Fixed Point Theory, 25(2024), No. 1, 379-398
DOI: 10.24193/fpt-ro.2024.1.24
http://www.math.ubbcluj.ro/~nodeacj/sfptcj.html

SEARCH OF MINIMAL METRIC STRUCTURE IN THE CONTEXT OF FIXED POINT THEOREM AND CORRESPONDING OPERATOR EQUATION PROBLEMS

JAYESH SAVALIYA*, DHANANJAY GOPAL**, SHAILESH KUMAR SRIVASTAVA*
AND VLADIMIR RAKOČEVIČ***

*Department of Mathematics,
Sardar Vallabhbhai National Institute of Technology, Surat, India
E-mail: jdsavaliya555@gmail.com, shaileshiitr2010@gmail.com

**Department of Mathematics, Guru Ghasidas Vishwavidyalaya,
Bilaspur, Chhattisgarh, India
E-mail: gopaldhananjay@yahoo.in

***Faculty of Sciences and Mathematics, University of Niš,
Višegradska, Serbia
E-mail: vrakoc@sbb.rs

Abstract. The paper contains a brief summary of the generalization of metrical structure regarding the fixed point theorem and corresponding operator equation problems. We observed that many researcher either tried to weaken the metrical structure, the contraction condition, or both. The idea behind this paper is to look for a minimal metrical structure to establish fixed point theorems. In this connection, we present new variants of the known fixed point theorem under non-triangular metric space (namely F-contraction, $(\mathcal{A}, \mathcal{S})$-contraction, (ψ, ϕ)-contraction). We also apply the obtain result in solving various types of operator equation problems. e.g., high-order fractional differential equation with non-local boundary conditions and non-linear integral equation problems.
Key Words and Phrases: Non-triangular metric, F-contraction, $(\mathcal{A}, \mathcal{S})$-contraction, (ψ, ϕ)-contraction.
2020 Mathematics Subject Classification: 47H10, 54H25.

1. Introduction

Definition 1.1. Let X be a non-empty set and a function $d : X \times X \to \mathbb{R}_0^+$. If d satisfies the following conditions:

(i) $d(x, x) = 0$, for all $x \in X$;
(ii) $d(x, y) = d(y, x)$, for all $x, y \in X$;
(iii) For each $x, y \in X$ and $\{x_n\} \subset X$ such that $\lim_{n\to\infty} d(x_n, x) = 0$, and $\lim_{n\to\infty} d(x_n, y) = 0$, then $x = y$.

Then, d is a non-triangular metric and (X, d) is known as non-triangular metric space.
Theorem 1.2. [2] *Let (X, d) be a JS-metric space, such that for each $x \in X$, the set $C(d, X, x)$, is non-empty. Then, (X, d) is a non-triangular metric space, where $C(d, X, x) = \{(x_n) | \lim_{n\to\infty} d(x_n\,, x) = 0\}$.*
Example 1.3. Let $X = \mathbb{R}$ and $d : X \times X \to [0\,, \infty)$ defined as follows:

$$d(x, y) = \begin{cases} |x - y|, & \text{if } x = 0 \,\text{or}\, y = 0 \,\text{or}\, x = y. \\ 1, & \text{otherwise.} \end{cases}$$

Then, the pair (X, d) is a non-triangular metric space but it is not a JS-metric space.
Example 1.4. Let $X = [0, \infty)$ and $d : X \times X \to \mathbb{R}_0^+$ is defined as

$$d(x, y) = \begin{cases} \dfrac{x+y}{x+y+1}, & \text{if } x \neq y; \\ 0, & \text{if } x = y \neq 0; \\ \dfrac{x}{2}, & \text{if } y = 0; \\ \dfrac{y}{2}, & \text{if } x = 0. \end{cases}$$

Then d is a non-triangular metric, but not a JS-metric.
Definition 1.5. (**Property C**) [2] Let (X, d) be a non-triangular metric space then d is said to satisfy property (C) if for any sequence (x_n) with $d(x_n, x) \to 0$, have $d(x_n, y) \to d(x, y)$ for every $y \in X$.
Definition 1.6. Let (X, d) be a complete metric space, then the mapping $f : X \to X$ is called **orbitally S-operator** if the following conditions hold:

- (S_1) The Picard sequence $\{x_n\}$ based on x_0, is asymptotically regular for some $x_0 \in X$,
- (S_2) For any two sub-sequences $x_{n(k)}$ and $x_{m(k)}$ of $\{x_n\}$ if $d(x_{n(k)}, x_{m(k)})$ converges to some limit $L \geq 0$ and $d(x_{n(k)}, x_{m(k)}) > L$ for all $k \in \mathbb{N}_0$, then $L = 0$, in which, $\{x_n\}$ is the Picard sequence of f based on $x_0 \in X$,
- (S_3) f is orbitally continuous.

Theorem 1.7. [5] *Let (X, d) be a $d-$complete non-triangular metric space, and let $f : X \to X$ be a map such that, f satisfies the (S_2) and (S_3) of Definition 1.6. If there exists $x_0 \in X$ such that $O(f, x_0)$ is bounded, then f has fixed point.*
Example 1.8. Let (X, d) be a non-triangular metric space as given in Example 1.4 and $f : X \to X$ is given as $fx = \dfrac{x}{2}$. Then the map f satisfies the condition of Theorem 1.7, and hence it has a fixed point at $x = 0$.

In, 2021 V. Rakočević et al. [8] noticed that non-triangular metric space is not a new concept, as it is nothing but symmetric space with the condition W3 (Hausdorffness of the topology induced by the symmetric structure). They introduced the non-triangular metric-like space which they claimed to be a generalization of non-triangular metric space.
Definition 1.8. (non-triangular metric-like space) [8] Let X be a non-empty set and $d : X \times X \to [0, \infty)$ be a mapping. Then (X, d) is said to be a non-triangular metric-like space if for every $x, y, z \in X$, the following conditions are satisfied:

(NTML-1) $d(x,y)=0 \implies x=y$;
(NTML-2) $d(x,y)=d(y,x)$;
(NTML-3) if $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} d(x_n,x) = d(x,x)$ and $\lim_{n\to\infty} d(x_n,y)=d(y,y)$, then $d(x,y)\le k\max\{d(x,x),d(y,y)\}$ for some fixed $k\ge 1$.

Remark 1.9. If $d(x,x)=0$ for all $x\in X$, then a non-triangular metric-like space is a non-triangular metric space. Notice, also that a function $d: X\times X\to \mathbb{R}_0^+$ satisfying (i) and (ii) in Definition 1.1 and the axiom (NTML-1) from above is called a semi-metric.

Example 1.10. [8] Let $X=\{0\}\cup\{\frac{1}{n}: n\ge 1\}$ and $d: X\times X\to\mathbb{R}_0^+$ be defined as

$$d(0,0)=1, d\left(\frac{1}{n},\frac{1}{n}\right)=3,\ d\left(0,\frac{1}{n}\right)=1+\frac{1}{n}=d\left(\frac{1}{n},0\right)\ \text{ for all }\ n\in\mathbb{N}$$

and

$$d\left(\frac{1}{n},\frac{1}{m}\right)=1-\frac{1}{n+m}=d\left(\frac{1}{m},\frac{1}{n}\right)\ \text{for all } n,m\ge 1.$$

Then (X,d) is a non-triangular metric-like space, for any $k\ge 1$.

Definition 1.11. [8] In a non-triangular metric-like space (X,d), the map $f: X\to X$ is said to have **property** S if for any Picard iterating sequence $\{f^n x_0\}, x_0\in X$, converging to $x\in X$ we have $d(x,fx)\le \limsup_{n\to\infty} d(f^n x_0, fx)$.

Theorem 1.12. [8] *Let (X,d) be a d-complete non-triangular metric-like space for some $k\ge 1$ and $f: X\to X$ be a mapping satisfying*

$$d(fx,fy)\le q\max\{d(x,y),d(x,fx),d(y,fy),d(x,fy),d(y,fx)\}$$

for all $x,y\in X$ and for some $q\in(0,1)$. If f has property S and for some $x_0\in X$ the orbit of f with respect to x_0 i.e. $O(f,x_0)$ is bounded then f has unique fixed point.

Remark 1.13. From Remark 3.10 of [8], in a non-triangular metric-like space (X,d), if a sequence $\{x_n\}$ is such that $\lim_{n\to\infty} d(x_n,x)=0=d(x,x)$ and $\lim_{n\to\infty} d(x_n,y)=0=d(y,y)$ for some $x,y\in X$, then $d(x,y)=0$,. That is $x=y$, i.e., every convergent sequence $\{x_n\}$ has a unique limit.

Now, from the proof of Theorem 1.12 in [8], the relation (3.2) implies $\lim_{n\to\infty} d(x_n,x)=d(x,x)=0$ and here the sequence $\{x_n\}$ is Picard sequence which has unique limit. Thus, every convergent sequence in non-triangular metric-like space has a unique limit, so by definition, it is a non-triangular metric space. Hence, in the context of fixed point theorems, the non-triangular metric-like space is nothing new. it but fall in the class of non-triangular metric space.

In [9], the authors defined the pseudo non-triangular metric in order to to generalize the non-triangular metric. However, in the context of a fixed point theorem, pseudo non-triangular coincide with non-triangular metric, and hence non-triangular metric space is the required minimal metrical structure for establishment of fixed point theorem.

2. Main results

In this section we re-establish some known fixed point theorems in the setting of non-triangular metric space, using F-contraction, $(\mathcal{A}, \mathcal{S})$- contraction and (ψ, ϕ)-contraction. The idea of the following lemma is well known and it was given by Boyd and Wong [1]. We modified the proof of this lemma, by dropping the use of the triangular inequality.

Lemma 2.1. *Let (X, d) be a non-triangular metric space and (x_n) be a sequence in X which is not Cauchy and $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$. Then there exist $\epsilon > 0$ and two sub-sequences (x_{n_k}) and (x_{m_k}) of (x_n) such that*

$$\lim_{n\to\infty} d(x_{n_k-1}, x_{m_k-1}) = \lim_{n\to\infty} d(x_{n_k}, x_{m_k}) = \epsilon. \tag{2.1}$$

Proof. In the non-triangular metric space, let the sequence (x_n) which is not Cauchy, then there exist $\epsilon > 0$ and two sub-sequences $\{x_{n_k}\}$ and $\{x_{m_k}\}$ of sequence $\{x_n\}$ such that

$$k < n_k < m_k \implies d(x_{n_k}, x_{m_k-1}) \leq \epsilon < d(x_{n_k}, x_{m_k}) \text{ for each } k \in \mathbb{N}. \tag{2.2}$$

since

$$n_k - 1 < m_k, d(x_{n_k-1}, x_{m_k-1}) \leq \epsilon, \text{ for every } k \in \mathbb{N};$$

$$\implies \lim_{k\to\infty} d(x_{n_k-1}, x_{m_k-1}) \leq \epsilon. \tag{2.3}$$

Also $n_k - 1 < m_k - 1, \implies \epsilon < d(x_{n_k-1}, x_{m_k-1})$, for every $k \in \mathbb{N}$

$$\implies \epsilon \leq \lim_{n\to\infty} d(x_{n_k-1}, x_{m_k-1}) \tag{2.4}$$

from (2.3) and (2.4)

$$\lim_{k\to\infty} d(x_{n_k-1}, x_{m_k-1}) = \epsilon. \tag{2.5}$$

Now from (2.2), $\epsilon < d(x_{n_k}, x_{m_k})$ for each $k \in \mathbb{N}$

$$\epsilon \leq \lim_{k\to\infty} d(x_{n_k}, x_{m_k}) \text{ for each } k \in \mathbb{N}. \tag{2.6}$$

Since $n_k < m_k + 1 \implies d(x_{n_k}, x_{m_k+1-1}) \leq \epsilon$ for every $k \in \mathbb{N}$

$$\lim_{k\to\infty} d(x_{n_k}, x_{m_k}) \leq \epsilon. \tag{2.7}$$

From (2.6) and (2.7)

$$\lim_{k\to\infty} d(x_{n_k}, x_{m_k}) = \epsilon. \tag{2.8}$$

Now finally from the equation (2.5) and (2.8) we can say that

$$\lim_{k\to\infty} d(x_{n_k-1}, x_{m_k-1}) = \lim_{k\to\infty} d(x_{n_k}, x_{m_k}) = \epsilon.$$

2.1. **F-contraction.** In 2012, Wardowski [12] introduced a new concept for contraction mappings called F-contraction by considering a class of real valued functions.

Definition 2.1.1. [12] Let $\mathcal{F}$ be the set of all functions $F : (0, \infty) \to \mathbb{R}$ satisfying the following conditions:

(F1) F is strictly increasing, i.e., for all $\alpha, \beta \in (0, \infty)$ such that $\alpha < \beta, F(\alpha) < F(\beta)$,

(F2) For each sequence $\{x_n\}$ of positive numbers

$$\lim_{n\to\infty} x_n = 0 \iff \lim_{n\to\infty} F(x_n) = -\infty,$$

(F3) There exists $k \in (0, 1)$ such that $\lim\limits_{\alpha\to 0^+} \alpha^k F(\alpha) = 0$.

Then a self mapping T of a metric space (X, d) is said to be F-contraction if there exist $F \in \mathcal{F}$ and $\tau > 0$ such that

$$\forall x, y \in X, d(Tx, Ty) > 0 \Rightarrow \tau + F(d(Tx, Ty)) \leq F(d(x, y)). \tag{2.9}$$

By condition $(F1)$ every F-contraction is a contractive mapping and hence it is continuous. From the Banach and Edelstein fixed point theorems, we know that every Banach contraction mapping on a complete metric space has a unique fixed point and every contractive mapping on a compact metric space has a unique fixed point. That is, passing from Banach to Edelstein fixed point theorem, when the class of mapping is expending by contractive condition, the structure of the space is restricted. Now, the following question arises naturally.

Is there any change of structure of the space while investigating the existence of the fixed points of F-contractions? Therefore, Wardowski proved the following result without restricting the structure of the space:

Theorem 2.1.2. [12] *Let (X, d) be a complete metric space and let $T : X \to X$ be F-contraction. Then T has a unique fixed point in X.*

Property 2.1.3. *Let (X, d) be a non-triangular metric space and $\{x_n\}$ be any given sequence such that $\lim\limits_{n\to\infty} d(x_n, x_{n+1}) = 0$. Then, for any two subsequence $\{x_{n_k}\}$ and $\{x_{m_k}\}$ of $\{x_n\}$, if $d(x_{n_k}, x_{m_k})$ converges to some limit $L \geq 0$ and $d(x_{n_k}, x_{m_k}) > L$ for all $k \in \mathbb{N}$, then $L = 0$.*

Property 2.1.4. *A non-triangular metric d is said to be continuous if $\lim\limits_{n\to\infty} d(x_n, x) = 0$ and $\lim\limits_{n\to\infty} d(y_n, y) = 0$ imply $\lim\limits_{n\to\infty} d(x_n, y_n) = d(x, y)$ where x_n, y_n are sequences in X and $x, y \in X$.*

In the following theorem we assume that the non-triangular metric d satisfies the Property 2.1.3 and Property 2.1.4.

Theorem 2.1.5. *Let (X, d) be a complete non-triangular metric space and $T : X \to X$ be an F-contraction. Then, T has a unique fixed point in X.*

Proof. First, let us observe that T has at most one fixed point. Indeed, if $x^\star \neq x^{\star\star} \in X$, with $Tx^\star = x^\star$ and $Tx^{\star\star} = x^{\star\star}$ then we get

$$\tau \leq F(d(x^\star, x^{\star\star})) - F(d(Tx^\star, Tx^{\star\star})) = 0,$$

which is contradiction as $\tau > 0$. In order to show that T has a fixed point let $x_0 \in X$ be arbitrary and fixed. We define a sequence $\{x_n\}$ in X as $x_{n+1} = Tx_n$ for every $n \in \mathbb{N}$. Denote $\gamma_n = d(x_{n+1}, x_n)$ for each natural number n. If there exists $n_0 \in \mathbb{N}$ for which $x_{n_0+1} = x_{n_0}$, then $Tx_{n_0} = x_{n_0}$ and the proof is finished.

Now suppose that $x_{n+1} \neq x_n$ for each $n \in \mathbb{N}$. Then $\gamma_n > 0$ for all $n \in \mathbb{N}$. Using (2.9), we get

$$F(\gamma_n) \leq F(\gamma_{n-1}) - \tau \leq F(\gamma_{n-2}) - 2\tau \leq ... \leq F(\gamma_0) - n\tau \tag{2.10}$$

From (2.10), we obtain $\lim_{n\to\infty} F(\gamma_n) = -\infty$ that together with $(F2)$ gives

$$\lim_{n\to\infty} \gamma_n = \lim_{n\to\infty} d(x_{n+1}, x_n) = 0. \tag{2.11}$$

Now, we are going to prove that sequence $\{x_n\}$ is Cauchy. By contradiction, we assume that $\{x_n\}$ is not a Cauchy sequence. Then by Lemma 2.1 for $\epsilon > 0$ and subsequence $\{x_{n_k}\}$ and $\{x_{m_k}\}$ of $\{x_n\}$ we get $\lim_{k\to\infty} d(x_{n_k}, x_{m_k}) = \epsilon$. Which contradicts that $\{x_n\}$ satisfies Property 2.1.3. Hence $\epsilon = 0$ and which implies sequence $\{x_n\}$ is a Cauchy sequence. Since X is complete there exist some $x^\star \in X$ such that $x_n \to x^\star$ as $n \to \infty$. Also, by the continuity of T, we have that $Tx_n \to Tx^\star$. Hence

$$0 = \lim_{n\to\infty} d(x_n, x_{n+1}) = \lim_{n\to\infty} d(x_n, Tx_n) = d(x^\star, Tx^\star),$$

which implies $Tx^\star = x^\star$. This completes the proof.

Example 2.1.6. Let $X = \{0, 1, 2\}$ and define the mapping $d : X \times X \to \mathbb{R}_0^+$ as

$$d(0,0) = d(1,1) = d(2,2) = 0$$

$$d(0,1) = 4, d(0,2) = 2, d(1,2) = 1,$$

Now define the mapping $T : X \to X$ define as $T(1) = 1, T(2) = 1$ and $T(0) = 2$ and $F(x) = \log(x)$, then T is a F- contraction with respect to map F and $\tau = \frac{1}{2}\log 2$, As

$$\frac{1}{2}\log(2) + \log(d(T(0), T(1))) = \frac{1}{2}\log(2) + \log(1) \leq \log(d(0,1)) = \log(4).$$

$$\frac{1}{2}\log(2) + \log(d(T(0), T(2))) = \frac{1}{2}\log(2) + \log(1) \leq \log(d(0,2)) = \log(2).$$

It's clear that the mapping d is a complete non-triangular metric on the set X, but not usual metric as $d(0,1) > d(0,2) + d(1,2)$.

Also the non-triangular metric d satisfies Property 2.1.3 and Property 2.1.4, so, by Theorem 2.1.5, the mapping T has a unique fixed point $T(1) = 1$.

Remark 2.1.7. By observing the Example 2.1.6 and Theorem 2.1.5 we shall conclude that Wardowski's theorem is improved significantly with respect to non-triangular metric.

2.2. $(\mathcal{A}, \mathcal{S})$ **contraction.** Throughout this section, we shall use $\mathcal{S}$ to denote a binary relation on X, which is nothing but a non-empty subset of the Cartesian product $X \times X$. The notation we shall use is that whenever $(x, y) \in \mathcal{S}$, we shall write $x\mathcal{S}y$. If, in addition to $x\mathcal{S}y$, $x \neq y$, we will denote it as $x\mathcal{S}^*y$. Two points are $\mathcal{S}$-comparable if $x\mathcal{S}y$ or $y\mathcal{S}x$. A binary relation $\mathcal{S}$ on X is reflexive if $x\mathcal{S}x$ for all $x \in X$. It is said to be transitive if whenever $x\mathcal{S}y$ and $y\mathcal{S}z$ for some $x, y, z \in X$, $x\mathcal{S}z$ holds. It is said to be antisymmetric if whenever $x\mathcal{S}y$ and $y\mathcal{S}x$ for any $x, y \in X$, $x = y$ holds. A preorder is a reflexive, transitive binary relation. If a binary relation is reflexive, transitive and antisymmetric, it can be termed as a partial order. ! Thus, a partial order is simply an antisymmetric preorder. $\mathcal{S}_X$ shall denote the trivial preorder given by $x\mathcal{S}_Xy$ for all $x, y \in X$.

Definition 2.2.1. ($\mathcal{S}$-nondecreasing sequence) [10] A sequence $\{x_n\} \subseteq X$ is $\mathcal{S}$-non-decreasing if $x_n \mathcal{S} x_{n+1}$ for all $n \in \mathbb{N}$.

Definition 2.2.2. ($\mathcal{S}$-increasing sequence [10]) A sequence $\{x_n\} \subseteq X$ is $\mathcal{S}$-increasing if $x_n \mathcal{S}^* x_{n+1}$ for all $n \in \mathbb{N}$.

Definition 2.2.3. ($(T, \mathcal{S})$-sequence [10]) Let $\{a_n\}$ and $\{b_n\}$ be two sequences of real numbers. We say that $\{(a_n, b_n)\}$ is a $(T, \mathcal{S})$-sequence if there exist two sequences $\{x_n\}$, $\{y_n\} \subseteq X$ such that

$$x_n \mathcal{S} y_n, a_n = d(Tx_n, Ty_n) > 0 \text{ and } b_n = d(x_n, y_n) > 0 \text{ for all } n \in \mathbb{N}.$$

If $\mathcal{S}$ is the trivial binary relation $\mathcal{S}_X$, then $\{(a_n, b_n)\}$ is called a T-sequence.

Definition 2.2.4. ($(\mathcal{A}, \mathcal{S})$-contraction [10]) Let (X, d) be a non-triangular metric space. Let $T : X \to X$ be a self-mapping into X and $\mathcal{S}$ be a binary relation on X. We will say that $T : X \to X$ is an $(\mathcal{A}, \mathcal{S})$-contraction if there exists a function $\varrho : A \times A \to \mathbb{R}$ such that T and ϱ satisfy the following conditions:

($\mathcal{A}_1$) $\operatorname{ran}(d) \subseteq A$.

($\mathcal{A}_2$) If $\{x_n\}$ is a Picard $\mathcal{S}$-non-decreasing sequence of T such that

$$x_n \neq x_{n+1}, \text{ and } \varrho(d(x_{n+1}, x_{n+2}), d(x_n, x_{n+1})) > 0 \text{ for all } n \in \mathbb{N},$$

then $\{x_n\}$ is asymptotically regular on (X, d), that is, $\{d(x_n, x_{n+1})\} \to 0$.

($\mathcal{A}_3$) If $\{(a_n, b_n)\} \subseteq A \times A$ is a $(T, \mathcal{S})$-sequence such that $\{a_n\}$ and $\{b_n\}$ converge to the same limit $L \geq 0$ and verifying that $L < a_n$ and $\varrho(a_n, b_n) > 0$ for all $n \in \mathbb{N}$, then $L = 0$.

($\mathcal{A}_4$) $\varrho(d(Tx, Ty), d(x, y)) > 0$ for all $x, y \in X$ such that $x\mathcal{S}^*y$ and $Tx\mathcal{S}^*Ty$.

The family of all $(\mathcal{A}, \mathcal{S})$-contractions from (X, d) into itself with respect to ϱ can be denoted by $\mathcal{A}_{X,d,\mathcal{S},\varrho,A}$, or, where no confusion is possible, by $\mathcal{A}_\varrho$. If $\mathcal{S}$ is the trivial binary relation $\mathcal{S}_x$, then T is called an $\mathcal{A}$-contraction.

In some cases, the following properties shall also be important to us.

($\mathcal{A}_2'$) If $x_1, x_2 \in X$ are two points such that $T^n x_1 \mathcal{S}^* T^n x_2$ and $\rho(d(T^{n+1}x_1, T^{n+1}x_2), d(T^n x_1, T^n x_2)) > 0$ for all $n \in \mathbb{N}$, then $\{d(T^n x_1, T^n x_2)\} \to 0$.

($\mathcal{A}_5$) If $\{(a_n, b_n)\}$ is a $(T, \mathcal{S})$-sequence such that $b_n \to 0$ and $\rho(a_n, b_n) > 0$ for all $n \in \mathbb{N}$, then $a_n \to 0$.

Remark 2.2.5. Condition $\mathcal{A}_1$ implies that A is a non-empty set.

Remark 2.2.6. Asymptotic regularity of a contractive mapping is a useful property in the study of fixed point theory. It is clear from Definition 2.2.4 that if T is an $(\mathcal{A}, \mathcal{S})$-contraction with respect to some function $\varrho : A \times A \to \mathbb{R}$ such that its Picard sequence is $\mathcal{S}$-non-decreasing, with $x_n \neq x_{n+1}$ and $\varrho(d(x_{n+1}, x_{n+2}), d(x_n, x_{n+1})) > 0$ for all $n \in \mathbb{N}$, then T is asymptotically regular at x_0, if x_0 is the initial point of the Picard sequence.

Example 2.2.7. We consider $X = \{0, 1, 2, 3\}$, endowed with the non-triangular metric $d : X \times X \to [0, \infty)$ defined as

$$d(x, y) = \begin{cases} \frac{3}{n}, & \frac{3}{n+1} < |x - y| \leq \frac{3}{n}. \\ 0, & x = y. \end{cases}$$

Also consider $\mathcal{S}$ to be the trivial binary relation $\mathcal{S}_X$ on X. So, $x\mathcal{S}y$ for all $x, y \in X$. Define $T : X \to X$ by

$$Tx = \begin{cases} 0, & x = \{0, 1, 2\}. \\ 1, & x = 3. \end{cases}$$

Before moving on to the main section of the proof, let us quickly verify that the considered (X, d) is a non-triangular metric space. (NT1) and (NT2) are obvious. Thus, we examine (NT3).

Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in $X = \{0, 1, 2, 3\}$ such that $\lim_{n\to\infty} d(x_n, x) = \lim_{n\to\infty} d(x_n, y) = 0$. We want to show that $x = y$. If $\{x_n\}_{n\in\mathbb{N}}$ is eventually constant, $\lim_{n\to\infty} d(x_n, x) = 0$ implies that $x_n = x$ for all $n \geq n_1$, for some $n_1 \in \mathbb{N}$. Similarly, $\lim_{n\to\infty} d(x_n, y) = 0$ implies that $x_n = y$, for all $n \geq n_2$, where $n_2 \in \mathbb{N}$. Thus, $x_n = x = y$, for all $n \geq N$, where $N = \max\{n_1, n_2\}$. Thus, $x = y$.

Now, upon examining the function d closer, we find that $\operatorname{ran}(d) = \{0, 1, 3\}$. Thus, $\{x_n\}_{n\in\mathbb{N}}$ is a convergent sequence if and only if it is eventually constant, for the current example. This is easy to see since for a convergent sequence, for all $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that whenever $n \geq n_0$, $d(x_n, x) < \epsilon$. In this case, if we consider $\epsilon = 1$, there must exist some $n_0 \in \mathbb{N}$ so that whenever $n \geq n_0$, $d(x_n, x) < 1 \implies d(x_n, x) = 0$. This means that for all $n \geq n_0$, $x_n = x$. In other words, $\{x_n\}_{n\in\mathbb{N}}$ is eventually constant. That eventually constant sequences are convergent is obvious. Thus, (NT3) holds, since we have already elaborated the case wherein $\{x_n\}_{n\in\mathbb{N}}$ is eventually constant.

Now that we have established that (X, d) is a non-triangular metric space, we choose a function $\varrho : A \times A \to \mathbb{R}$ to be able to move ahead with the proof. Let it be defined as $\varrho(t, s) = 1$, for all $s, t \in A = [0, \infty)$.

It is clear that $\operatorname{ran}(d) = \{0, 1, 3\} \subseteq A$. Thus, $(\mathcal{A}_1)$ holds. For our choice of ϱ, $(\mathcal{A}_4)$ is also very obvious. We check the other two conditions.

For $(\mathcal{A}_2)$, if $\{x_n\} \subseteq X$ is a Picard sequence of T, such that $x_n \neq x_{n+1}$ and $\varrho(d(x_{n+1}, x_{n+2}), d(x_n, x_{n+1}))$, for all $n \in \mathbb{N}$, we want to show that $\{x_n\}_{n\in\mathbb{N}}$ is asymptotically regular. Here, if $x_0 \in X$, $x_1 = Tx_0 \in \{0, 1\} \implies x_2 = 0$. Thus, $x_n = 0$, for all $n \geq 2$. This means that the hypothesis $x_n \neq x_{n+1}$ is not satisfied, which means that $(\mathcal{A}_2)$ is vacuously true. However, it can be seen that $\{d(x_n, x_{n+1})\} \to 0$, nevertheless.

We check $(\mathcal{A}_3)$. We consider a $(T, \mathcal{S})$-sequence $\{(a_n, b_n)\}_{n\in\mathbb{N}}$ such that $\{a_n\} \to L$ and $\{b_n\} \to L$ for some $L \geq 0$. We further assume $L < a_n$ and $\varrho(a_n, b_n) > 0$, for all $n \in \mathbb{N}$. By definition, there exist $\{x_n\}_{n\in\mathbb{N}}$ and $\{y_n\}_{n\in\mathbb{N}}$ in X such that $a_n = d(Tx_n, Ty_n) > 0$ and $b_n = d(x_n, y_n) > 0$. Now, $a_n = d(Tx_n, Ty_n) \in \{0, 1\}$, Since $a_n > 0, a_n = 1$ for all $n \in \mathbb{N}$, which implies that $\lim_{n\to\infty} a_n = 1 = L$. However, this is a contradiction, since the hypothesis requires that $L < a_n$. Thus, $(\mathcal{A}_3)$ is also vacuously true.

Thus, all the four conditions for T to an $(\mathcal{A}, \mathcal{S})$-contraction hold.

Next, we state the two fixed point results based on theorems presented in [10], whose beauty lies in the fact that neither of the two makes any use of the triangle inequality. Thus, they happen to hold well in the context of non-triangular metric spaces as well. However, they do rely on uniqueness of limits of convergent sequences, which is a natural characteristic of non-triangular metric spaces.

Theorem 2.2.8. *Let (X, d) be a non-triangular metric space equipped with a transitive binary relation $\mathcal{S}$ and let $T : X \to X$ be an $\mathcal{S}$-non-decreasing $(\mathcal{A}, \mathcal{S})$-contraction with respect to $\varrho : A \times A \to \mathbb{R}$. Suppose that $T(X)$ is $(\mathcal{S}, d)$-strictly-increasing-precomplete and there exists a point $x_0 \in X$ such that $x_0 \mathcal{S} T x_0$. Assume that, at least, one of the following conditions is fulfilled:*

(a) T is $\mathcal{S}$-strictly-increasing-continuous.
(b) (X, d) is $\mathcal{S}$-strictly-increasing-regular and condition $(\mathcal{A}_5)$ holds.
(c) (X, d) is $\mathcal{S}$-strictly-increasing-regular and $\varrho(t, s) \le s - t$ for all $t, s \in A \cap (0, \infty)$.

Then the Picard sequence of T based on x_0 converges to a fixed point of T.
In particular, T has, at least, a fixed point.

Proof. Let $x_0 \in X$ be a point such that $x_0 S T x_0$ and let $x_{n+1} = T x_n$ be the Picard sequence of T based on x_0. If there exists some $n_0 \in \mathbb{N}$ such that $x_{n_0+1} = x_{n_0}$, then x_{n_0} is a fixed point of T, and $\{x_n\}$ converges to such point. On the contrary case, assume that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$. As T is S-non-decreasing and $x_0 \mathcal{S} T x_0 = x_1$, then $x_n \mathcal{S} x_{n+1}$ for all $n \in \mathbb{N}$, and as S is transitive, then

$$x_n \mathcal{S} x_m \quad \text{for all } n, m \in \mathbb{N} \text{ such that } n < m. \tag{2.12}$$

In fact, as $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$, then

$$x_n \mathcal{S}^* x_{n+1} \quad \text{and} \quad T x_n \mathcal{S}^* T x_{n+1} \quad \text{for all } n \in \mathbb{N}. \tag{2.13}$$

Let consider the sequence $\{d(x_n, x_{n+1})\}$. Taking into account (13) and the fact that T is an $(\mathcal{A}, \mathcal{S})$-contraction, condition $(\mathcal{A}_4)$ implies that, for all $n \in \mathbb{N}$,

$$\varrho(d\left(T^{n+1}x_0, T^{n+2}x_0\right), d\left(T^n x_0, T^{n+1}x_0\right)) = \varrho\left(d(Tx_n, Tx_{n+1}), d(x_n, x_{n+1})\right) > 0.$$

Applying $(\mathcal{A}_2)$ we deduce that $\{x_n = T^n x_0\}$ is an asymptotically regular sequence on (X, d), that is, $\{d(x_n, x_{n+1})\} \to 0$. Let us show that $\{x_n\}$ is an $\mathcal{S}$-strictly-increasing sequence. Indeed, in view of (12), assume that there exists $n_0, m_0 \in \mathbb{N}$ such that $n_0 < m_0$ and $x_{n_0} = x_{m_0}$. If $p_0 = m_0 - n_0 \in \mathbb{N} \diagdown \{0\}$, then $x_{n_0} = x_{n_0 + k\, p_0}$ for all $k \in \mathbb{N}$. In particular, the sequence $\{d(x_n, x_{n+1})\}$ contains the constant subsequence

$$\{d(x_{n_0+k\,p_0}, x_{n_0+k\,p_0+1}) = d(x_{n_0}, x_{n_0+1}) > 0\}_{k\in\mathbb{N}},$$

which contradicts the fact that $\{d(x_n, x_{n+1})\} \to 0$. This contradiction guarantees that $x_n \neq x_m$ for all $n \neq m$, so $x_n \mathcal{S}^* x_m$ for all $n, m \in \mathbb{N}$ such that $n < m$, that is, $\{x_n\}$ is an $\mathcal{S}$-strictly-increasing sequence. Next we show that $\{x_n\}$ is a Cauchy sequence reasoning by contradiction. If $\{x_n\}$ is not a Cauchy sequence, then by lemma (2.1) there exist $\epsilon_0 > 0$ and two sub-sequences $\{x_{n(k)}\}$ and $\{x_{m(k)}\}$ of $\{x_n\}$ such that

$$\begin{aligned} k \le n(k) < m(k), \quad d(x_{n(k)}, x_{m(k)-1}) \le \varepsilon_0 < d(x_{n(k)}, x_{m(k)}) \quad \text{for all } k \in \mathbb{N}, \\ \lim_{k\to\infty} d(x_{n(k)}, x_{m(k)}) = \lim_{k\to\infty} d(x_{n(k)-1}, x_{m(k)-1}) = \varepsilon_0. \end{aligned}$$

Let $L = \varepsilon_0 > 0$, $\{a_k = d(x_{n(k)}, x_{m(k)})\} \to L$ and $\{b_k = d(x_{n(k)-1}, x_{m(k)-1})\} \to L$. Clearly, $\{(a_k, b_k)\}$ is a $(T, \mathcal{S})$-sequence. Since $L = \varepsilon_0 < d(x_{n(k)}, x_{m(k)}) = a_k$ and

$$\begin{aligned}\varrho\left(a_k, b_k\right) &= \varrho\left(d(x_{n(k)}, x_{m(k)}), d(x_{n(k)-1}, x_{m(k)-1})\right) \\ &= \varrho\left(d(Tx_{n(k)-1}, Tx_{m(k)-1}), d(x_{n(k)-1}, x_{m(k)-1})\right) > 0\end{aligned}$$

for all $k \in \mathbb{N}$, condition $(\mathcal{A}_3)$ guarantees that $\varepsilon_0 = L = 0$, which is a contradiction. As a consequence, $\{x_n\}$ is a Cauchy sequence. Since $\{x_n\}_{n\geq 1} \subseteq T(X)$ and $T(X)$ is $(\mathcal{S}, d)$-strictly-increasing-pre-complete, there is a subset $Z \subset X$ such that $T(X) \subseteq Z \subseteq X$ and Z is $(\mathcal{S}, d)$ -strictly-increasing-complete. In particular, as $\{x_n\}$ is an $\mathcal{S}$-strictly-increasing and Cauchy sequence, there exists $z \in Z \subseteq X$ such that $\{x_n\} \to z$. Let us show that z is a fixed point of T distinguishing three cases.

Case 1. Assume that T is $\mathcal{S}$-strictly-increasing-continuous.
In this case, $\{x_{n+1} = Tx_n\} \to Tz$, so $Tz = z$.

Case 2. Assume that $\mathcal{S}$-strictly-increasing-regular and condition $(\mathcal{A}_5)$ holds. In this case, as $\{x_n\}$ is an $\mathcal{S}$-strictly-increasing sequence such that $\{x_n\} \to z$, it follows that

$$x_n \mathcal{S} z \quad \text{for all } n \in \mathbb{N}. \tag{2.14}$$

Since T is $\mathcal{S}$-non-decreasing,

$$Tx_n \mathcal{S} Tz \quad \text{for all } n \in \mathbb{N}. \tag{2.15}$$

Let $a_n = d(x_{n+1}, Tz) = d(Tx_n, Tz)$ and $b_n = d(x_n, z)$ for all $n \in \mathbb{N}$.
Clearly, $\{b_n\} \to 0$. Notice that

$$b_n = 0 \ \Rightarrow \ a_n = 0 \tag{2.16}$$

because

$$b_n = 0 \quad \Leftrightarrow \quad x_n = z \quad \Rightarrow \quad x_{n+1} = Tx_n = Tz \quad \Leftrightarrow \quad a_n = 0.$$

Let consider the set

$$\Omega = \{n \in \mathbb{N} : a_n = 0\} = \{n \in \mathbb{N} : d(x_{n+1}, Tz) = 0\}.$$

Subcase 2.1. Assume that Ω is finite. In this case, there exists $n_0 \in \mathbb{N}$ such that $d(x_{n+1}, Tz) = a_n > 0$ for all $n \geq n_0$. By (16), $d(x_n, z) = b_n > 0$ for all $n \geq n_0$. In this case, $\{(a_n, b_n)\}_{n\geq n_0}$ is a $(T, \mathcal{S})$- sequence. In particular $x_n \neq z$ and $Tx_n \neq Tz$ for all $n \geq n_0$. By (14) and (15), we deduce that $x_n \mathcal{S}^* z$ and $Tx_n \mathcal{S}^* Tz$ for all $n \geq n_0$. It follows from $(\mathcal{A}_4)$ that

$$\varrho(a_n, b_n) = \varrho\left(d(Tx_n, Tz), d(x_n, z)\right) > 0 \quad \text{for all } n \geq n_0.$$

As a consequence, as $(\mathcal{A}_5)$ holds, we conclude that $\{a_n = d(x_{n+1}, Tz)\} \to 0$, that is, $\{x_{n+1}\} \to Tz$, which guarantees that $Tz = z$.

Subcase 2.2. Assume that Ω is not finite. In this case, there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that

$$d(x_{n(k)+1}, Tz) = 0 \quad \text{for all } k \in \mathbb{N}.$$

Hence $x_{n(k)+1} = Tz$ for all $k \in \mathbb{N}$. Since $\{x_n\} \to z$ and $\left\{x_{n(k)+1}\right\} \to Tz$, then $Tz = z$.

Case 3. Assume that $\mathcal{S}$-strictly-increasing-regular and $\varrho(t,s) \leq s-t$ for all $t,s \in A \cap (0,\infty)$. Then item(b) is applicable.

In any case, we conclude that z is a fixed point of T.

Theorem 2.2.9. *Under the hypotheses of Theorem (2.2.8), we further assume that the following properties hold:*

(a) Condition $(\mathcal{A}_2')$ *holds (refer to Definition 2.2.4).*
(b) For all $x, y \in Fix(T)$*, which is the set of all fixed points of* T *in* X*, there exists* $z \in X$ *such that* z *is* $\mathcal{S}$*-comparable to both* x *and* y*.*

Then, T *has a unique fixed point.*

Proof. Let $x, y \in \mathrm{Fix}(T)$ be two fixed points of T. By hypothesis, there exists $z_0 \in X$ such that z_0 is, at the same time, $\mathcal{S}$-comparable to x and $\mathcal{S}$-comparable to y. Let $\{z_n\}$ be the Picard sequence of T based on z_0, that is, $z_{n+1} = Tz_n$ for all $n \in \mathbb{N}$. We will prove that $x = y$ by showing that $\{z_n\} \to x$ and $\{z_n\} \to y$. We first use x, but the same reasoning is valid for y.

Since z_0 is $\mathcal{S}$-comparable to x, assume that $z_0\mathcal{S}x$. As T is $\mathcal{S}$-non-decreasing, $z_n\mathcal{S}x$ for all $n \in \mathbb{N}$. If there exists $n_0 \in \mathbb{N}$ such that $z_{n_0} = x$, then $z_n = x$ for all $n \geq n_0$. In particular $\{z_n\} \to x$ and the proof is finished. On the contrary case, assume that $z_n \neq x$ for all $n \in \mathbb{N}$. Therefore $z_n\mathcal{S}^*x$ and $Tz_n\mathcal{S}^*Tx$ for all $n \in \mathbb{N}$. Using the contrary condition $(\mathcal{A}_4)$, for all $n \in \mathbb{N}$,

$$0 < \varrho(d(Tz_n, Tx), d(z_n, x)) = \varrho(d(T^{n+1}z_0, T^{n+1}x), d(T^n z_0, T^n x)).$$

It follows from $(\mathcal{A}_2')$ that $\{d(T^n z_0, T^n x)\} \to 0$ that is $\{z_n\} \to x$.

2.3. **Fixed point theorems for** (ψ, ϕ)**-contractions.** In this section first we see some lemmas which will be used to prove our main result.

Lemma 2.3.1. *Let* $\psi : (0,\infty) \to \mathbb{R}$*. Then the following conditions are equivalent:*
(i) $\inf_{t>\epsilon} \psi(t) > -\infty$ *for every* $\epsilon > 0$*.*
(ii) $\liminf_{t\to\epsilon} \psi(t) > -\infty$ *for every* $\epsilon > 0$*.*
(iii) $\lim_{n\to\infty} \psi(t_n) = -\infty$ *implies* $\lim_{n\to\infty} t_n = 0$*.*

Proof. $(i) \implies (ii)$: Let (i) hold and $\inf_{t>\epsilon} \psi(t) = A$ for some $\epsilon > 0$. Then $\psi(t) \geq A$ for every $t > \epsilon$. Therefore, $\liminf_{t\to\epsilon} \psi(t) \geq A$, i.e., (ii) holds.
$(ii) \implies (iii)$: Let (ii) hold and $\lim_{n\to\infty} \psi(t_n) = -\infty$ for a sequence $(t_n) \subset (0,\infty)$. Assume that (t_n) does not converge to zero. Then there exists $\epsilon > 0$ and a subsequence (t_{n_k}) such that $t_{n_k} > \epsilon$ for every $k \geq 1$. Since $\lim_{n\to\infty} \psi(t_n) = -\infty$ implies $\lim_{k\to\infty} \psi(t_{n_k}) = -\infty$, we conclude that $\liminf_{t\to\epsilon} \psi(t) = -\infty$ which is a contradiction to (ii). Hence, $\lim_{n\to\infty} (t_n) = 0$ which means that (iii) holds.
$(iii) \implies (i)$: Let (iii) hold. Assume that $\inf_{t>\epsilon} \psi(t) = -\infty$ for some $\epsilon > 0$. Then there exists a sequence $(t_n) \subset (0,\infty)$ such that $t_n > \epsilon$ for every $n \geq 1$ and $\lim_{n\to\infty} \psi(t_n) = -\infty$. From (iii), we get that $\lim_{n\to\infty} t_n = 0$ which contradicts $t_n > \epsilon$. Therefore, (i) holds.

We are going to provide a fixed point theorem for a self-mapping T on a complete non-triangular metric space (X, d) satisfying a contractive-type condition

Definition 2.3.2. Contractive type condition:

$$\psi(d(Tx,Ty)) \leq \phi(d(x,y)) \text{ for all } x,y \in X \text{ with } d(Tx,Ty) > 0, \tag{2.17}$$

where $\psi,\phi:(0,\infty)\to\mathbb{R}$ are two functions such that $\phi(t)<\psi(t)$ for $t>0$.

Idea of following lemmas is taken from the [7].

Lemma 2.3.3. *Let (X,d) be a non-triangular metric space and T be a self-mapping on X satisfying condition (2.17), where the functions $\psi,\phi:(0,\infty)\to\mathbb{R}$ are such that*
(i) $\phi(t)<\psi(t)$ for any $t>0$;
(ii) $\inf\limits_{t>\epsilon}\psi(t)>-\infty$ for any $\epsilon>0$.
Suppose also that at least one of the following conditions holds:
(iii) ψ is non-decreasing and $\limsup\limits_{t\to\epsilon}\phi(t)<\psi(\epsilon)$ for any $\epsilon>0$;
(iv) if $(\psi(t_n))$ and $(\phi(t_n))$ are convergent sequences with the same limit and $(\psi(t_n))$ is a strictly decreasing, then $\lim\limits_{n\to\infty}t_n=0$.
Then T is an asymptotically regular mapping.

Lemma 2.3.4. *Let (X,d) be a metric space and T be a self-mapping on X satisfying condition (2.17) with the functions $\psi,\phi:(0,\infty)\to\mathbb{R}$ which satisfy at least one of the following conditions:*

(i) *ψ is non-decreasing, $\phi<\psi$, and $\limsup\limits_{t\to\epsilon}\phi(t)<\psi(\epsilon)$ for any $\epsilon>0$;*
(ii) *$\limsup\limits_{t\to\epsilon}\phi(t)<\liminf\limits_{t\to\epsilon}\psi(t)$ for any $\epsilon>0$.*

If T is asymptotically regular at a point $x\in X$, then $(T^n x)$ is a Cauchy sequence.

Proof. Let T be an asymptotically regular mapping at a point $x\in X$. Assume that the sequence $(T^n x)$ is not Cauchy. Set $x_n=T^n x$ for each $n\geq 0$. Let ψ and ϕ satisfy condition (i). It follows from Lemma (2.1) that there exist $\epsilon>0$ and two subsequences (x_{n_k}) and (x_{m_k}) of (x_n) such that the limits (2.2) and (2.3) hold. It follows from (2.3) that $d(x_{n_{k+1}},x_{m_{k+1}})>\epsilon$ for all $k\geq 1$. Applying (2.17) with $x=x_{n_k}$ and $y=x_{m_k}$, we get

$$\psi(d(x_{n_{k+1}},x_{m_{k+1}}))\leq\phi(d(x_{n_k},x_{m_k})). \tag{2.18}$$

for all $k\geq 1$. We set $\alpha_k=d(x_{n_{k+1}},x_{m_{k+1}})$ and $\beta_k=d(x_{n_k},x_{m_k})$. Then 2.18 takes the form

$$\psi(\alpha_k)\leq\phi(\beta_k). \tag{2.19}$$

Hence, taking into account that $\phi<\psi$, we obtain

$$\psi(\alpha_k)\leq\phi(\beta_k)<\psi(\beta_k).$$

From this and monotonicity of ψ, we deduce that $\alpha_k<\beta_k$. Then it follows from 2.2 that $\alpha_k\to\epsilon$ and $\beta_k\to\epsilon$. Taking the limit superior in 2.19, we get

$$\psi(\epsilon)=\lim_{k\to\infty}\psi(\alpha_k)\leq\limsup_{k\to\infty}\phi(\beta_k)\leq\limsup_{t\to\epsilon}\phi(t)$$

which is a contradiction to the third part of condition (i). Let ψ and ϕ satisfy condition (ii). Note that we have proved (2.19) without using condition (i). It follows from (2.2) that $\alpha_k\to\epsilon$ and $\beta_k\to\epsilon$. From (2.19), we get

$$\liminf_{t\to\epsilon}\psi(t)\leq\liminf_{k\to\infty}\psi(\alpha_k)\leq\limsup_{k\to\infty}\phi(\beta_k)\leq\limsup_{t\to\epsilon}\phi(t)$$

which is a contraction to (ii). Therefore $(T^n x)$ is Cauchy.

Lemma 2.3.5. *Let (X, d) be a non-triangular metric space which satisfies the property C and T be a self-mapping on X satisfy condition (2.17), with the functions $\psi, \phi : (0, \infty) \to \mathbb{R}$ which satisfy at least one of the following conditions:*

(i) *ψ is non-decreasing and $\phi(t) < \psi(t)$ for any $t > 0$;*
(ii) *$\limsup_{t\to 0} \phi(t) < \liminf_{t\to\epsilon} \psi(t)$ for any $\epsilon > 0$.*

If $\lim_{n\to\infty} T^n x = \xi$ for some $x \in X$, then ξ is a fixed point of T.

Proof. Let $x_n = T^n x$, then $(T^n x)$ converges to a point $\xi \in X$. As $d(T^n x, \xi) \to 0$ as $n \to \infty$ and d satisfies the property C, $d(T^n x, T\xi) \to d(\xi, T\xi)$. If $d(T^n x, T\xi) = 0$ for infinitely many values of n, then $d(\xi, T\xi) = 0$. This means $T\xi = \xi$, that is ξ is a fixed point of map T.

Now suppose that $d(T^n x, T\xi) > 0$ holds for infinitely many values of n. Then applying (2.17) with $x = x_n$ and $y = \xi$, we conclude that

$$\psi(d(x_{n+1}, T\xi)) \leq \phi(d(x_n, \xi)) \tag{2.20}$$

holds for any values of n. Let ψ and ϕ satisfy condition (i) and (ii). Then it follows from (2.20) that

$$\psi(d(x_{n+1}, T\xi)) \leq \phi(d(x_n, \xi)) < \psi(d(x_n, \xi)).$$

This implies that $d(x_{n+1}, T\xi) < d(x_n, \xi)$. Taking the limit $n \to \infty$, we get $d(\xi, T\xi) \leq 0$ which implies that ξ is a fixed point of T. Let ψ and ϕ satisfy condition (ii). From (2.20), we have

$$\psi(\alpha_n) \leq \phi(\beta_n) \tag{2.21}$$

for infinitely many values of n, where $\alpha_n = d(x_{n+1}, T\xi)$ and $\beta_n = d(x_n, \xi)$. Obviously, $\alpha_n \to \epsilon$(by property C) and $\beta_n \to 0$ as $n \to \infty$, where $\epsilon = d(\xi, T\xi)$. It follows from (2.21) that

$$\liminf_{t\to\epsilon} \psi(t) \leq \liminf_{n\to\infty} \psi(\alpha_n) \leq \limsup_{n\to\infty} \phi(\alpha_n) \leq \limsup_{t\to 0} \psi(t),$$

If $\epsilon > 0$, then the last inequality is a contradiction to condition (ii). Hence, $d(\xi, T\xi) = 0$, which implies that ξ is a fixed point of T.

Now we are ready to state and prove the theorem for self-mappings that satisfy contractive-type condition (2.17).

Theorem 2.3.6. *Let (X, d) be a complete non-triangular metric space which satisfy the property C and $T : X \to X$ be a mapping satisfying condition (2.17), where the functions $\psi, \phi : (0, \infty) \to \mathbb{R}$ satisfy the following conditions:*

(i) *ψ is non-decreasing;*
(ii) *$\phi(t) < \psi(t)$ for any $t > 0$;*
(iii) *$\limsup_{t\to\epsilon} \phi(t) < \psi(\epsilon)$ for any $\epsilon > 0$.*

Then T has a unique fixed point $\xi \in X$ and the iterative sequence (Tx_n) converges to ξ for every $x \in X$.

Proof. It follows from conditions (i)-(iii) and Lemma 2.3.3 that T is asymptotically regular. Let x be an arbitrary point in X. It follows from conditions (i)-(iii) and Lemma 2.3.4 that the sequence $\{T^n x\}$ is Cauchy. Since the space X is complete, then $\{T^n x\}$ converges to a point $\xi \in X$. Then by conditions (i),(ii) and Lemma 2.3.5 we conclude that ξ is a fixed point of T. The uniqueness of the fixed point follows trivially from conditions (2.17) and (ii).

Example 2.3.6. Let $X = [0,1]$ and $d : X \times X \to [0,\infty)$ be defined by

$$d(x,y) = \begin{cases} \mid x-y \mid, \text{ if } x \neq \frac{1}{2}, y \neq \frac{1}{2}; \\ \dfrac{x+2}{3}, \text{ if } y = \frac{1}{2}, x \neq \frac{1}{2}; \\ \dfrac{y+2}{3}, \text{ if } x = \frac{1}{2}, y \neq \frac{1}{2}; \\ 0, \text{ if } x = y = \frac{1}{2}. \end{cases}$$

Let $T : X \to X$ be defined as $Tx = \dfrac{x}{3}$. Consider the functions $\phi, \psi : (0,\infty) \to (0,\infty)$ defined by $\phi(x) = 12x$ and $\psi(x) = 14x$. It's easy to verify that the given mapping d is a complete non-triangular metric on a set X at it has the property C. Now, for checking the map T satisfies the condition (2.17), we required few cases as:

Case (i). $x \neq \dfrac{1}{2}, y \neq \dfrac{1}{2}$. For this case

$$\psi(d(Tx,Ty)) = 14\left|\frac{x-y}{3}\right| \leq 12|x-y| = \phi(d(x,y)),$$

i.e. In this case we obtain $\psi(d(Tx,Ty)) \leq \phi(d(x,y))$.

Case (ii). $x = \dfrac{1}{2}, y \neq \dfrac{1}{2}$. For this case

$$\psi(d(Tx,Ty)) = 14d(\frac{1}{6},y) = 14\left|\frac{1}{6} - y\right|,$$

$$\phi(d(x,y)) = 12d\left(\frac{1}{2},y\right) = 12\frac{y+2}{3} = 4(y+2).$$

Now for each $y \in X$, $14|\dfrac{1}{6} - y| \leq 4(y+2)$, which implies that

$$\psi(d(Tx,Ty)) \leq \phi(d(x,y)).$$

As here $d(x,y) = d(y,x)$, for all the $x \in X, T$ satisfies the condition (2.17). Also, the maps T, ϕ and ψ satisfy all the conditions of Theorem 2.3. Hence T has a unique fixed point $\xi \in X$.

Corollary 2.3.7. *Let (X,d) be a complete non-triangular metric space which satisfies property C and the map $T : X \to X$ satisfies the condition (2.17), where the functions $\psi, \phi : (0,\infty) \to \mathbb{R}$ are defined by $\psi(x) = x$ and $\phi(x) = kx$, for some $k < 1$ and all $x \in X$. Then T has a unique fixed point in the set X.*

Theorem 2.3.8. *Let (X,d) be a non-triangular metric space with property C. Let $T : X \to X$ be a mapping satisfying the condition (2.17), where the functions $\psi, \phi : (0,\infty) \to \mathbb{R}$ satisfy the following conditions:*

(i) $\phi(t) < \psi(t)$ *for any* $t > 0$*;*
(ii) $\inf\limits_{t>\epsilon} \psi(t) > -\infty$ *for any* $\epsilon > 0$*;*

(iii) *if* $(\psi(t_n))$ *and* $(\phi(t_n))$ *are convergent sequences with the same limit and* $(\psi(t_n))$ *is strictly decreasing, then* $t_n \to 0$ *as* $n \to \infty$*;*
(iv) $\limsup_{t\to\epsilon} \psi(t) < \liminf_{t\to\epsilon} \psi(t)$ *or* $\limsup_{t\to\epsilon} \phi(t) < \liminf_{t\to\epsilon} \psi(t)$ *for any* $\epsilon > 0$*;*
(v) T *has closed graph or* $\limsup_{t\to 0} \phi(t) < \liminf_{t\to\epsilon} \psi(t)$ *for any* $\epsilon > 0$.

Then T *has a unique fixed point* $\xi \in X$ *and the iterative sequence* $(T^n x)$ *converges to* ξ*, for every* $x \in X$.

Corollary 2.3.9. *Let* (X, d) *be a non-triangular metric space and let* $T : X \to X$ *be a mapping satisfying condition (2.17), where the functions* $\psi, \phi : (0,\infty) \to \mathbb{R}$ *satisfy the following conditions:*

(i) $\phi(t) < \psi(t)$ *for any* $t > 0$*;*
(ii) ψ *is lower semi-continuous and* ϕ *is upper semi-continuous;*
(iii) *if* $(\psi(t_n))$ *and* $(\phi(t_n))$ *are convergent sequences with the same limit and* $(\psi(t_n))$ *is a strictly decreasing, then* (t_n) *is a bounded sequence;*
(iv) T *has closed graph or* $\limsup_{t\to 0} \phi(t) < \psi(\epsilon)$ *for any* $\epsilon > 0$.

Then T *has a unique fixed point* $\xi \in X$ *and the iterative sequence* $(T^n x)$ *converges to* ξ *for every* $x \in X$.

3. Applications to operator equations

3.1. **Application to high-order fractional differential equations with non-local Boundary conditions.** Motivated by [11], we investigate the existence of a unique solution for a class of high-order fractional differential equations with non-local boundary conditions given by

$$\begin{cases} {}^C D_t^{\alpha} u + f(t, u(t)) = 0, \quad t \in (0,1), \\ u'(0) = u''(0) = \cdots = u^{(m-1)} = 0, \quad u(0) = \lambda \int_0^1 u(s)\mathrm{d}s, \end{cases} \tag{3.1}$$

where $m - 1 < \alpha \le m, m \in \mathbb{N}$, $0 < \lambda < 1$, ${}^C D_t^{\alpha} u$ is the Caputo fractional derivative and $f : (0,1) \times \mathbb{R} \to \mathbb{R}$ is a given continuous function.

Theorem 3.1.1. *Let* $m - 1 < \alpha \le m$*. Assume* $y \in C[0,1]$*. Then, the problem (3.1) has a unique solution* $u \in C[0,1]$*, given by*

$$u(t) = \int_0^1 G(t,s) y(s)\mathrm{d}s, \tag{3.2}$$

where

$$G(t,s) = \begin{cases} \dfrac{(1-s)^{\alpha} - (1-\lambda)\alpha(t-s)^{\alpha-1}}{(1-\lambda)\Gamma(\alpha+1)}, & 0 \le s \le t \le 1; \\ \dfrac{(1-s)^{\alpha}}{(1-\lambda)\Gamma(\alpha+1)}, & 0 \le t \le s \le 1. \end{cases} \tag{3.3}$$

Consider $C[0,1]$, the Banach space of all continuous functions from $[0,1]$ to $\mathbb{R}$, equipped with the norm $||u|| = \sup_{x\in[0,1]} |u(x)| + \sup_{x\in[0,1]} |u'(x)|$. Replacing $y(x)$ by

$f(t, u(t))$ in Theorem 3.1, an operator $T : C[0,1] \to C[0,1]$ associated with problem (3.1) can be defined as

$$Tu(t) = \int_0^1 G(t,s)f(s,u(s))\mathrm{d}s :\equiv -\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}f(s,u(s))\mathrm{d}s + \mu\int_0^1 (1-s)^{\alpha}f(s,u(s))\mathrm{d}s, \tag{3.4}$$

where

$$\mu = \frac{1}{(1-\lambda)\Gamma(\alpha+1)}. \tag{3.5}$$

Now, by Theorem 3.1, the fixed points of operator T are exactly the solutions of problem (3.1). Therefore, it remains to investigate the fixed points of operator T, by using Corollary 2.3.

We are now ready to prove the main theorem. For computational convenience, we put

$$R = \frac{1}{\Gamma(\alpha+1)} + \frac{\mu}{\alpha+1}.$$

Let $B_r = \{u \in C[0,1] \ : \ \|u\| \le r\}$, where $r \ge \dfrac{4N_f R}{1-4L_f R}$ with $N_f = \sup\limits_{t\in[0,1]} |f(t,0)|$. and $d(u,v) = \|u-v\|^2$, then it is clear that (B_r, d) is a complete non-triangular metric space but not a classical metric space.

Theorem 3.1.2. *Assume that the following contraction condition holds: There exists $L_f > 0$, such that*

$$\forall t \in [0,1], u, v \in \mathbb{R} : |f(t,u) - f(t,v)| \le L_f|u-v|. \tag{3.6}$$

Then, if $L_f R < \dfrac{1}{2}$, then the BVP (3.1) has a unique solution in B_r.

Proof. First, we show that $T : B_r \to B_r$ i.e. $T(B_r) \subset B_r$. Take $u \in B_r$, $t \in [0,1]$. This means $||u|| = \sup\limits_{x\in[0,1]} |u(x)| + \sup\limits_{x\in[0,1]} |u'(x)| \le r$.

We need to show that $||Tu|| = \sup\limits_{x\in[0,1]} |Tu(x)| + \sup\limits_{x\in[0,1]} |Tu'(x)| \le r$.

$$\begin{aligned}|Tu(t)| \le& \frac{1}{\Gamma(\alpha)}\int_0^t |t-s|^{\alpha-1}|f(s,u(s))|\mathrm{d}s + \mu\int_0^1 |1-s|^{\alpha}|f(s,u(s))|\mathrm{d}s\\ \le& \frac{1}{\Gamma(\alpha+1)}t^{\alpha}\|f\| + \frac{\mu}{\alpha+1}\|f\|\\ \le& \|f\|\left(\frac{1}{\Gamma(\alpha+1)} + \frac{\mu}{\alpha+1}\right).\end{aligned} \tag{3.7}$$

In a similar manner

$$|Tu'(t)| \le \|f\|\left(\frac{1}{\Gamma(\alpha+1)} + \frac{\mu}{\alpha+1}\right). \tag{3.8}$$

By (3.7) and (3.8) we get

$$\|Tu\| = \sup_{t\in[0,1]} |Tu(t)| + \sup_{t\in[0,1]} |Tu'(t)| \leq \|f\|R + \|f\|R = 2\|f\|R, \tag{3.9}$$

However

$$\begin{aligned} |f(t,u(t))| \leq & |f(t,u(t)) - f(t,0)| + |f(t,0)| \\ \leq & L_f(|u(t)|) + |f(t,0)| \\ \leq & L_f\|u\| + N_f. \end{aligned} \tag{3.10}$$

Similarly

$$|f(t,u'(t))| \leq L_f\|u\| + N_f. \tag{3.11}$$

From (3.10) and (3.11), we get

$$\|f\| \leq 2\{L_f\|u\| + N_f\} \leq 2\{L_f r + N_f\}. \tag{3.12}$$

Therefore, by the inequalities (3.9) and (3.12), we get

$$\|Tu\| \leq 2(L_f r + N_f)2R \leq 4\{L_f r + N_f\}. \tag{3.13}$$

So, by choosing sufficient $r \geq 4(L_f r + N_f)R$ we get $T(B_r) \subset B_r$. Next, we show that T is a contraction. Notice that, for arbitrary $u, v \in C[0,1]$, we have

$$\begin{aligned} |(Tu)(t) - (Tv)(t)| \leq & \frac{1}{\Gamma(\alpha)} \int_0^t |t-s|^{\alpha-1} |f(s,u(s)) - f(s,v(s))| \mathrm{d}s + \mu \\ & \int_0^1 |1-s|^{\alpha} |f(s,u(s)) - f(s,v(s))| \mathrm{d}s \\ & \leq L_f|u-v| \left[\frac{1}{\Gamma(\alpha+1)} + \frac{\mu}{\alpha+1}\right] \\ & \leq L_f\|u-v\|R. \end{aligned} \tag{3.14}$$

Similarly

$$\begin{aligned} |(Tu')(t) - (Tv')(t)| \quad & \leq L_f|u'-v'| \left[\frac{1}{\Gamma(\alpha+1)} + \frac{\mu}{\alpha+1}\right] \\ \leq L_f\|u-v\|R. & \end{aligned} \tag{3.15}$$

From 3.14 and 3.15 we get

$$\|Tu - Tv\| \leq 2L_f\|u-v\|R \tag{3.16}$$

If we square inequality (3.16), then we get

$$\begin{aligned} & \|Tu - Tv\|^2 \leq 4L_f^2\|u-v\|^2R^2 \\ \implies & d(Tu,Tv) \leq 4L_f^2R^2d(u,v). \end{aligned} \tag{3.17}$$

Now define $\psi(u) = u$ and $\phi(u) = (2L_fR)^2u$, where $L_fR < \dfrac{1}{2}$ which implies $\psi(d(Tu,Tv)) \leq \phi(d(u,v))$. Hence T, ϕ and ψ satisfy all the assumptions of the Corollary 2.3.7. Hence the problem (3.1) has a unique solution in B_r.

Example 3.1.3. Consider the following fractional differential equation:

$$\begin{cases} {}^{C}D_t^{3.5}u + \beta\left(\dfrac{1+u(t)}{1+cos^2(t)}\right) = 0, \quad t \in (0,1) \text{ and } \beta > 0, \\ u'(0) = u''(0) = u^{'''} = 0, \quad u(0) = \dfrac{1}{3}\displaystyle\int_0^1 u(s)\mathrm{d}s, \end{cases} \tag{3.18}$$

As we can see $\alpha = 3.5$, $\lambda = \dfrac{1}{3}$ and $f(t,u(t)) = \beta\left(\dfrac{1+u(t)}{1+cos^2(t)}\right)$ It is easy to observe that

$$\begin{aligned} |f(t,u(t)) - f(t,v(t))| &= \left|\beta\left(\frac{1+u(t)}{1+cos^2(t)}\right) - \beta\left(\frac{1+v(t)}{1+cos^2(t)}\right)\right| \\ &\leq \beta\left(\left|\frac{|u(t)|}{1+cos^2(t)} - \frac{|v(t)|}{1+cos^2(t)}\right|\right) \leq \beta\,|u(t)-v(t)|\,. \end{aligned}$$

Therefore by setting $L_f = \beta$, the hypothesis (3.6) of Theorem 3.1.2 is satisfied. By routine calculations we can find $\mu = \dfrac{8}{35\sqrt{\pi}}$, $R = \dfrac{64}{315\sqrt{\pi}}$ and $N_f = \sup\limits_{t\in[0,1]} |f(t,0)| = \beta$. Now, if we consider $\beta < \dfrac{315\sqrt{\pi}}{128}$, then the $L_f R < \dfrac{1}{2}$ holds. Thus, we get the result concerning the existence of the unique solution in the ball $\beta_r = \{u \in C[0,1] : \|u\| \leq r\}$, where $r \geq \dfrac{4\beta R}{1-4\beta R}$. As an example, if we set $\beta = 3$, then the unique solution belongs to

$$\beta_r = \left\{u \in C[0,1] : \|u\| \leq \frac{768}{315\sqrt{\pi} - 768}\right\}.$$

3.2. **Application to non-linear integral equation.** In this section we present an application of our fixed point results for non-linear integral equations.
Let $X = C[0,1]$ and $d : X \times X \to \mathbb{R}^+$ define by $d(x,y) = \|x-y\|_\infty^2$. Now consider the integral equation

$$x(t) = g(t) + \int_0^1 k(t,s)f(s,x(s))\mathrm{d}s \tag{3.19}$$

and let $T : X \to X$, defined by

$$Tx(t) = g(t) + \int_0^1 k(t,s)f(s,x(s))\mathrm{d}s,$$

where f is a function on $[0,1]\times C[0,1] \to \mathbb{R}$, and $k : [0,1]\times[0,1] \to \mathbb{R}_0^+$ is a continuous map.
For smooth calculation we take $\mu = \max\limits_{t\in[0,1]} \displaystyle\int_0^1 k(t,s)\mathrm{d}s$.

Theorem 3.2.1. *Assume that the following contraction condition holds for the mapping f: for some $L_f > 0$, we have*

$$|f(t,u) - f(t,v)| \leq L_f|u-v|, \forall t \in [0,1], u,v \in \mathbb{R}. \tag{3.20}$$

Then, if $L_f\mu < 1$, then non-linear integral equation (3.19) has a unique solution.

Proof. For the arbitrary $x, y \in C[0,1]$, we have

$$\begin{aligned}|(Tx)(t)-(Ty)(t)| &\leq \int_0^1 k(t,s)|f(s,x(s))-f(s,y(s))|\mathrm{d}s\\ &\leq L_f|x-y|\int_0^1 k(t,s)\mathrm{d}s\\ &\leq L_f|x-y|\mu. \end{aligned} \tag{3.21}$$

Taking supremum on the both the side we get

$$\|Tx-Ty\| \leq L_f\mu\|x-y\|. \tag{3.22}$$

Now, define $\psi(x)=x$ and $\phi(x)=L_f\mu x$, where $L_f\mu<1$. Then T,ϕ and ψ satisfies all the assumption of Corollary 2.3.7. Hence, the equation (3.19) has a unique solution.

4. Conclusion

Motivated by the rich literature on fixed point theorems and applications, in this paper, we possibly provide the answer to the question:

What would be the minimal metric structure to prove fixed point theorems for contractive type mapping?

In connection to above question, from observing many abstract metric structure in the book [3] and [6] some how we realized that non-triangular metric space is the minimal metric structure to establish fixed point theorems for various contractions (we have established new fixed point theorems for F-contraction, $(\mathcal{A},\mathcal{S})$-contraction,and (ψ,ϕ)-contraction).

On the other hand, a careful study of solution procedure of operator equation problems reveals that the method of application of fixed point theorem to operator equations consists of following main steps:

(i) Since integrals are easier to handle than differential, first the given operator equation is converted into an equivalent equation via theory of differential and integral calculus and then obtained integral equation is written in the form of corresponding equation in a suitable metric space.

(ii) Depending upon the nature of nonlinear involved in a operator equation, a fixed point theorem on a suitable metric space is used to prove the existence of solution for the so obtained equivalent operator equation which theory implies the existence results for the operator equation.

In the present article, we demonstrates the applicability of fixed point theorems proved under weaker metrical structure in solving (i) high-order fractional differential equations with non-local boundary conditions (ii) nonlinear integral equation.

One can apply the technique used in Corollary 2.3.7 for proving the existence and uniqueness of solutions of various mathematical models(differential, integral, ordinary and partial differential equations, variational inequalities), the same technique could be applied in other fields e.g. Steady-state temperature distribution, Chemical reactions, Neutron transport theory, Economic theory, Game theory, Optimal control theory, Fractals, etc.

Acknowledgements. We are indebted to the editor for their valuable comments and suggestions which contributed to the improvement of the paper.

References

[1] D.W. Boyd, J.S. Wong, *On nonlinear contractions*, Proc. Am. Math. Soc., **20**(1969), 458-464.

[2] A. Deshmukh, D. Gopal, *Topology of non-triangular metric spaces and related fixed point results*, Filomat, **35**(2020), no. 11, 3557-3570.

[3] D. Gopal, P. Agarwal, P. Kumam, *Metric Structures and Fixed Point Theory*, New York, Taylor and Francis (CRC Press), ISBN:9780367689148, https://doi.org/10.1201/9781003139607

[4] M. Jleli, B. Samet, *A generalized metric space and related fixed point theorems*, Fixed Point Theory Applications, (2015).

[5] F. Khojasteh, H. Khandani, *Scrutiny of some fixed point results by S-operators without triangular inequality*, Math. Slovaca, **70**(2020), no. 2, 467-476.

[6] W. Kirk, N. Shahzad, *Fixed Point Theory in Distance Spaces*, Cham, Switzerland, Springer International Publishing, 2014.

[7] P.D. Proinov, *Fixed point theorems for generalized contractive mappings in metric spaces*, J. Fixed Point Theory Appl., **22**(2020), no. 1, 1-27.

[8] V. Rakočević, K. Roy, M. Saha, *Wardowski and Ćirić type fixed point theorems over non-triangular metric spaces*, Quaest. Math., **45**(2022), no. 1, 1-11.

[9] J. Savaliya, D. Gopal, S.K. Srivastava, *Some discussion on generalizations of metric spaces in fixed point perspective*, Int. J. Nonlinear Anal. Appl. (in press).

[10] N. Shahzad, A.F. Roldán López de Hierro, F. Khojasteh, *Some new fixed point theorems under $(\mathcal{A}, \mathcal{S})$-contractivity conditions*, Rev. R. Acad. Cienc. Exactas Fis. Nat. Ser. A Mat., **111**(2017), 307-324.

[11] E. Shivanian, *Error estimate and stability analysis on the study of a high-order nonlinear fractional differential equation with Caputo-derivative and integral boundary condition*, J. Comput. Appl. Math., **41**(2022), no. 8, 395.

[12] D. Wardowski, *Fixed points of a new type of contractive mappings in complete metric spaces*, Fixed Point Theory Appl., **2012**(2012), no. 1, 1-6.

Received: February 10, 2022; Accepted: November 3, 2022.