u_{\lambda_n}-u\rangle \\
&= \frac{a}{p}\left[u_{\lambda_n}-u\right]_{s,p}^{p} + \frac{b}{p(k+1)}\left[u_{\lambda_n}-u\right]_{s,p}^{p(k+1)} + \frac{1}{p}\int_{\mathbb{R}^N} V_{\lambda_n}(x)\left|u_{\lambda_n}-u\right|^p \\
&\quad - \frac{1}{q}\int_{\mathbb{R}^N} H(x)\left|u_{\lambda_n}-u\right|^q - \frac{a}{q}\left[u_{\lambda_n}-u\right]_{s,p}^{p} - \frac{b}{q}\left[u_{\lambda_n}-u\right]_{s,p}^{p(k+1)} \\
&\quad - \frac{1}{q}\int_{\mathbb{R}^N} V_{\lambda_n}(x)\left|u_{\lambda_n}-u\right|^p + \frac{1}{q}\int_{\mathbb{R}^N} H(x)\left|u_{\lambda_n}-u\right|^q \\
&= \left(\frac{a}{p}-\frac{a}{q}\right)\left[u_{\lambda_n}-u\right]_{s,p}^{p} + \left(\frac{b}{p(k+1)}-\frac{b}{q}\right)\left[u_{\lambda_n}-u\right]_{s,p}^{p(k+1)} \\
&\quad + \left(\frac{1}{p}-\frac{1}{q}\right)\int_{\mathbb{R}^N} V_{\lambda_n}(x)\left|u_{\lambda_n}-u\right|^p \\
&\geq \left(\frac{1}{p}-\frac{1}{q}\right)\left(a\left[u_{\lambda_n}-u\right]_{s,p}^{p} + \int_{\mathbb{R}^N} V_{\lambda_n}(x)\left|u_{\lambda_n}-u\right|^p\right) \\
&\geq \left(\frac{1}{p}-\frac{1}{q}\right)\min\{a,1\}\left\|u_{\lambda_n}-u\right\|_{W^{s,p}}^{p},
\end{aligned}$$

which implies that $\|u_{\lambda_n}-u\|_{W^{s,p}}^{p} \to 0$ as $n\to\infty$. Therefore, the proof of Theorem 1.2 is completed. □

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