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CYCLIC PROJECTION METHODS FOR SOLVING THE SPLIT COMMON ZERO POINT PROBLEM IN BANACH SPACES

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Abstract. We study the split common zero point problem with multiple output sets (SCZPPMOS, for short) in Banach spaces. We introduce two new cyclic projection algorithms for finding a solution to SCZPPMOS and establish the strong convergence of the sequences generated by them.
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1. Introduction

Let E and F be two Banach spaces and let (P_1) and (P_2) be two given problems on E and F, respectively. Let $T:\ E \to F$ be a given operator which we call the transfer mapping. The split problem corresponding to (P_1) and (P_2) on the two Banach spaces E and F is to find an element $x \in E$ such that x is a solution to Problem (P_1) and its image under the transfer mapping T is a solution to Problem (P_2). We denote this split problem by (P). This model problem was first introduced by Censor and Elfving [12] in 1994 for modeling certain inverse problems. More precisely, they considered the following split feasibility problem (SFP): Find an element in a given nonempty, closed and convex subset of a real Hilbert space such that its image under a given transfer mapping belongs to a given nonempty, closed and convex subset of the image space. It is well known by now that the SFP plays an important role in medical image reconstruction and in signal processing (see, for example, [6, 7]). Since then, the SFP has attracted the attention of many mathematicians, who have proposed and studied many algorithms and iterative methods for solving it. See, for example, [6, 7, 9, 11, 12, 10, 24, 31, 44, 45, 46, 47] and references therein.

As a matter of fact, several problems of the SFP type have been studied. We mention, for instance, the multiple-set SFP (MSSFP) (see, for example, [13, 21]), the split common fixed point problem (SCFPP) (see, for example, [15, 22, 33, 26, 30]), the split variational inequality problem (SVIP) (see, for example, [14, 16]) and the split common null point problem (SCNPP) (see, for example, [8, 25, 27, 28, 35, 36, 37, 42, 39, 41, 40]).

In 2020 Reich and Tuyen [25] studied a general case of Problem (P) and proposed a model split feasibility type problem. More precisely, they considered the following model problem: Let $E_1, E_2, \dots, E_N$ be Banach or Hilbert spaces and let the transfer mappings $T_i: E_i \to E_{i+1}$, $i = 1, 2, \dots, N-1$, be given. Suppose that (P_i), $i = 1, 2, \dots, N$, are N given problems on E_i, respectively. The general case of Problem (P) is to find an element x in E_1 such that x is a solution to (P_1), $T_1(x)$ is a solution to (P_2), ..., and $T_{N-1}(T_{N-2}(...T_2(T_1(x))))$ is a solution to (P_N). This problem is denoted by (GP).

Next, Reich et al. [24, 31, 28] introduced and studied the following model of the split feasibility problem with multiple output sets in different image spaces. Let E and E_i, $i = 1, 2, \dots, N$, be Banach or Hilbert spaces and let the transfer mappings $T_i: E \to E_i$, $i = 1, 2, \dots, N$, be given. Suppose that (P_0) and (P_i), $i = 1, 2, \dots, N$, are given $N+1$ problems on E and E_i, respectively. Then the problem is to find an element x in E such that x is a solution to (P_0), and $T_i(x)$ is a solution to (P_i) for all $i = 1, 2, \dots, N$. We denote this problem by GPMOS. It is not difficult to see that Problem GPMOS is a general case of Problem (GP) (see, for example, [24, Remark 1.1]). Some other results regarding this type of problem can be found in [20, 29, 32].

In this paper, we study the model of Problem GPMOS where E is a uniformly convex and smooth Banach space, E_i, $i = 1, 2, \dots, N$, are smooth Banach spaces, and (P_0) and (P_i), $i = 1, 2, \dots, N$, are the problems of finding a zero point of a maximal monotone operator on E and E_i, respectively. More precisely, we study Problem SCZPPMOS. By using the definition of an enlargement of a maximal monotone operator (see, for example, [5]), we propose two new projection algorithms for finding a solution to Problem SCZPPMOS (see Section 3). Our algorithms do not depend on the norm of the transfer mappings. In Section 4 we introduce two applications of our main results to solving the split minimum point problem and the split feasibility problem with multiple output sets. Finally, in Section 5, we implement a numerical example and compare the effectiveness of the proposed algorithms with some previous results.

2. Preliminaries

Let E be a real Banach space with norm $\|\cdot\|$ and let E^* be its dual. We denote by $\langle x, f\rangle$ the value of $f \in E^*$ at the point $x \in E$. When $\{x_n\}$ is a sequence in E, we use the symbols $x_n \to x$ and $x_n \rightharpoonup x$ to denote the strong convergence and the weak convergence of the sequence $\{x_n\}$ to x, respectively.

Let J_E denote the normalized duality mapping from E into 2^{E^*} given by

$$J_E(x) = \left\{f \in E^*: \ \langle x, f\rangle = \|x\|^2 = \|f\|^2\right\}, \ \forall x \in E.$$

Remark 2.1. In any Banach space, we have $J_E(x) = \partial(\|x\|^2/2)$ for all $x \in E$, where $\partial(\|x\|^2/2)$ is the subdifferential of the function $\|x\|^2/2$ (see, for example, [17, Example 2.9, page 16]). In a Hilbert space H it is easy to see that $J_H(x) = x$ for all $x \in H$ (see, for example, [17, Proposition 4.8, page 29]).

We always use S_E to denote the unit sphere of a Banach space E, that is,

$$S_E = \{x \in E : \ \|x\| = 1\}.$$

A Banach space E is said to be strictly convex if for all $x, y \in S_E$ with $x \neq y$, we have $\|x + y\| < 2$ or equivalently, $\|(1-t)x + ty\| < 1$ for all $t \in (0, 1)$.

A Banach space E is said to be uniformly convex (see, for example, [17, 19]) if for any $\varepsilon \in (0, 2]$ and for all $x, y \in E$ with $\|x\| \leq 1, \ \|y\| \leq 1, \ \|x - y\| \geq \varepsilon$, there exists a positive real number $\delta = \delta(\varepsilon) > 0$ such that $\|x + y\|/2 \leq 1 - \delta$.

Remark 2.2. If E is a uniformly convex Banach space and if $\{x_n\}$ and $\{y_n\}$ are two sequences in E such that

$$\lim_{n\to\infty} \|x_n\| = \lim_{n\to\infty} \|y_n\| = d \geq 0 \text{ and } \lim_{n\to\infty} \frac{\|x_n + y_n\|}{2} = d,$$

then $\lim_{n\to\infty} \|x_n - y_n\| = 0$.

Recall that a Banach space E is said to have the Kadec-Klee property if for any sequence $\{x_n\} \subset E$ such that $\|x_n\| \to \|x\|$ and $x_n \rightharpoonup x$ as $n \to \infty$, we have $x_n \to x$ as $n \to \infty$. It is well known that every uniformly convex Banach space has the Kadec-Klee property (see, for example, [17, Proposition 2.8] or [23]).

A Banach space E is said to be smooth if for each x in S_E there exists a unique linear functional $j_x \in E^*$ such that $\langle x, j_x\rangle = \|x\|$ and $\|j_x\| = 1$ (see, for example, [1, Definition 2.6.1, page 91]).

Next, we recall several properties of the normalized duality mapping J_E of a real Banach space E (see, for example, [1, 17, 18]):

i) E is reflexive if and only if J_E is surjective;
ii) If E is smooth or E^* is strictly convex, then J_E is single-valued;
iii) If E is a smooth, strictly convex and reflexive Banach space, then J_E is a single-valued bijection;
iv) If E^* is uniformly convex, then J_E is uniformly continuous on each bounded subset of E.

It is also known that, if E is a smooth, strictly convex and reflexive Banach space, and C is a nonempty, closed and convex subset of E, then for each $x \in E$, there exists a unique point $z \in C$ such that $\|x - z\| = \inf_{y\in C} \|x - y\|$. The mapping $P_C : \ E \to C$ defined by $P_C x = z$ for all $x \in E$ is called the metric projection from E onto C.

Let $A : \ E \rightrightarrows E^*$ be an operator. The effective domain of A is denoted by $D(A)$, that is, $D(A) := \{x \in E : \ Ax \neq \emptyset\}$. Recall that A is called a monotone operator if $\langle x - y, u - v\rangle \geq 0$ for all $x, y \in D(A)$ and for all $u \in Ax, \ v \in A(y)$. The graph of A is denoted by $Gr(A)$. It is defined by $Gr(A) := \{(x, u) \in E \times E^* : \ x \in D(A), u \in A(x)\}$. A monotone operator A on E is called maximal monotone if its graph is not properly contained in the graph of any other monotone operator on E. It is known that if A is a maximal monotone operator on E, and if E is a uniformly convex and smooth Banach space, then $R(J_E + rA) = E^*$ for all $r > 0$, where $R(J_E + rA)$ is the range of $J_E + rA$ (see, for example, [4], [2, Theorem 1.7.13, page 57]). For each $x \in E$ and $r > 0$, there exists a unique point $x_r \in E$ such that $0 \in J_E(x_r - x) + rAx_r$. We define a mapping Q_r^A by $Q_r^A x := x_r$. The mapping Q_r^A is called the metric resolvent of A.

The zero point set of a maximal monotone operator A is defined as follows: $\mathrm{Zer}(A) := \{z \in E : 0 \in Az\}$. It is known that $\mathrm{Zer}(A)$ is a closed and convex subset of E (see, for example, [2, Corollary 1.4.10, page 31]).

Let $A : \rightrightarrows E^*$ be a maximal monotone operator. In [5], for each $\varepsilon \geq 0$, Burachik and Svaiter defined $A^\varepsilon(x)$, an ε-enlargement of A, as follows:

$$A^\varepsilon x := \{u \in E^* : \langle y - x, v - u\rangle \geq -\varepsilon \quad \forall y \in E,\ v \in Ay\}.$$

It is easy to see that $A^0 x = Ax$ and if $0 \leq \varepsilon_1 \leq \varepsilon_2$, then $A^{\varepsilon_1} x \subseteq A^{\varepsilon_2} x$ for any $x \in E$ (see, for example, [5, Lemma 3.1]). The use of elements in A^ε instead of A allows us an extra degree of freedom which is very useful in various applications.

The following lemmas are needed in the sequel for the proof of our main theorems.

Lemma 2.3. (see, for example, [1, Theorem 2.8.17, page 105]) *Let E be a Banach space. Then the following statements are equivalent:*

i) *E is uniformly convex.*

ii) *For any $1 < k < \infty$ and $r > 0$, there exists a strictly increasing convex function $g_r : \mathbb{R}_+ \to \mathbb{R}_+$ such that $g_r(0) = 0$ and*

$$\|tx + (1-t)y\|^k \leq t\|x\|^k + (1-t)\|y\|^k - t(1-t)g_r(\|x - y\|)$$

for all $t \in [0,1]$ and for all $x, y \in E$ with $\max\{\|x\|, \|y\|\} \leq r$.

Lemma 2.4. (see, for example, [19, Proposition 3.4, page 13]) *Let E be a smooth, strictly convex and reflexive Banach space. Let C be a nonempty, closed and convex subset of E, and let $x_1 \in E$ and $z \in C$. Then the following conditions are equivalent:*

i) *$z = P_C x_1$;*

ii) *$\langle y - z, J_E(x_1 - z)\rangle \leq 0 \quad \forall y \in C$.*

Lemma 2.5. (see, for example, [5, Proposition 3.4]) *The graph of $A^\varepsilon : \mathbb{R}_+ \times E \rightrightarrows E^*$ is demiclosed, that is, the statements below hold:*

i) *If the sequence $\{x_n\} \subset E$ converges strongly to x_0, the sequence $\{u_n \in A^{\varepsilon_n} x_n\}$ converges weakly to u_0 in E^* and the sequence $\{\varepsilon_n\} \subset \mathbb{R}_+$ converges to ε, then $u_0 \in A^\varepsilon x_0$;*

ii) *If the sequence $\{x_n\} \subset E$ converges weakly to x_0, the sequence $\{u_n \in A^{\varepsilon_n} x_n\}$ converges strongly to u_0 in E^* and the sequence $\{\varepsilon_n\} \subset \mathbb{R}_+$ converges to ε, then $u_0 \in A^\varepsilon x_0$.*

Lemma 2.6. (see, for example, [1, Theorem 1.9.10, page 39]) *Let E be a Banach space and let $\{x_n\}$ be a sequence in E. Suppose that $\{x_n\}$ converges weakly to some point $x \in E$. Then we have $\|x\| \leq \liminf_{n\to\infty} \|x_n\|$.*

3. Main results

Let E be a uniformly convex and smooth Banach space, and let E_i, $i = 1, 2, \ldots, N$, be smooth Banach spaces. Let $A : E \rightrightarrows E^*$ and $A_i : E_i \rightrightarrows E_i^*$, $i = 1, 2, \ldots, N$, be maximal set-valued operators. Let $T_i : E \to E_i$, $i = 1, 2, \ldots, N$, be bounded linear operators. Assume that

$$\Omega := \mathrm{Zer}(A) \bigcap \big(\cap_{i=1}^N T_i^{-1}(\mathrm{Zer}(A_i)) \big) \neq \emptyset.$$

We consider the following problem:

$$\text{Find an element in } \Omega. \tag{SCZPPMOS}$$

Assume that $\{\mu_n\}$ is a sequence of positive real numbers and that $\{\varepsilon_n\}$ is a sequence of nonnegative real numbers. We study the strong convergence of the proposed algorithms in this paper under the following conditions on the parameters μ_n and ε_n.

(C1) $\inf_n\{\mu_n\} = \mu > 0$;
(C2) $\lim_{n\to\infty} \varepsilon_n\mu_n = 0$.

Let $\mathcal{N} = \{0, 1, \dots, N\}$. We recall that a mapping $\mathrm{Ind}: \mathbb{N} \to \mathcal{N}$ is called an index control mapping if for each $i \in \mathcal{N}$, there is a natural number M_i such that

$$i \in \{\mathrm{Ind}(n), \mathrm{Ind}(n+1), \dots, \mathrm{Ind}(n+M_i-1)\} \quad \forall n \in \mathbb{N}.$$

Example 3.1. Let $\mathcal{N} = \{0, 1, 2, \dots, N\}$.

The mapping $\mathrm{Ind}: \mathbb{N} \to \mathcal{N}$ defined by

$$\mathrm{Ind}(n) = n \mod (N+1) \quad \forall n \in \mathbb{N}$$

is an index control mapping (see, for example, [3]).

3.1. **Hybrid projection algorithm.** Let $E_0 = E$, $A_0 = A$, and let $T_0 = I^E$ be the identity operator on E. Below we propose an algorithm for solving Problem (SCZPPMOS).

Algorithm 1 The hybrid projection algorithm for solving Problem (SCZPPMOS)

For any initial guess $x_0 \in E$, define the sequence $\{x_n\}$ as follows:
Step 1 Compute $y_n = T_{\mathrm{Ind}(n)}x_n$.
Step 2 Find an element $z_n \in E_{\mathrm{Ind}(n)}$ such that

$$0 \in J_{E_{\mathrm{Ind}(n)}}(z_n - y_n) + \mu_n A^{\varepsilon_n}_{\mathrm{Ind}(n)}(z_n). \tag{3.1}$$

Step 3 Define the subsets C_n and Q_n by

$$C_n = \{z \in E : \langle T_{\mathrm{Ind}(n)}z - z_n, J_{E_{\mathrm{Ind}(n)}}(y_n - z_n)\rangle \le \mu_n\varepsilon_n\},$$
$$Q_n = \{z \in E : \langle z - x_n, J_E(x_0 - x_n)\rangle \le 0\}.$$

Step 4 Compute $x_{n+1} = P_{C_n\cap Q_n}x_0,\ n \ge 0$, and go to Step 1.

The strong convergence of the sequences generated by Algorithm 1 is established in the following theorem.

Theorem 3.2. *If the sequences $\{\mu_n\}$ and $\{\varepsilon_n\}$ satisfy conditions* (C1) *and* (C2)*, then the sequence $\{x_n\}$ generated by Algorithm 1 converges strongly to $P_\Omega x_0$.*

Proof. We divide the proof of this theorem into seven claims as follows.
Claim 1. The sequence $\{x_n\}$ is well defined.

First, we show that C_n and Q_n are closed half-spaces of E.

Indeed, we rewrite the definitions of the subsets C_n and Q_n in the following forms:

$$\begin{aligned} C_n &= \{z \in E : \langle T_{\mathrm{Ind}(n)}z, J_{E_{\mathrm{Ind}(n)}}(y_n - z_n)\rangle \le \langle z_n, J_{E_{\mathrm{Ind}(n)}}(y_n - z_n)\rangle + \mu_n\} \\ &= \{z \in E : \langle z, T^*_{\mathrm{Ind}(n)}J_{E_{\mathrm{Ind}(n)}}(y_n - z_n)\rangle \le \langle z_n, J_{E_{\mathrm{Ind}(n)}}(y_n - z_n)\rangle + \mu_n\varepsilon_n\} \end{aligned}$$

and

$$Q_n = \{z \in E : \langle z, J_{E_{\mathrm{Ind}(n)}}(x_0 - x_n)\rangle \le \langle x_n, J_{E_{\mathrm{Ind}(n)}}(x_0 - x_n)\rangle\}.$$

Now it is clear that C_n and Q_n are indeed closed half-spaces of E.

We next show that $\Omega \subset C_n \cap Q_n$ for all $n \ge 0$. Indeed, take any $p \in \Omega$. It follows from (3.1) that

$$\frac{1}{\mu_n} J_{E_{\mathrm{Ind}(n)}}(y_n - z_n) \in A^{\varepsilon_n}_{\mathrm{Ind}(n)}(z_n). \tag{3.2}$$

Thus, using (3.2), $0 \in A_{\mathrm{Ind}(n)}T_{\mathrm{Ind}(n)}p$ and the definition of $A^{\varepsilon_n}_{\mathrm{Ind}(n)}$, we obtain

$$\langle T_{\mathrm{Ind}(n)}p - z_n, -\frac{1}{\mu_n} J_{E_{\mathrm{Ind}(n)}}(y_n - z_n)\rangle \ge -\varepsilon_n.$$

This is equivalent to

$$\langle T_{\mathrm{Ind}(n)}p - z_n, J_{E_{\mathrm{Ind}(n)}}(y_n - z_n)\rangle \le \varepsilon_n \mu_n,$$

which implies that $p \in C_n$.

It is clear that $Q_0 = E$ and hence $\Omega \subset Q_0$. Suppose that $\Omega \subset Q_n$ for some $n \ge 0$. It follows from $x_{n+1} = P_{C_n \cap Q_n} x_0$, $p \in \Omega \subset C_n \cap Q_n$ and Lemma 2.4 that

$$\langle p - x_{n+1}, J_{E_{\mathrm{Ind}(n)}}(x_0 - x_{n+1})\rangle \le 0.$$

This implies that $p \in Q_{n+1}$ and hence $\Omega \subset Q_{n+1}$. So, using mathematical induction, we conclude that $\Omega \subset Q_n$ for all $n \ge 0$. Combining this with $\Omega \subset C_n$ for all $n \ge 0$, we obtain that $\Omega \subset C_n \cap Q_n$ for all $n \ge 0$. Thus $C_n \cap Q_n$ is a nonempty, closed and convex subset of E, and hence the metric projection of x_0 onto $C_n \cap Q_n$ always exists. Therefore the sequence $\{x_n\}$ is well defined, as claimed.

Claim 2. The sequence $\{x_n\}$ is bounded.

Fix an element $p \in \Omega \subset Q_n$. It follows from the definition of Q_n and Lemma 2.4 that $x_n = P_{Q_n} x_0$. Thus, using the definition of the metric projection, we have

$$\|x_n - x_0\| \le \|p - x_0\|, \ \forall n \ge 0. \tag{3.3}$$

This implies that the sequence $\{x_n\}$ is bounded, as claimed.

Claim 3. There exists the finite limit $\lim_{n\to\infty} \|x_n - x_0\| = l$.

It follows from $x_{n+1} = P_{C_n \cap Q_n} x_0 \in Q_n$ and the definition of Q_n that

$$\begin{aligned} 0 &\ge \langle x_{n+1} - x_n, J_{E_{\mathrm{Ind}(n)}}(x_0 - x_n)\rangle \\ &= \langle x_{n+1} - x_0 + x_0 - x_n, J_{E_{\mathrm{Ind}(n)}}(x_0 - x_n)\rangle \\ &= \langle x_{n+1} - x_0, J_{E_{\mathrm{Ind}(n)}}(x_0 - x_n)\rangle + \|x_n - x_0\|^2. \end{aligned}$$

This implies that

$$\|x_n - x_0\|^2 \le \langle x_0 - x_{n+1}, J_{E_{\mathrm{Ind}(n)}}(x_0 - x_n)\rangle \le \|x_{n+1} - x_0\| \|x_n - x_0\|,$$

which yields that $\|x_n - x_0\| \le \|x_{n+1} - x_0\|$. Combining this with the boundedness of the sequence $\{x_n\}$, we infer that there exists the finite limit $\lim_{n\to\infty} \|x_n - x_0\| = l$, as claimed.

Claim 4. The sequence $\{x_n\}$ is asymptotically regular, that is,

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0.$$

It follows from the facts that x_n and x_{n+1} belong to Q_n, and from the convexity of Q_n that $\dfrac{x_n + x_{n+1}}{2}$ also belongs to Q_n. Since $x_n = P_{Q_n}x_0$, we have

$$\|x_n - x_0\| \le \left\| \frac{x_n + x_{n+1}}{2} - x_0 \right\| \le \frac{1}{2}\Big(\|x_n - x_0\| + \|x_{n+1} - x_0\| \Big).$$

Combining this with $\lim_{n\to\infty} \|x_n - x_0\| = l$, we obtain

$$\left\| \frac{x_n + x_{n+1}}{2} - x_0 \right\| \to l,$$

that is,

$$\lim_{n\to\infty} \frac{\|(x_n - x_0) + (x_{n+1} - x_0)\|}{2} = l.$$

Thus, using Remark 2.2, we can deduce that

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0,$$

as claimed.

Claim 5. $\lim_{n\to\infty} \|z_n - y_n\| = 0.$

It follows from the fact that $x_{n+1} = P_{C_n \cap Q_n} x_0 \in C_n$ and the definition of C_n that

$$\langle T_{\mathrm{Ind}(n)} x_{n+1} - z_n, J_{E_{\mathrm{Ind}(n)}}(y_n - z_n) \rangle \le \mu_n \varepsilon_n.$$

This is equivalent to

$$\begin{aligned}
\mu_n\varepsilon_n &\ge \langle T_{\mathrm{Ind}(n)} x_{n+1} - y_n + y_n - z_n, J_{E_{\mathrm{Ind}(n)}}(y_n - z_n) \rangle \\
&= \langle T_{\mathrm{Ind}(n)} x_{n+1} - T_{\mathrm{Ind}(n)} x_n, J_{E_{\mathrm{Ind}(n)}}(y_n - z_n) \rangle + \|z_n - y_n\|^2 \\
&\ge -\frac{1}{2}(\|T_{\mathrm{Ind}(n)} x_{n+1} - T_{\mathrm{Ind}(n)} x_n\|^2 + \|z_n - y_n\|^2) + \|z_n - y_n\|^2 \\
&\ge -\frac{1}{2}(\|T_{\mathrm{Ind}(n)}\|^2 \|x_{n+1} - x_n\|^2 + \|z_n - y_n\|^2) + \|z_n - y_n\|^2.
\end{aligned}$$

This, in its turn, implies that

$$\|z_n - y_n\|^2 \le \|T_{\mathrm{Ind}(n)}\|^2 \|x_{n+1} - x_n\|^2 + 2\mu_n\varepsilon_n. \tag{3.4}$$

It now follows from (3.4), $\|x_{n+1} - x_n\| \to 0$ (see Step 4) and $\mu_n\varepsilon_n \to 0$ (assumption (C2)) that

$$\lim_{n\to\infty} \|z_n - y_n\| = 0, \tag{3.5}$$

as claimed.

Claim 6. All weak cluster points of the sequence $\{x_n\}$ belong to Ω.

Indeed, suppose that q is an arbitrary cluster point of the sequence $\{x_n\}$. Then there exists a subsequence $\{x_{m_n}\}$ of $\{x_n\}$ such that $\{x_{m_n}\}$ converges weakly to q.

We now claim that $q \in \Omega$. Indeed, for any $i \in \mathcal{N} = \{0, 1, \ldots, N\}$, there exists a natural number M_i such that

$$i \in \{\mathrm{Ind}(m_n), \mathrm{Ind}(m_n + 1), \ldots, \mathrm{Ind}(m_n + M_i - 1)\} \text{ for all } n.$$

We can remove some elements of the subsequence $\{x_{m_n}\}$, if necessary, to obtain a new subsequence, which is also denoted by $\{x_{m_n}\}$, such that $m_{n+1} \geq m_n + M_i$. Then there is another subsequence $\{x_{p_n}\}$ of $\{x_n\}$, where

$$m_n \leq p_n \leq m_n + M_i - 1 < m_{n+1} \leq p_{n+1},\ i = \mathrm{Ind}(p_n).$$

We have

$$\|x_{p_n} - x_{m_n}\| \leq \sum_{l=m_n}^{m_n+M_i-2} \|x_{l+1} - x_l\| \leq (M_i - 1) \max_{m_n \leq l \leq m_n+M_i-2} \|x_{l+1} - x_l\|.$$

Combining this with $\|x_{n+1} - x_n\| \to 0$, we see that $x_{p_n} - x_{m_n} \to 0$. It now follows from $x_{m_n} \rightharpoonup q$ that $x_{p_n} \rightharpoonup q$. Since T_i is a bounded linear operator, we also have $T_i x_{p_n} \rightharpoonup T_i q$.

It follows from (3.5) that $\|z_{p_n} - y_{p_n}\| \to 0$. Combining this with $y_{p_n} = T_i x_{p_n} \rightharpoonup T_i q$, we infer that $z_{p_n} \rightharpoonup T_i q$. It now follows from (3.1), the fact that $\mathrm{Ind}(p_n) = i$ for all n and condition (C1) that

$$A_i^{\varepsilon_{p_n}}(z_{p_n}) \ni \frac{1}{\mu_n} J_{E_i}(y_{p_n} - z_{p_n}) \to 0.$$

Applying Lemma 2.5 ii) to the sequences $\{z_{p_n}\}$ and $\{\frac{1}{\mu_n} J_{E_i}(y_{p_n} - z_{p_n}) \in A_i^{\varepsilon_{p_n}}(z_{p_n})\}$, we conclude that $T_i q \in \mathrm{Zer}(A_i)$. Since $i \in \{0, 1, \ldots, N\}$ is arbitrary, we can infer that $T_i q \in \mathrm{Zer}(A_i)$ for all $i = 0, 1, \ldots, N$, that is, $q \in \Omega$, as claimed.

Claim 7. $\lim_{n\to\infty} x_n = P_\Omega x_0$.

Suppose $\{x_{k_n}\}$ is a subsequence of $\{x_n\}$ such that $x_{k_n} \rightharpoonup q$. It follows form Claim 6 that $q \in \Omega$.

Now, let $x^\dagger = P_\Omega x_0$. From (3.3), the fact that $x_0 - x_{k_n} \rightharpoonup x_0 - q$ and Lemma 2.6, it follows that

$$\begin{aligned}\|x_0 - x^\dagger\| &\leq \|x_0 - q\| \\ &\leq \liminf_{n\to\infty} \|x_0 - x_{k_n}\| \\ &\leq \limsup_{n\to\infty} \|x_0 - x_{k_n}\| \\ &\leq \|x_0 - x^\dagger\|.\end{aligned}$$

This implies that $\lim_{n\to\infty} \|x_0 - x_{k_n}\| = \|x_0 - q\| = \|x_0 - x^\dagger\|$. Using the definition of $x^\dagger$, we obtain $x^\dagger = q$. Since E is a uniformly convex Banach space, it has the Kadec-Klee property. Using now the facts that $\lim_{n\to\infty} \|x_0 - x_{k_n}\| = \|x_0 - q\|$ and $x_0 - x_{k_n} \rightharpoonup x_0 - q$, we infer that $x_{k_n} \to q = x^\dagger$. Using the uniqueness of $x^\dagger$, we can immediately deduce that $x_n \to x^\dagger$, as claimed.

This completes the proof. □

In Theorem 3.2, if $\varepsilon_n = 0$ for all n, then we obtain the following algorithm for solving Problem (SCZPPMOS).

Corollary 3.3. *If condition* (C1) *is satisfied, then the sequence* $\{x_n\}$ *generated by Algorithm 2 converges strongly to* $P_\Omega x_0$.

Algorithm 2 The hybrid projection algorithm for solving Problem (SCZPPMOS) with $\varepsilon_n = 0$

For any initial guess $x_0 \in E$, define the sequence $\{x_n\}$ as follows:
Step 1 Compute $y_n = T_{\mathrm{Ind}(n)} x_n$.
Step 2 Compute $z_n = Q_{\mu_n}^{A_{\mathrm{Ind}(n)}}(y_n)$.
Step 3 Define the subsets C_n and Q_n by

$$C_n = \{z \in E : \langle T_{\mathrm{Ind}(n)} z - z_n, J_{E_{\mathrm{Ind}(n)}}(y_n - z_n)\rangle \leq 0\},$$
$$Q_n = \{z \in E : \langle z - x_n, J_E(x_0 - x_n)\rangle \leq 0\}.$$

Step 4 Compute $x_{n+1} = P_{C_n \cap Q_n} x_0,\ n \geq 0$, and go to Step 1.

3.2. **Shrinking projection algorithm.** Let $E_0 = E$, $A_0 = A$, and let $T_0 = I^E$ be the identity operator on E. We now introduce the following algorithm for solving Problem (SCZPPMOS).

Algorithm 3 The shrinking projection algorithm for solving Problem (SCZPPMOS)

For any initial guess $x_0 \in E$, let $C_0 = E$ and define the sequence $\{x_n\}$ as follows:
Step 1 Compute $y_n = T_{\mathrm{Ind}(n)} x_n$.
Step 2 Find an element z_n such that

$$0 \in J_{E_{\mathrm{Ind}(n)}}(z_n - y_n) + \mu_n A_{\mathrm{Ind}(n)}^{\varepsilon_n}(z_n).$$

Step 3 Define the subset C_{n+1} by

$$C_{n+1} = \{z \in C_n : \langle T_{\mathrm{Ind}(n)} z - z_n, J_{E_{\mathrm{Ind}(n)}}(y_n - z_n)\rangle \leq \mu_n \varepsilon_n\}.$$

Step 4 Compute $x_{n+1} = P_{C_{n+1}} x_0,\ n \geq 0$, and go to Step 1.

The strong convergence of Algorithm 3 is established in the following theorem.

Theorem 3.4. *If the sequences $\{\mu_n\}$ and $\{\varepsilon_n\}$ satisfy conditions* (C1) *and* (C2)*, then the sequence $\{x_n\}$ generated by Algorithm 3 converges strongly to $P_\Omega x_0$.*

Proof. We divide the proof of this theorem into five claims as follows.
Claim 1. The sequence $\{x_n\}$ is well defined

First, we show that C_n is a closed and convex subset of E for each n. We prove this by mathematical induction. Indeed, it is easy to see that $C_0 = E$ is a closed and convex set. Suppose now that C_n is a closed and convex set for some $n \geq 0$. It follows from

$$\begin{aligned} C_{n+1} = C_n \cap \{z \in E : \ & \langle z, T_{\mathrm{Ind}(n)}^* J_{E_{\mathrm{Ind}(n)}}(y_n - z_n)\rangle \\ & \leq \langle z_n, J_{E_{\mathrm{Ind}(n)}}(y_n - z_n)\rangle + \mu_n \varepsilon_n\} \end{aligned}$$

that C_{n+1} is a closed and convex subset of E. Thus, we can conclude that C_n is a closed and convex subset of E for all $n \geq 0$.

We next prove that $\Omega \subset C_n$ for all $n \geq 0$. Obviously, $C_0 = E \supset \Omega$. Suppose that $\Omega \subset C_n$ for some $n \geq 0$. Take any $p \in \Omega$, that is, $T_i p \in \mathrm{Zer}(A_i)$ for all $i = 0, 1, \ldots, N$.

It follows from $0 \in A_{\mathrm{Ind}(n)}(T_{\mathrm{Ind}(n)}p)$,

$$\frac{1}{\mu_n} J_{E_{\mathrm{Ind}(n)}}(y_n - z_n) \in A^{\varepsilon_n}_{\mathrm{Ind}(n)}(z_n)$$

and the definition of $A^{\varepsilon_n}_{\mathrm{Ind}(n)}$ that

$$\langle T_{\mathrm{Ind}(n)}p - z_n, -\frac{1}{\mu_n} J_{E_{\mathrm{Ind}(n)}}(y_n - z_n)\rangle \geq -\varepsilon_n.$$

This is equivalent to

$$\langle T_{\mathrm{Ind}(n)}p - z_n, J_{E_{\mathrm{Ind}(n)}}(y_n - z_n)\rangle \leq \varepsilon_n \mu_n.$$

Combining this with $p \in \Omega \subset C_n$, we obtain

$$p \in C_{n+1} = C_n \cap \{z \in E : \ \langle T_{\mathrm{Ind}(n)}z - z_n, J_{E_{\mathrm{Ind}(n)}}(y_n - z_n)\rangle \leq \mu_n \varepsilon_n\}.$$

This implies that $\Omega \subset C_{n+1}$. Thus, using mathematical induction, we infer that $\Omega \subset C_n$ for all $n \geq 0$.

Consequently, for each n, C_n is a nonempty, closed and convex subset of E, and hence the nearest point projection of x_0 onto C_n always exists, that is, the sequence $\{x_n\}$ is well defined, as claimed.

Claim 2. The sequence $\{x_n\}$ is bounded. Indeed, for each $p \in \Omega$, it follows from $p \in C_n$ and $x_n = P_{C_n}x_0$ that

$$\|x_n - x_0\| \leq \|p - x_0\|. \tag{3.6}$$

This implies that the sequence $\{x_n\}$ is bounded, as claimed.

Claim 3. The sequence $\{x_n\}$ converges strongly to some point $q \in E$.

For every $m \geq n$, it follows from $x_m = P_{C_m}x_0 \in C_m \subset C_n$, $x_n = P_{C_n}x_0 \in C_n$ and the convexity of C_n that $(x_n + x_m)/2 \in C_n$. Let $r = \sup_n\{\|x_n - x_0\|\} < \infty$. Using $x_n = P_{C_n}x_0$ and applying Lemma 2.3 to $k = 2$, $t = 1/2$, and $x = x_n - x_0$ and $y = x_m - x_0$, we see that there exists a strictly increasing convex function $g_r : \mathbb{R}_+ \to \mathbb{R}_+$ such that $g_r(0) = 0$ and

$$\begin{aligned}
\|x_n - x_0\|^2 &\leq \left\|\frac{x_n + x_m}{2} - x_0\right\|^2 \\
&= \left\|\frac{1}{2}(x_n - x_0) + \frac{1}{2}(x_{n+1} - x_0)\right\|^2 \\
&\leq \frac{1}{2}\|x_n - x_0\|^2 + \frac{1}{2}\|x_{n+1} - x_0\|^2 - \frac{1}{4} g_r(\|x_m - x_n\|).
\end{aligned}$$

This implies that

$$\frac{1}{2} g_r(\|x_m - x_n\|) \leq \|x_m - x_0\|^2 - \|x_n - x_0\|^2. \tag{3.7}$$

This, in its turn, implies that $\|x_{n+1} - x_0\| \geq \|x_n - x_0\|$ for all $n \geq 0$, that is, the sequence $\{\|x_n - x_0\|\}$ is decreasing. Combining this with the boundedness of the sequence $\{x_n\}$, we infer that there exists the finite limit $\lim_{n\to\infty} \|x_n - x_0\| = l$. Thus, it follows from (3.7) and the properties of the function g_r that $\|x_{n+1} - x_n\| \to 0$ as $n, m \to \infty$. Hence, $\{x_n\}$ is a Cauchy sequence. Since E is a Banach space, it follows that the sequence $\{x_n\}$ converges strongly to some point $q \in E$, as claimed.

Claim 4. $\lim_{n\to\infty} \|z_n - y_n\| = 0$.

It follows from the strong convergence of the sequence $\{x_n\}$ that $\|x_{n+1} - x_n\| \to 0$. So, by an argument similar to the one employed in the proof of Claim 5 in Theorem 3.2, we can now finish the proof of this claim.

Claim 5. $\lim_{n\to\infty} x_n = P_\Omega x_0$.

Using the fact that $x_n \to q$ and an argument similar to the one used in the proof of Claim 6 of Theorem 3.2, we obtain that $q \in \Omega$. It now follows from (3.6) that

$$\|x_0 - q\| \leq \|x_0 - p\|, \ \forall p \in \Omega.$$

This inequality implies that $q = x^\dagger = P_\Omega x_0$, as claimed.

This completes the proof of the theorem. □

Remark 3.5. We can prove the strong convergence of the sequence $\{x_n\}$ generated by Algorithm 3 in another way as follows. Since $C_{n+1} \subset C_n$ and $C_n \supset \Omega \neq \emptyset$ for all $n \geq 0$, there exists the limit of the sequence $\{C_n\}$ in the sense of Mosco and $M\text{-}\lim_{n\to\infty} C_n = C = \cap_{n=1}^{\infty} C_n$ (see, for example, [43, Remark 2.1]). Thus the sequence $\{x_n\}$ generated by $x_n = P_{C_n} x_0$ converges strongly to $q = P_C x_0$ (see, for example, [38, Theorem 3.2]).

In Theorem 3.4, if $\varepsilon_n = 0$ for all n, then we have the following algorithm for solving Problem (SCZPPMOS).

Algorithm 4 The shrinking projection algorithm for solving Problem (SCZPPMOS) with $\varepsilon_n = 0$

For any initial guess $x_0 \in E$, let $C_0 = E$ and define the sequence $\{x_n\}$ as follows:

Step 1 Compute $y_n = T_{\mathrm{Ind}(n)} x_n$.

Step 2 Compute $z_n = Q_{\mu_n}^{A_{\mathrm{Ind}(n)}}(y_n)$.

Step 3 Define the subset C_{n+1} by

$$C_{n+1} = \{z \in C_n : \ \langle T_{\mathrm{Ind}(n)} z - z_n, J_{E_{\mathrm{Ind}(n)}}(y_n - z_n)\rangle \leq 0\}.$$

Step 4 Compute $x_{n+1} = P_{C_{n+1}} x_0, \ n \geq 0$, and go to Step 1.

Corollary 3.6. *If condition* (C1) *is satisfied, then the sequence* $\{x_n\}$ *generated by Algorithm 4 converges strongly to* $P_\Omega x_0$.

4. Applications

In this section we introduce some applications of our main results. They involve the split minimum point problem and the split feasibility problem with multiple output sets.

4.1. **Split minimum point problem.** Let E be a Banach space and let $f : E \to (-\infty, \infty]$ be a proper, lower semicontinuous and convex function. Recall that the subdifferential of f is a set-valued mapping $\partial f : E \rightrightarrows E^*$ which is defined by

$$\partial f(x) := \{g \in E^* : \ f(y) - f(x) \geq \langle y - x, g\rangle \ \ \forall y \in E\}$$

for all $x \in E$. It is well known that ∂f is a maximal monotone operator (see, for example, [34]) and that $x_0 \in \arg\min_E f(x)$ if and only if $\partial f(x_0) \ni 0$.

Now, let E be a uniformly convex and smooth Banach space and let E_i, $i = 1, 2, \ldots, N$, be smooth Banach spaces. Let $f : E \to (-\infty, \infty]$ and $f_i : E_i \to (-\infty, \infty]$, $i = 1, 2, \ldots, N$, be proper, lower semicontinuous and convex functions. Let $T_i : E \to E_i$, $i = 1, 2, \ldots, N$, be bounded linear operators. Suppose that

$$\Omega^{SMPP} := \arg\min_E f(x) \bigcap \left(\cap_{i=1}^N T_i^{-1}(\arg\min_{E_i} f_i(x)) \right) \neq \emptyset.$$

We consider the following problem:

$$\text{Find an element in } \Omega^{SMPP}. \tag{4.1}$$

Let $E_0 = E$, $f_0 = f$ and $T_0 = I^E$. It follows from Algorithm 1 and Algorithm 3 that we have the following two algorithms for solving Problem (4.1).

Algorithm 5 The hybrid projection algorithm for solving Problem (4.1)

For any initial guess $x_0 \in E$, define the sequence $\{x_n\}$ as follows:
Step 1 Compute $y_n = T_{\mathrm{Ind}(n)} x_n$.
Step 2 Define an element z_n by

$$0 \in J_{E_{\mathrm{Ind}(n)}}(z_n - y_n) + \mu_n \partial^{\varepsilon_n} f_{\mathrm{Ind}(n)}(z_n).$$

Step 3 Define the subsets C_n and Q_n by

$$C_n = \{z \in E : \langle T_{\mathrm{Ind}(n)} z - z_n, J_{E_{\mathrm{Ind}(n)}}(y_n - z_n) \rangle \le 0\},$$
$$Q_n = \{z \in E : \langle z - x_n, J_E(x_0 - x_n) \rangle \le 0\}.$$

Step 4 Compute $x_{n+1} = P_{C_n \cap Q_n} x_0,\ n \ge 0$, and go to Step 1.

Algorithm 6 The shrinking projection algorithm for solving Problem (4.1)

For any initial guess $x_0 \in E$, let $C_0 = E$ and define the sequence $\{x_n\}$ as follows:
Step 1 Compute $y_n = T_{\mathrm{Ind}(n)} x_n$.
Step 2 Define an element z_n by

$$0 \in J_{E_{\mathrm{Ind}(n)}}(z_n - y_n) + \mu_n \partial^{\varepsilon_n} f_{\mathrm{Ind}(n)}(z_n).$$

Step 3 Define the subset C_{n+1} by

$$C_{n+1} = \{z \in C_n : \langle T_{\mathrm{Ind}(n)} z - z_n, J_{E_{\mathrm{Ind}(n)}}(y_n - z_n) \rangle \le 0\}.$$

Step 4 Compute $x_{n+1} = P_{C_{n+1}} x_0,\ n \ge 0$, and go to Step 1.

Remark 4.1. If $\varepsilon_n = 0$ for all n, then the element z_n in Step 2 of Algorithm 5 and Algorithm 6 is defined by

$$z_n = \arg\min_{E_{\mathrm{Ind}(n)}} \left\{ f_{\mathrm{Ind}(n)}(y) + \frac{1}{2\mu_n} \|y - y_n\|^2 \right\}.$$

The strong convergence of the sequences generated by Algorithm 5 and Algorithm 6 is established in the following theorem.

Theorem 4.2. *If the sequences* $\{\mu_n\}$ *and* $\{\varepsilon_n\}$ *satisfy conditions* (C1) *and* (C2)*, then the sequence* $\{x_n\}$ *generated by either Algorithm 5 or Algorithm 6 converges strongly to* $P_{\Omega^{SMPP}}x_0$*.*

4.2. **Split feasibility problem.** Let C be a nonempty, closed and convex subset of E. Let i_C be the indicator function of C, that is,

$$i_C(x) = \begin{cases} 0, & \text{if } x \in C, \\ \infty, & \text{if } x \notin C. \end{cases}$$

It is not difficult to see that i_C is a proper, semicontinuous and convex function. Therefore its subdifferential ∂i_C is a maximal monotone operator (see, for example, [34]). It is known that

$$\partial i_C(u) = N(u, C) = \{f \in E^* : \ \langle y - u, f\rangle \le 0 \ \ \forall y \in C\},$$

where $N(u, C)$ is the normal cone of C at u.

We denote the metric resolvent of ∂i_C by $Q_r^{\partial i_C}$, where $r > 0$. Suppose $u = Q_r^{\partial i_C} x$ for $x \in E$, that is,

$$\frac{J_E(x-u)}{r} \in \partial i_C(u) = N(u, C).$$

Then we have

$$\langle y - u, J_E(x - u)\rangle \le 0$$

for all $y \in C$. Using Lemma 2.4, we see that $u = P_C x$.

Let E be a uniformly convex and smooth Banach space, and let E_i, $i = 1, 2, \dots, N$, be smooth Banach spaces. Let L and L_i be nonempty, closed and convex subsets of E and E_i, $i = 1, 2, \dots, N$, respectively. Let $T_i : \ E \to E_i$, $i = 1, 2, \dots, N$, be bounded linear operators. Suppose that

$$\Omega^{SFP} := L \bigcap \big(\cap_{i=1}^N T_i^{-1}(L_i)\big) \ne \emptyset.$$

We consider the following problem:

$$\text{Find an element in } \Omega^{SFP}. \tag{4.2}$$

Let $E_0 = E$, $L_0 = L$, and $T_0 = I^E$. Using Algorithm 3 and Algorithm 4, we arrive at the following two algorithms for solving Problem (4.2).

The strong convergence of the sequence $\{x_n\}$ generated by either Algorithm 7 or 8 is provided by the following theorem.

Theorem 4.3. *The sequence* $\{x_n\}$ *generated by either Algorithm 7 or Algorithm 8 converges strongly to* $P_{\Omega^{SFP}}x_0$*.*

Algorithm 7 The hybrid projection algorithm for solving Problem (SCZPPMOS) with $\varepsilon_n = 0$

For any initial guess $x_0 \in E$, define the sequence $\{x_n\}$ as follows:
Step 1 Compute $y_n = T_{\mathrm{Ind}(n)} x_n$.
Step 2 Compute $z_n = P_{L_{\mathrm{Ind}(n)}}(y_n)$.
Step 3 Define the subsets C_n and Q_n by

$$C_n = \{z \in E : \langle T_{\mathrm{Ind}(n)} z - z_n, J_{E_{\mathrm{Ind}(n)}}(y_n - z_n)\rangle \leq 0\},$$
$$Q_n = \{z \in E : \langle z - x_n, J_E(x_0 - x_n)\rangle \leq 0\}.$$

Step 4 Compute $x_{n+1} = P_{C_n \cap Q_n} x_0,\ n \geq 0$, and go to Step 1.

Algorithm 8 The shrinking projection algorithm for solving Problem (SCZPPMOS) with $\varepsilon_n = 0$

For any initial guess $x_0 \in E$, let $C_0 = E$ and define the sequence $\{x_n\}$ as follows:
Step 1 Compute $y_n = T_{\mathrm{Ind}(n)} x_n$.
Step 2 Compute $z_n = P_{L_{\mathrm{Ind}(n)}}(y_n)$.
Step 3 Define the subset C_{n+1} by

$$C_{n+1} = \{z \in C_n : \langle T_{\mathrm{Ind}(n)} z - z_n, J_{E_{\mathrm{Ind}(n)}}(y_n - z_n)\rangle \leq 0\}.$$

Step 4 Compute $x_{n+1} = P_{C_{n+1}} x_0,\ n \geq 0$, and go to Step 1.

5. Numerical experiments

In this section our algorithms are implemented in MATLAB 14a running on DESKTOP-9RLTPS0, Intel(R) Core(TM) i5-10210U CPU @ 1.60GHz with 2.11 GHz and 8GB RAM.

We consider the split feasibility problem with multiple output sets in a finite dimensional Euclidean space and compare our new algorithms (Algorithm 7 and Algorithm 8) with our previous algorithms (Algorithms (1.5) and (1.6) in [33], and the algorithms defined by Corollaries 4.3 and 4.4 in [28]).

Let L_i, $i = 0, 1, 2, 3$, be closed and convex subsets of $\mathbb{R}^{100}$, $\mathbb{R}^{200}$, $\mathbb{R}^{300}$ and $\mathbb{R}^{400}$, respectively. Suppose that the subset L_i is defined by $L_i = \{x \in \mathbb{R}^{100(i+1)} : \langle a_i, x\rangle \leq b_i\}$, where the coordinates of the vector a_i and the real number b_i are chosen randomly in the closed intervals $[10, 50]$ and $[0, 0.5]$, respectively, for all $i = 0, 1, 2, 3$.

Let $T_i : \mathbb{R}^{100} \to \mathbb{R}^{100(i+1)}$, $i = 1, 2, 3$, be bounded linear operators the elements of their representing matrices are generated randomly in the closed interval $[-5, 5]$.

Since $0 \in \Omega^{SFP}$, it is easy to check that $\Omega^{SFP} = L_0 \cap (\cap_{i=1}^{3} T_i^{-1}(L_i)) \neq \emptyset$.

The control parameters for each algorithm (iterative method) are chosen as follows.

- Algorithm A: Algorithm 7.
- Algorithm B: Algorithm 8.
- Algorithm C (The iterative method (1.5) in [24]):

$$\gamma_n = \frac{1}{5(\|T_1\|^2 + \|T_2\|^2 + \|T_3\|^2)},\ \text{for all } n \geq 0.$$

TABLE 1. Table of numerical results

	$\varepsilon = 10^{-3}$		$\varepsilon = 10^{-4}$	
Algorithms	Σ_n	n	Σ_n	n
Algorithm A	8.118826×10^{-4}	59	6.354513×10^{-5}	71
Algorithm B	2.340168×10^{-25}	4	2.340168×10^{-25}	4
Algorithm C	9.916995×10^{-4}	1040	9.851878×10^{-5}	1228
Algorithm D	9.980770×10^{-4}	1105	9.861496×10^{-5}	1159
Algorithm E	9.719251×10^{-4}	134	9.847986×10^{-5}	152
Algorithm F	9.110554×10^{-4}	112	9.909748×10^{-5}	132

- Algorithm D (The iterative method (1.6) in [24]):

$$\gamma_n = \frac{1}{5(\|T_1\|^2 + \|T_2\|^2 + \|T_3\|^2)}, \ \alpha_n = (n+1)^{-0.5}$$

  for all $n \geq 0$, and $f(x) = 0.85x$ for all $x \in \mathbb{R}^{10}$.
- Algorithm E (The iterative method in Corollary 4.3 of [28]): $\beta_{i,n} = 0.25$, $\theta_{i,n} = 10^{-3}$, $\alpha_n = (n+1)^{-0.5}$ for all $i = 0, 1, 2, 3$ and for all $n \geq 0$, and $\mathfrak{f}(x) = 0.85x$ for all $x \in \mathbb{R}^{10}$.
- Algorithm F (The iterative method in Corollary 4.4 of [28]): $\theta_{i,n} = 10^{-3}$, $\alpha_n = (n+1)^{-0.5}$ for all $i = 0, 1, 2, 3$ and for all $n \geq 0$, and $\mathfrak{f}(x) = 0.85x$ for all $x \in \mathbb{R}^{10}$.

Using the initial guess u_0, the coordinates of which are randomly generated in the closed interval $[-50, 50]$, and the stopping rule $\Sigma_n < \varepsilon$ to stop the iterative process, where

$$\Sigma_n = \frac{1}{4}(\|u_n - P_{L_0}u_n\|^2 + \|T_1u_n - P_{L_1}T_1u_n\|^2 + \|T_2u_n - P_{L_2}T_2u_n\|^2 + \|T_3u_n - P_{L_3}T_3u_n\|^2)$$

and ε is a given error, we obtain the following table of numerical results.

The behavior of the function Σ_n in Table 1 is described in the figures below (Figure 1 and Figure 2).

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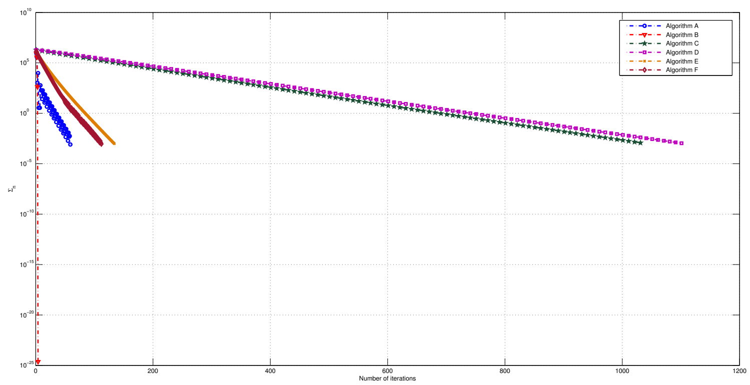

FIGURE 1. The behavior of Σ_n with the stopping rule $\Sigma_n < 10^{-3}$

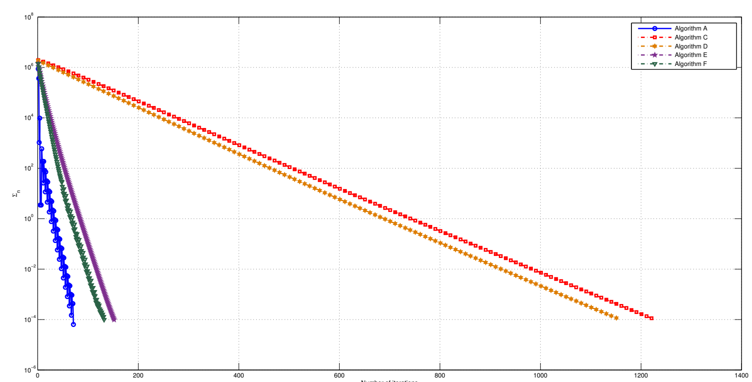

FIGURE 2. The behavior of Σ_n with the stopping rule $\Sigma_n < 10^{-4}$ without Algorithm B (Algorithm 8)

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