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A THEOREM OF CARISTI-KIRK TYPE FOR b-METRIC SPACES

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Abstract. The renowned Caristi-Kirk theorem states that a metric space is complete if and only if every Caristi mapping on it has a fixed point. In this note we extend this result to the setting of b-metric spaces.
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1. Introduction

Caristi proved in [4] his celebrated fixed point theorem that every Caristi mapping on a complete metric space has a fixed point, where a self mapping T of a metric space (X, d) is called a Caristi mapping if there exists a lower semicontinuous function $\varphi : X \to [0, \infty)$ such that $d(x, Tx) \leq \varphi(x) - \varphi(Tx)$ for all $x \in X$.

Caristi's fixed point theorem has, among other appealing properties, that it generalizes the Banach contraction principle, and also allows to obtain a characterization of complete metric spaces. This last fact was proved by Kirk in [13]. Consequently, we have the following important result, named by several authors as the Caristi-Kirk theorem.

Theorem 1.1. (Caristi-Kirk's theorem) *A metric space is complete if and only if every Caristi mapping on it has a fixed point.*

The main goal of this note is to obtain an extension of Caristi's fixed point theorem to the framework of b-metric spaces in such a way that it generalizes the b-metric version of the Banach contraction principle and also allows us to characterize completeness of b-metric spaces.

By $\mathbb{R}$, $\mathbb{R}^+$, $\mathbb{N}$ and ω we shall denote the set of real numbers, the set of non-negative real numbers, the set of positive integer numbers and the set of non-negative integer numbers, respectively.

We now remind those concepts and properties that will be useful in the rest of the paper, where our basic reference for general topology is [10] and for b-metric spaces it is [14, Chapter 12].

We point out that several authors have reinvented over time the notion of a b-metric space under different approaches and designations (a detailed compilation can be found in [5, page 134]). Here we adjust that notion as given by Czerwik in [7].

By a b-metric space we mean a triple (X, d, s), where X is a set, s is a real number with $s \geq 1$, and $d : X \times X \to \mathbb{R}^+$ is a function fulfilling the following conditions for all $x, y, z \in X$:

(b1) $d(x, y) = 0$ if and only if $x = y$;

(b2) $d(x, y) = d(y, x)$;

(b3) $d(x, y) \leq s(d(x, z) + d(z, y))$.

If (X, d, s) is a b-metric space, the function d is called a b-metric on X. Of course d is a metric on X if $s = 1$.

Remark 1.2. Distinguished and well-known examples of b-metric spaces may be found, for instance, in [3, 14, 19]. In particular, if (X, d) is a metric space, and K and β are real constants with $K > 0$ and $\beta > 1$, then (X, d_b, s) is a b-metric space, where $s = 2^{\beta-1}$ and d_b is the b-metric on X given by $d_b(x, y) = Kd(x, y)^\beta$ for all $x, y \in X$ (see e.g. [19, Example 2.2] and [14, Example 12.2]).

As in the metric case, a b-metric d on a set X induces in a natural way a topology on X. Indeed, for each $x \in X$ and $r > 0$ put $B(x, r) = \{y \in X : d(x, y) < r\}$, and let

$$\tau_d = \{A \subseteq X : \text{for each } x \in A \text{ there is } r > 0 \text{ such that } B(x, r) \subseteq A\}.$$

It is easily seen that τ_d is a topology on X. In fact, this topology can be induced by the uniformity which has a countable base the sets of the form

$$U_n = \{(x, y) \in X \times X : d(x, y) < 1/n\},$$

for all $n \in \mathbb{N}$. Consequently τ_d is a metrizable topology.

In fact, the study of the topological properties and the problem of the completion of b-metric spaces, as well as an extensive and deep development of the fixed point theory for these spaces, has received the care of many authors (see e.g. [1, 2, 3, 5, 8, 9, 12, 14] and the references therein).

It is interesting to note that, contrarily to the classical metric case, the balls $B(x, r)$ are not necessarily τ_d-open sets and the b-metric d is not continuous, in general (see e.g. [2, Example 3.9], [14, Section 12.4]). However, for each $x \in X$ and $r > 0$ we have $x \in IntB(x, r)$ (see e.g. [10, Corollary 8.1.3]), which implies that a sequence $(x_n)_{n\in\omega}$ in X τ_d-converges to $x \in X$ if and only if $\lim_n d(x, x_n) = 0$.

In the sequel, if $(x_n)_{n\varepsilon\omega}$ is a sequence in a b-metric space (X, d, s) that τ_d-converges to $x \in X$ we will simply write $x_n \to x$ or $d(x, x_n) \to 0$ if no confusion arises.

We finally recall that, exactly as in the metric case, a sequence $(x_n)_{n\varepsilon\omega}$ in a b-metric space (X, d, s) is said to be a Cauchy sequence if for each $\varepsilon > 0$ there is $n_\varepsilon \in \omega$ such that $d(x_n, x_m) < \varepsilon$ for all $n, m \geq n_{\varepsilon.}$ (X, d, s) is said to be complete if every Cauchy sequence is τ_d-convergent.

Remark 1.3. Let (X, d) be a metric space and let (X, d_b, s) the b-metric space constructed in Remark 1.2. Then, it is clear that (X, d_b, s) is complete if and only if (X, d) is complete.

2. The results

Various authors have investigated the problem of extending Caristi's fixed point theorem to the realm of b-metric spaces [3, 8, 12, 14, 16]. In particular, Dung and Hang presented in [8, Example 2.8] an instance of a complete b-metric space (X, d, s), with $s = 8$, for which there is a self mapping T of X and a lower semicontinuous and bounded from below function $\varphi : X \to \mathbb{R}$ such that $d(x, Tx) \leq \varphi(x) - \varphi(Tx)$ for all $x \in X$, but T has no any fixed point. This key example shows that for obtaining a suitable extension of Caristi's fixed point theorem to b-metric spaces is necessary to add some ingredient or include some appropriate modification on the conditions of the self mapping or on the contractivity condition provided by the function φ. In this direction, our approach takes as a starting point the following relevant theorem obtained by Miculescu and Mihail in [16]. (Recall that a self mapping of a b-metric space (X, d, s) is continuous provided that it is continuous from the topological space (X, τ_d) into itself.)

Theorem 2.1. ([16, Theorem 3.1]). *Let T be a continuous self mapping of a complete b-metric space (X, d, s), $\varphi : X \to \mathbb{R}^+$ and $r > 1$ such that $d(x, Tx) \leq \varphi(x) - r\varphi(Tx)$ for all $x \in X$. Then T has a fixed point.*

We also need two definitions.

Definition 2.2. Let(X, d, s) be a b-metric space. We say that a function $\varphi : X \to \mathbb{R}^+$ is 0-lower semicontinuous (0-lsc in short) provided that the next condition holds:

If $x_n \to x$ and $\lim_n \varphi(x_n) = 0$, then $\varphi(x) = 0$.

Remark 2.3. Obviously, every lower semicontinuous function $\varphi : X \to \mathbb{R}^+$ is 0-lsc. On the other hand, it is not hard to found examples of 0-lsc functions that are not lower semicontinuous. For instance, let $d_{\mathbb{R}}$ be the usual metric on $\mathbb{R}$ and $\varphi : \mathbb{R} \to \mathbb{R}^+$ defined by $\varphi(x) = x$ if $x \in [0, 1]$ and $\varphi(x) = 0$ otherwise. Then φ is 0-lsc but not lower semicontinuous for the (complete) metric space $(\mathbb{R}, d_{\mathbb{R}})$.

Note that the contraction condition of Theorem 2.1 implies that $\varphi(u)$=0 whenever u is a fixed point of T. In order to avoid this restriction we slightly modify that contraction condition in Definition 2.4 below.

Definition 2.4. A self mapping T of a b-metric space (X, d, s), with $s > 1$, is said to be a b-Caristi mapping if there exist a constant $r \in (1, s]$ and a 0-lsc function $\varphi : X \to \mathbb{R}^+$ such that for each $x \in X$,

$$d(x, Tx) > 0 \implies d(x, Tx) \leq \varphi(x) - r\varphi(Tx).$$

Theorem 2.5. *Let (X, d, s) be a complete b-metric space with $s > 1$. Then, every b-Caristi mapping of X has a fixed point.*

Proof. Let T be a b-Caristi mapping of X. Then, there are an $r \in (1, s]$ and a 0-lsc function $\varphi : X \to \mathbb{R}^+$ such that for each $x \in X$,

$$d(x, Tx) > 0 \implies d(x, Tx) \leq \varphi(x) - r\varphi(Tx). \tag{2.1}$$

Fix $x_0 \in X$ and put $x_n := T^n x_0$ for all $n \in \omega$. If $T^n x_0 = T^{n+1} x_0$ for some $n \in \omega$, $T^n x_0$ is a fixed point of T. So, in the sequel we assume that $T^n x_0 \neq T^{n+1} x_0$ for all $n \in \omega$.

Then $r\varphi(x_{n+1}) < \varphi(x_n)$ for all $n \in \omega$, and hence $(r^n \varphi(x_n))_{n\in\omega}$ is a strictly decreasing sequence in $\mathbb{R}^+$. Therefore $\lim_n r^n \varphi(x_n) = L \geq 0$ where $L = \inf_{n\in\omega} r^n \varphi(x_n)$. Since $\lim_n r^n = \infty$ we get $\lim_n \varphi(x_n) = 0$.

For each $n \in \omega$ we have:

$$\sum_{k=0}^{n} r^k d(x_k, x_{k+1}) \leq r\varphi(x_0),$$

which implies that the series $\sum_{k=0}^{\infty} r^k d(x_k, x_{k+1})$ is convergent and by [16, Corollary 2.8], $(x_n)_{n\in\omega}$ is a Cauchy sequence.

Let $y \in X$ be such that $d(y, x_n) \to 0$. Since φ is 0-lsc and $\lim_n \varphi(x_n) = 0$, we get $\varphi(y) = 0$. If $y \neq Ty$, we deduce from the contraction condition (2.1) that $\varphi(y) > r\varphi(Ty) > \varphi(Ty)$, and thus $\varphi(Ty) < 0$, a contradiction. Hence $y = Ty$. This concludes the proof. □

Next we shall apply Theorem 2.5 to deduce a full b-metric generalization of the Banach contraction principle. This generalization was obtained in [8, Theorem 2.1] as a consequence of a b-metric version of the well-known Matkowski's fixed point theorem [15, Theorem 1.2], established by Czerwik in [6, Theorem 1]. Kajántó and Lukács observed in [11] that the original proof of [6, Theorem 1] has an inaccuracy; then, they gave a correct proof, validating thereby [6, Theorem 1] and, consequently, also [8, Theorem 2.1] (another (correct) proof of Czerwik's theorem was given by Miculescu and Mihail in [17]).

Theorem 2.6. ([8, Theorem 2.1]). *Let (X, d, s) be a complete b-metric space with $s > 1$. Let T be a self mapping of X such that there is a constant $c \in (0, 1)$ satisfying $d(Tx, Ty) \leq cd(x, y)$ for all $x, y \in X$. Then T has a unique fixed point.*

Proof. Let $r = \min\{s, (1 + c)/2c\}$ and let $\varphi : X \to \mathbb{R}^+$ defined as $\varphi(x) = 2d(x, Tx)/(1-c)$ for all $x \in X$. Observe that $r \in (1, s]$. Moreover φ is 0-lsc on (X, d, s). Indeed, suppose $d(x, x_n) \to 0$ and $\lim_n \varphi(x_n) = 0$. Since , for each $n \in \omega$,

$$\begin{aligned} d(x, Tx) &\leq sd(x, x_n) + s^2(d(x_n, Tx_n) + d(Tx_n, Tx)) \\ &\leq sd(x, x_n) + \frac{s^2(1-c)}{2}\varphi(x_n) + cs^2 d(x_n, x), \end{aligned}$$

we deduce that $d(x, Tx) = 0$, i.e., $\varphi(x) = 0$.

Next we show that $r\varphi(Tx) \leq \varphi(x) - d(x, Tx)$ for all $x \in X$. Indeed, we have

$$\begin{aligned} r\varphi(Tx) &= \frac{2r}{1-c} d(Tx, T^2x) \leq \frac{1+c}{c(1-c)} d(Tx, T^2x) \\ &\leq \frac{1+c}{1-c} d(x, Tx) = \frac{2}{1-c} d(x, Tx) - d(x, Tx) \\ &= \varphi(x) - d(x, Tx). \end{aligned}$$

Thus, all conditions of Theorem 2.5 are fulfilled and hence T has a fixed point. Finally, its uniqueness follows immediately from the contraction condition. □

Next we give an easy example where we can apply Theorem 2.5 for an $r \in (1, s)$ and a 0-lsc function φ but not for s and for such φ.

Example 2.7. Let (X, d, s) be the complete b-metric space where $X := \mathbb{R}^+$, $s = 2$, and d is the b-metric on X given by $d(x, y) = |x - y|^2$ for all $x, y \in X$ (see Remarks 1.2 and 1.3).

Let T be the self mapping of X given by $Tx = cx$, with c constant such that $c \in (0, 1)$. By Theorem 2.6, T has a unique fixed point. Let $r_c = \min\{2, (1 + c)/2c\}$ and $\varphi : X \to \mathbb{R}^+$ defined by $\varphi(x) = 2d(x, Tx)/(1 - c)$ for all $x \in X$. In accordance to the proof of Theorem 2.6 we obtain $d(x, Tx) \leq \varphi(x) - r_c\varphi(Tx)$ for all $x \in X$.

Choose any $c \in (0, 1)$ such that $1 + c < 4c^2$, i.e. $(1 + \sqrt{17})/8 < c < 1$. Then $r_c = (1 + c)/2c < (1 + c)/2c^2 < 2$, and we have

$$\begin{aligned} d(x, Tx) &= (1 - c)^2x^2 > 2(1 - c)x^2 - 4(1 - c)c^2x^2 \\ &= \varphi(x) - 2\varphi(Tx), \end{aligned}$$

for all $x \in X\backslash\{0\}$.

Therefore, the conditions of Theorem 2.5 are satisfied for r_c and φ but not for s and φ.

The following is an example where we can apply Theorem 2.5 but not Theorems 2.1 and 2.6.

Example 2.8. Let (X, d, s) be the b-metric space where $X := \mathbb{R}^+$, $s = 2$, and d is the b-metric on X given by $d(x, y) = (\max\{x, y\})^2$ for all $x, y \in X$ with $x \neq y$.

We first note that (X, d, s) is complete because the only non-eventually constant Cauchy sequences are those that τ_d-converges to 0 (equivalently, those that converges to 0 for the usual topology on X).

Let T be the self mapping of X given by $T0 = 0$, $Tx = 1$ for all $x \in (0, 4)$, and $Tx = x^{1/2}$ for all $x \geq 4$. Note that T has two fixed points: 0 and 1. Moreover it is not continuous at $x = 0$.

Now define a function $\varphi : X \to \mathbb{R}^+$ by $\varphi(0) = 2$, $\varphi(x) = 1$ for all $x \in (0, 4)\backslash\{1\}$, $\varphi(1) = 0$, and $\varphi(x) = 2x^2$ for all $x \geq 4$.

We going to check that T is a b-Caristi mapping on (X, d, s) (for this φ and for $r = 2$).

Indeed φ is a 0-lsc function because if $x_n \to x$ with $\lim_n \varphi(x_n) = 0$, then $x_n = 1$ eventually, so $x = 1$, and $\varphi(1) = 0$. Note also that φ is not lower semicontinuous because $1/n \to 0$ and, nevertheless, $\varphi(0) = 2$ and $\varphi(1/n) = 1$ for all $n \in \mathbb{N}$.

Finally, take $x \in X$ such that $d(x, Tx) > 0$. If $x \in (0, 4)\backslash\{1\}$ we get

$$d(x, Tx) = d(x, 1) = 1 = \varphi(x) - 2\varphi(1) = \varphi(x) - 2\varphi(Tx),$$

and if $x \geq 4$ we distinguish two cases:

Case 1. $Tx \geq 4$. Then

$$d(x, Tx) = d(x, x^{1/2}) = x^2 \leq 2x^2 - 4x = \varphi(x) - 2\varphi(x^{1/2}) = \varphi(x) - 2\varphi(Tx).$$

Case 2. $Tx < 4$. Then $Tx = x^{1/2} > 2$, so $\varphi(Tx) = 1$, and hence

$$d(x, Tx) = x^2 < 2x^2 - 2 = \varphi(x) - 2\varphi(Tx).$$

Therefore, all conditions of Theorem 2.5 are satisfied. However, we cannot apply Theorems 2.1 and 2.6 because T is not continuous.

We now prove our main result.

Theorem 2.9. *A b-metric space (X, d, s), with $s > 1$, is complete if and only if every b-Caristi mapping on it has a fixed point.*

Proof. The "only if" part follows from Theorem 2.5.

To prove the "if" part we shall reason by contradiction. Let (X, d, s), with $s > 1$, be a non-complete b-metric space for which every b-Caristi mapping has a fixed point.

Since (X, d, s) is not complete, there exists a Cauchy sequence $(x_n)_{n\in\omega}$ of distinct points in X that does not converge. Hence, there is a subsequence $(x_{n(k)})_{k\in\omega}$ of $(x_n)_{n\in\omega}$ such that $d(x_{n(k)}, x_m) < 2^{-(k+1)}$ for all $k \in \omega$ and $m \geq n(k)$. So, in particular, $d(x_{n(k)}, x_{n(k+1)}) < 2^{-(k+1)}$ for all $k \in \omega$.

Obviously $(x_{n(k)})_{k\in\omega}$ does not converge because $(x_n)_{n\in\omega}$ is a non-convergent Cauchy sequence.

We proceed to construct a b-Caristi mapping free of fixed points, which will yields a contradiction.

Put $F := \{x_{n(k)} : k \in \omega\}$, and define a self mapping T of X, free of fixed points, as follows:

$Tx_{n(k)} = x_{n(k+1)}$ for all $k \in \omega$, and $Tx = x_{n(0)}$ whenever $x \in X\backslash F$.

Now fix $r \in (1, s]$ with $r < 2$ and define a function $\varphi : X \to \mathbb{R}^+$ by

$$\varphi(x_{n(k)}) = 2^{-k}/(2 - r)$$

for all $k \in \omega$, and $\varphi(x) = d(x, x_{n(0)}) + r/(2 - r)$ for all $x \in X\backslash F$.

We show that T is a b-Caristi mapping on X.

We first note that φ is 0-lsc because if $y_j \to y$, we get that $y_j \in X\backslash F$ eventually (in fact $y_j \in X\backslash(F \cup \{x_n : x_n \in X\backslash F\})$ eventually), and thus $\varphi(y_j) > r/(2 - r)$ eventually.

Finally, for each $k \in \omega$ we have

$$\begin{aligned} d(x_{n(k)}, Tx_{n(k)}) &= d(x_{n(k)}, x_{n(k+1)}) < 2^{-(k+1)} \\ &= \frac{2^{-k}}{2-r} - r\frac{2^{-(k+1)}}{2-r} \\ &= \varphi(x_{n(k)}) - r\varphi(Tx_{n(k)}), \end{aligned}$$

and for each $x \in X\backslash F$,

$$d(x, Tx) = d(x, x_{n(0)}) = \varphi(x) - r\varphi(x_{n(0)}) = \varphi(x) - r\varphi(Tx).$$

Therefore T is a b-Caristi mapping of X. This concludes the proof. □

Remark 2.10. Theorem 2.1 has the advantage that the function φ is not necessarily 0-lsc. However, it does not allow to characterize the completeness of b-metric spaces, which follows from Remark 1.2 and the example given by Suzuki and Takahashi in [18] of a non-complete metric space for which every continuous self mapping has a fixed point.

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