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NONEMPTINESS AND BOUNDEDNESS OF THE SET OF ZEROS OF A MONOTONE OPERATOR IN HADAMARD SPACES

SAJAD RANJBAR

Department of Mathematics, Higher Education Center of Eghlid, Eghlid, Iran
E-mail: sranjbar@eghlid.ac.ir, sranjbar74@yahoo.com

Abstract. In this paper, we investigate the nonemptiness and boundedness of the set of zeros of a monotone operator in Hadamard spaces. Two coercivity conditions R1 and R2 are proposed. The equivalence between the nonemptiness of the set of zeros of a monotone operator and coercivity condition R1 is established. Moreover, it is shown that coercivity condition R2 is a sufficient and necessary condition for the boundedness of the set of zeros of a monotone operator. Some applications in convex minimization and fixed point theory are also presented to support the main results.
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1. Problem description

The concept of monotonicity is a valuable tool in the study of problems associated with optimization, equilibrium point, variational inequality, convex analysis, partial differential equations, game theory, etc. In particular, the inclusion problem

$$\text{finding} \quad x \in \mathbb{D}(A) \;\; \text{such that} \;\; 0 \in A(x), \tag{1.1}$$

plays a fundamental role in the monotone operator theory.

There are basically two problems associated with the monotone inclusion problem (1.1), as follows.

- To give conditions for the existence and boundedness of solutions to monotone inclusion problem (1.1).
- To design algorithms for approximating solutions of the monotone inclusion problem (1.1), whenever those exist.

The design of algorithms to approximate the zeros of monotone operators has always been of interest to many authors in Hilbert spaces, Banach spaces, Hadamard spaces, etc.; see, for example, [7, 9, 14, 16, 19, 21, 24, 26, 28, 29, 30] and references therein. The conditions for the existence and boundedness of solutions to monotone inclusion problem (1.1) have been investigated by many authors but not to the extent of designing an algorithm to approximate the solutions of the monotone inclusion problem (1.1), (see, [3, 9, 14, 19, 22, 33]). Recently, Zhang et al. [33], using the coercivity conditions introduced in [6] and [11], proved that the coercivity condition A (see, [33, Section 2]) is a necessary and sufficient condition for a solution of the inclusion problem (1.1) in the setting of Hilbert spaces. They also showed the coercivity condition B (see, [33, Section 2]) is equivalent to the boundedness of the solution set of the inclusion problem (1.1) if the domain of the maximal monotone operator is convex. Very recently, in the setting of Hadamard manifolds, Ansari and Babu [3] considered the coercivity conditions (I) and (II) (see, [3, Section 3]) which respectively are equivalent to the nonemptiness of solution set and the boundedness of solution set of the inclusion problem (1.1).

Motivated and inspired by the research going on in this direction, we propose the coercivity conditions **R1** and **R2** (see Section 3 below) in Hadamard spaces which are studied in [3, 33]. The equivalence between nonemptiness of the solution set of the inclusion problem (1.1) and the coercivity condition **R1** is established. Moreover, we show the coercivity condition **R2** is a sufficient and necessary condition for the boundedness of the solution set of the inclusion problem (1.1). Some applications in convex minimization and fixed point theory are also presented to support the main results.

Our results improve and extend the previous corresponding results. These results improve and generalize the results of Ansari and Babu [3] and Zhang et al. [33] from Hilbert spaces and Hadamard manifolds to Hadamard spaces and also the convexity condition on the domain of the maximal monotone operator in [33, Theorem 4.1] is removed, (see Section 5 below).

2. Basic definitions and preliminaries

Let (X, d) be a metric space, $x, y \in X$ and $I = [0, d(x, y)]$. A geodesic path connecting x to y in X is an isometry $c : I \longrightarrow X$ such that $c(0) = x$, $c(d(x, y)) = y$ and $d(c(a), c(b)) = |a - b|$ for all $a, b \in I$. The image of a geodesic c is called a geodesic segment connecting x and y. When it is unique, this geodesic is denoted by $[x, y]$. We denote the unique point $z \in [x, y]$ such that $d(x, z) = td(x, y)$ and $d(y, z) = (1 - t)d(x, y)$, by $(1 - t)x \oplus ty$, where $0 \leq t \leq 1$. The metric space (X, d) is called a geodesic space if x and y are joined by a geodesic, for each $x, y \in X$. The (X, d) is said to be uniquely geodesic if there is exactly one geodesic segment connecting x and y for each $x, y \in X$. A subset K of X is called convex if $[x, y] \subseteq K$ for all $x, y \in K$.

Definition 2.1. [4, 8] A non-positive curvature metric space or a CAT(0) space (in honour of E. **C**artan, A.D. **A**lexandrov and V.A. **T**oponogov) is a geodesic space (X, d) which comes up to the following (CN) inequality:

$$d^2(tx \oplus (1 - t)y, z) \leq td^2(x, z) + (1 - t)d^2(y, z) - t(1 - t)d^2(x, y), \qquad (2.1)$$

for all $x, y, z \in X$ and $t \in [0, 1]$.

In particular, if x, y, z, w are points in X and $t \in [0, 1]$, then

- $d(tx \oplus (1 - t)y, z) \leq td(x, z) + (1 - t)d(y, z)$,
- $d(tx \oplus (1 - t)y, tz \oplus (1 - t)w) \leq td(x, z) + (1 - t)d(y, w)$,

It is known that a CAT(0) space is a uniquely geodesic space.

A complete CAT(0) space is called a **Hadamard space**. A Hadamard space X is a flat Hadamard space if and only if the inequality in (2.1) is an equality. Every closed convex subset of a Hilbert space is a flat Hadamard space. For other equivalent definitions and basic properties, we refer the reader to the standard texts such as [4, 8, 10, 15, 18]. The following are the main examples of Hadamard spaces:
Hilbert spaces, Hadamard manifolds (i.e. simply connected complete Riemannian manifolds with non-positive sectional curvature which can be of infinite dimension), $\mathbb{R}$-trees as well as examples that have been built out of given Hadamard spaces such as closed convex subsets, direct products, warped products, L^2-spaces, direct limits, and Reshetnyak's gluing (see [31], Section 3). Berg and Nikolaev [5] have introduced the concept of *quasilinearization* for the CAT(0) space X. They denoted a pair $(a,b) \in X \times X$ by $\overrightarrow{ab}$ and called it a *vector*. Then the quasilinearization map $\langle .,. \rangle$: $(X \times X) \times (X \times X) \to \mathbb{R}$ is defined by

$$\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle = \tfrac{1}{2}(d^2(a,d) + d^2(b,c) - d^2(a,c) - d^2(b,d)), \qquad (a,b,c,d \in X).$$

It can be easily verified that $\langle \overrightarrow{ab}, \overrightarrow{ab} \rangle = d^2(a,b)$, $\langle \overrightarrow{ba}, \overrightarrow{cd} \rangle = -\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle$ and $\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle = \langle \overrightarrow{ae}, \overrightarrow{cd} \rangle + \langle \overrightarrow{eb}, \overrightarrow{cd} \rangle$ are satisfied for all $a,b,c,d,e \in X$. Also, we can formally add compatible vectors, more precisely $\overrightarrow{ac} + \overrightarrow{cb} = \overrightarrow{ab}$, for all $a,b,c \in X$.

Lemma 2.1. [32] *Let (X,d) be a CAT(0) space. For any $t \in [0,1]$ and $u,v \in X$, let $z = tu \oplus (1-t)v$. Then, for all $x,y \in X$:*

(i) $\langle \overrightarrow{zx}, \overrightarrow{zy} \rangle \leq t\langle \overrightarrow{ux}, \overrightarrow{zy} \rangle + (1-t)\langle \overrightarrow{vx}, \overrightarrow{zy} \rangle$
(ii) $\langle \overrightarrow{zx}, \overrightarrow{uy} \rangle \leq t\langle \overrightarrow{ux}, \overrightarrow{uy} \rangle + (1-t)\langle \overrightarrow{vx}, \overrightarrow{uy} \rangle$

We say that X satisfies the Cauchy-Schwartz inequality if

$$\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle \leq d(a,b)d(c,d), \qquad (a,b,c,d \in X).$$

Berg and Nikolaev [5] have proved the following result.

Theorem 2.1. [5, Corollary 3] *A geodesic metric space is a CAT(0) space if and only if it satisfies the Cauchy-Schwartz inequality.*

Ahmadi Kakavandi and Amini [2] have introduced the concept of dual space of a Hadamard space X, based on a work of Berg and Nikolaev [5], as follows.
Consider the map $\Theta : \mathbb{R} \times X \times X \to C(X,\mathbb{R})$ defined by

$$\Theta(t,a,b)(x) = t\langle \overrightarrow{ab}, \overrightarrow{ax} \rangle, \qquad (t \in \mathbb{R},\ a,b,x \in X),$$

where $C(X,\mathbb{R})$ is the space of all continuous real-valued functions on X. Then the Cauchy-Schwartz inequality implies that $\Theta(t,a,b)$ is a Lipschitz function with Lipschitz semi-norm $L(\Theta(t,a,b)) = |t|d(a,b)$, $(t \in \mathbb{R},\quad a,b \in X)$, where

$$L(\varphi) = sup\left\{\frac{\varphi(x) - \varphi(y)}{d(x,y)} : x,y \in X, x \neq y\right\}$$

is the Lipschitz semi-norm for any function $\varphi : X \to \mathbb{R}$. A pseudometric D on $\mathbb{R} \times X \times X$ is defined by

$$D((t,a,b),(s,c,d)) = L(\Theta(t,a,b) - \Theta(s,c,d)), \qquad (t,s \in \mathbb{R},\ a,b,c,d \in X).$$

For a Hadamard space (X, d), the pseudometric space $(\mathbb{R} \times X \times X, D)$ can be considered as a subspace of the pseudometric space of all real-valued Lipschitz functions $(Lip(X, \mathbb{R}), L)$.

Lemma 2.2. [2, Lemma 2.1] *$D((t, a, b), (s, c, d)) = 0$ if and only if*

$$t\langle \overrightarrow{ab}, \overrightarrow{xy}\rangle = s\langle \overrightarrow{cd}, \overrightarrow{xy}\rangle, \text{ for all } x, y \in X.$$

By Lemma 2.2, D induces an equivalence relation on $\mathbb{R} \times X \times X$, where the equivalence class of (t, a, b) is

$$[t\overrightarrow{ab}] = \{s\overrightarrow{cd} : D((t, a, b), (s, c, d)) = 0\}.$$

The set $X^* = \{[t\overrightarrow{ab}] : (t, a, b) \in \mathbb{R} \times X \times X\}$ is a metric space with metric $D([t\overrightarrow{ab}], [s\overrightarrow{cd}]) := D((t, a, b), (s, c, d))$, which is called the dual space of (X, d). It is clear that $[\overrightarrow{aa}] = [\overrightarrow{bb}]$ for all $a, b \in X$. Fix $o \in X$, we write $\mathbf{0} = [\overrightarrow{oo}]$ as the zero of the dual space. In [2], it is shown that the dual of a closed and convex subset of Hilbert space H with nonempty interior is H and $t(b - a) \equiv [t\overrightarrow{ab}]$ for all $t \in \mathbb{R},\ a, b \in H$. Note that X^* acts on $X \times X$ by

$$\langle x^*, \overrightarrow{xy}\rangle = t\langle \overrightarrow{ab}, \overrightarrow{xy}\rangle, \quad (x^* = [t\overrightarrow{ab}] \in X^*, x, y \in X).$$

Also, we use the following notation:

$$\langle \alpha x^* + \beta y^*, \overrightarrow{xy}\rangle := \alpha\langle x^*, \overrightarrow{xy}\rangle + \beta\langle y^*, \overrightarrow{xy}\rangle, \quad (\alpha, \beta \in \mathbb{R},\ x, y \in X,\ x^*, y^* \in X^*).$$

Lemma 2.3. [17, Lemma 3.3] *Let X be a Hadamard space with dual X^*, then*

$$|\langle x^* - y^*, \overrightarrow{yx}\rangle| \le D(x^*, y^*)d(x, y), \qquad \text{for all } x, y \in X,\ x^*, y^* \in X^*.$$

A notion of convergence in Hadamard spaces is Δ-convergence which was introduced by Lim [25] and has been studied by many authors (e.g. [1, 2, 13] and references therein).

Ahmadi Kakavandi [1] proved the following characterization for Δ-convergence.

Theorem 2.2. [1, Theorem 2.6] *Let (X, d) be a Hadamard space, (x_n) be a sequence in X and $x \in X$. Then (x_n) Δ-converges to x if and only if*

$$\limsup_{n\to\infty}\langle \overrightarrow{xx_n}, \overrightarrow{xy}\rangle \le 0, \quad \text{for all } \ y \in X.$$

Obviously, by the Cauchy-Schwartz inequality, strong convergence implies Δ-convergence.

Lemma 2.4. [13] *Every bounded sequence in a complete CAT(0) space has a Δ-convergence subsequence.*

Proposition 2.3. [23, Proposition 5.2] *If a sequence (x_n) in a Hadamard space (X, d), Δ-converges to $x \in X$, then*

$$x \in \textstyle\bigcap_{k=1}^{\infty} \overline{conv}\{x_k, x_{k+1}, ...\},$$

where $\overline{conv}(A) = \bigcap\{B : B \supseteq A,$ where B is closed and convex$\}$ for any $A \subset X$.

Proposition 2.3 states that every closed convex subset of X is Δ-closed.

Definition 2.2. Let K be a nonempty subset of an Hadamard space X and $T : K \longrightarrow K$ be a mapping. The mapping T is called

- Nonexpansive if $d(Tx, Ty) \leq d(x, y)$, for all $x, y \in K$.
- Firmly nonexpansive if $d^2(Tx, Ty) \leq \langle \overrightarrow{TxTy}, \overrightarrow{xy} \rangle$, for all $x, y \in K$.

The fixed point set of T is denoted by $F(T)$, (i.e. $F(T) = \{x \in K : x = Tx\}$). It is well-known that if T is a nonexpansive mapping on subset K of CAT(0) space X then $F(T)$ is closed and convex.

Remark 2.1. From the Cauchy-Schwartz inequality, it is clear that the class of nonexpasive mappings includes the class of firmly nonexpansive mappings.

Lemma 2.5. [12] *Let X be a complete CAT(0) space and $T : X \longrightarrow X$ be a nonexpansive mapping, then the conditions that (x_n) Δ-converges to x and $(d(x_n, Tx_n))$ converges strongly to 0, imply that $x = Tx$.*

Definition 2.3. Let X be a Hadamard space with dual space X^*. The multi-valued operator $A : X \to 2^{X^*}$ with domain $\mathbb{D}(A) := \{x \in X : A(x) \neq \emptyset\}$, range $\mathbb{R}(A) := \bigcup_{x \in X} Ax$, $A^{-1}(x^*) := \{x \in \mathbb{D}(A) : x^* \in Ax\}$ and graph $gra(A) := \{(x, x^*) \in X \times X^* : x \in \mathbb{D}(A), x^* \in Ax\}$, is called

- monotone if for any $x, y \in \mathbb{D}(A)$ and for all $x^* \in Ax, y^* \in Ay$,
$$\langle x^* - y^*, \overrightarrow{yx} \rangle \geq 0,$$
- pseudomonotone if for any $x, y \in \mathbb{D}(A)$ and for all $x^* \in Ax, y^* \in Ay$,
$$\langle x^*, \overrightarrow{xy} \rangle \geq 0 \quad \text{implies} \quad \langle y^*, \overrightarrow{yx} \rangle \leq 0.$$

In the following, we present some properties of the resolvent operator of a monotone operator in CAT(0) spaces which were given in [21].

Definition 2.4. [21, Definition 3.4] Let X be a Hadamard space with dual X^*, $\lambda > 0$ and $A : X \to 2^{X^*}$ be a multi-valued operator. The resolvent and the Yosida approximation of order λ of the operator A are the multi-valued mappings $J_\lambda^A : X \to 2^X$ and $A_\lambda : X \to 2^{X^*}$ defined respectively by $J_\lambda^A(x) := \{z \in X \mid [\frac{1}{\lambda}\overrightarrow{zx}] \in Az\}$ and $A_\lambda(x) := \{[\frac{1}{\lambda}\overrightarrow{yx}] \mid y \in J_\lambda^A(x)\}$.

Theorem 2.4. [21, Theorem 3.9] *Let X be a CAT(0) space and $A : X \to 2^{X^*}$. Suppose J_λ^A and A_λ are respectively the resolvent and the Yosida approximation of order λ of the operator A. We have:*

(i) *For any $\lambda > 0$, $\mathbb{R}(J_\lambda^A) \subset \mathbb{D}(A)$, $F(J_\lambda^A) = A^{-1}(\mathbf{0}) = A_\lambda^{-1}(\mathbf{0})$, where $\mathbb{R}(J_\lambda^A)$ is the range of J_λ^A.*
(ii) *If J_λ^A is single-valued, then A_λ is single-valued and $A_\lambda(x) \subset A(J_\lambda^A(x))$.*
(iii) *If A is monotone, then J_λ^A is a single-valued and firmly nonexpansive mapping.*
(iv) *If A is monotone, then A_λ is a monotone operator.*
(v) *If A is monotone and $0 < \lambda \leq \mu$, then $d^2(J_\lambda^A x, J_\mu^A x) \leq \frac{\mu - \lambda}{\mu + \lambda} d^2(x, J_\mu^A x)$, which implies $d(x, J_\lambda^A x) \leq 2d(x, J_\mu^A x)$.*

Now, we present the following remark which is essential for getting to main results.

Remark 2.2. Let X be a Hadamard space with dual X^*, $A : X \to 2^{X^*}$ be a monotone operator and $\lambda > 0$. Then

- **(i)** Remark 2.1, Lemma 2.5 and parts (i) and (iii) of Theorem 2.4 imply that if the sequence (x_n) is Δ-convergent to x and $(d(x_n, J_\lambda^A x_n))$ converges strongly to 0, then $x \in A^{-1}(\mathbf{0})$.
- **(ii)** Definition 2.4 and parts (ii) and (iii) of Theorem 2.4 imply that if $x \in \mathbb{D}(J_\lambda^A)$, then $[\frac{1}{\lambda}\overrightarrow{J_\lambda^A(x)x}] \in A(J_\lambda^A(x))$.
- **(iii)** Since the fixed point set of a nonexpansive mapping is closed and convex, Remark 2.1 and part (i) of Theorem 2.4 imply $A^{-1}(\mathbf{0})$ is closed and convex. Therefore, by Proposition 2.3, $A^{-1}(\mathbf{0})$ is Δ-closed.

3. Coercivity conditions

Let (X, d) be a Hadamard space with dual X^* and $A : X \to 2^{X^*}$ be an operator with domain $\mathbb{D}(A)$. Suppose $\delta > 0$ and $z \in X$. We denote by $B_\delta[z]$ the closed ball at z with radius δ and $B_\delta(z)$ the open ball at z with radius δ which are defined by

$$B_\delta[z] := \{x \in X : d(x, z) \leq \delta\} \quad \text{and} \quad B_\delta(z) := \{x \in X : d(x, z) < \delta\}.$$

We now propose the following coercivity conditions for the operator A:

- **R1**: there exists $\delta > 0$ and $z \in X$ such that for every $x \in \mathbb{D}(A) - B_\delta[z]$, there exists $y \in \mathbb{D}(A) \cap B_{d(x,z)}(z)$ satisfying $\inf_{x^* \in Ax}\langle x^*, \overrightarrow{yx}\rangle \geq 0$.
- **R2**: there exists $\delta > 0$ and $z \in X$ such that for every $x \in \mathbb{D}(A) - B_\delta[z]$, there exists $y \in \mathbb{D}(A) \cap B_{d(x,z)}(z)$ satisfying $\sup_{y^* \in Ay}\langle y^*, \overrightarrow{yx}\rangle > 0$.

Proposition 3.1. *Let X be a Hadamard space with dual X^* and $A : X \to 2^{X^*}$ be a pseudomonotone operator. Then* **R2** *implies* **R1**.

Proof. If **R2** holds, then there exists $\delta > 0$ and $z \in X$ such that for every $x \in \mathbb{D}(A) - B_\delta[z]$, there exists $y \in \mathbb{D}(A) \cap B_{d(x,z)}(z)$ satisfying $\sup_{y^* \in Ay}\langle y^*, \overrightarrow{yx}\rangle > 0$. By pseudomonotonicity of A, we get $\sup_{x^* \in Ax}\langle x^*, \overrightarrow{xy}\rangle \leq 0$ which implies $\inf_{x^* \in Ax}\langle x^*, \overrightarrow{yx}\rangle \geq 0$. Hence **R1** is established. $\square$

Remark 3.1. Let (X, d) be a Hadamard space with dual X^* and $A : X \to 2^{X^*}$ be an operator with domain $\mathbb{D}(A)$. Then

- Clearly, monotonicity of the operator A implies pseudomonotonicity of A. Therefore, Proposition 3.1 implies that if A is a monotone operator, then **R2** implies **R1**.
- If $\mathbb{D}(A) = X$, then, obviously, the coercivity conditions (I) and (II) in [3, Section 3] imply **R1** and **R2**, respectively.
- It can be easily seen that the coercivity conditions A and B in [33, Section 2] imply **R1** and **R2**, respectively.

4. Existence of solutions

Let X be a Hadamard space with dual X^*. We say that the operator $A : X \to 2^{X^*}$ satisfies the *range condition* if for every $\lambda > 0$, $\mathbb{D}(J_\lambda^A) = X$, (see [21]). Minty in [27] proved that if A is a maximal monotone operator on the Hilbert space H then $R(I + \lambda A) = H$, $\forall\lambda > 0$, where I is the identity operator. Thus, every maximal monotone operator A on a Hilbert space satisfies the range condition. As considered in [3, 24], if A is a maximal monotone operator on a Hadamard manifold, then A satisfies the range condition. For presenting some examples of monotone operators that satisfy the range condition in CAT(0) spaces, we refer to [21, Sections 5 and 6].

The following lemma proves the demiclosedness of the monotone operator A in Hadamard spaces.

Lemma 4.1. *Let X be a Hadamard space with dual X^* and $A : X \to 2^{X^*}$ be a monotone operator that satisfies the range condition. Suppose $(x_n, x_n^*) \in gra(A)$ for all $n \in \mathbb{N}$ such that (x_n) is Δ-convergent to $p \in X$ and (x_n^*) converges to $\mathbf{0} \in X^*$ with respect to the metric D. Then $p \in A^{-1}(\mathbf{0})$.*

Proof. Let $\lambda > 0$ be fixed. Set $z_n = J_\lambda^A(x_n)$, for all $n \in \mathbb{N}$. Therefore $[\frac{1}{\lambda}\overrightarrow{z_n x_n}] \in A(z_n)$, for all $n \in \mathbb{N}$. If there exists $M > 0$ such that $z_n = x_n$ for all $n > M$, then part **(iii)** of Remark 2.2 implies $p \in A^{-1}(\mathbf{0})$. Otherwise, by monotonecity of A, we get

$$0 \leq \langle x_n^* - [\frac{1}{\lambda}\overrightarrow{z_n x_n}], \overrightarrow{z_n x_n}\rangle.$$

This, together with Lemma 2.3, yields that

$$\tfrac{1}{\lambda} d^2(z_n, x_n) \leq D(x_n^*, \mathbf{0}) d(z_n, x_n),$$

from where

$$\tfrac{1}{\lambda} d(z_n, x_n) \leq D(x_n^*, \mathbf{0}).$$

Therefore, by the assumptions, we obtain

$$\lim_{n\to\infty} d(J_\lambda^A(x_n), x_n) = \lim_{n\to\infty} d(z_n, x_n) = 0.$$

Now, by part **(i)** of Remark 2.2, we conclude $p \in A^{-1}(\mathbf{0})$. □

In the following theorem, we prove that the coercivity condition **R1** is a sufficient and necessary condition for the nonemptiness of the solution set of the inclusion problem (1.1).

Theorem 4.1. *Let X be a Hadamard space with dual X^* and $A : X \to 2^{X^*}$ be a monotone operator satisfying the range condition. Then the coercivity condition* **R1** *holds if and only if $A^{-1}(\mathbf{0}) \neq \emptyset$, (i.e. there exists $p \in \mathbb{D}(A)$ satisfying $0 \in A(p)$).*

Proof. Assume that the coercivity condition **R1** holds. Then there exists $\delta > 0$ and $z \in X$ such that for every $x \in \mathbb{D}(A) - B_\delta[z]$, there exists $y \in \mathbb{D}(A) \cap B_{d(x,z)}(z)$ satisfying $\inf_{x^* \in Ax}\langle x^*, \overrightarrow{yx}\rangle \geq 0$. By the range condition and part (iii) of Theorem 2.4, we can construct the sequence (x_n) in X such that $x_n = J_n^A(z)$, for all $n \in \mathbb{N}$. Then $[\frac{1}{n}\overrightarrow{x_n z}] \in A(x_n)$, for all $n \in \mathbb{N}$ and $x_n \in \mathbb{D}(A)$, for all $n \in \mathbb{N}$. Now, we show that the sequence (x_n) is bounded. For this, suppose there exists $s \in \mathbb{N}$ such that $\delta < d(x_s, z)$. Then, $x_s \in \mathbb{D}(A) - B_\delta[z]$. Using the coercivity condition **R1**, there exists $y \in \mathbb{D}(A) \cap B_{d(x_s,z)}(z)$ such that $0 \leq \inf_{x^* \in Ax_s}\langle x^*, \overrightarrow{yx_s}\rangle$. This together with $[\frac{1}{s}\overrightarrow{x_s z}] \in A(x_s)$ implies

$$0 \leq \frac{1}{s}\langle \overrightarrow{x_s z}, \overrightarrow{yx_s}\rangle. \tag{4.1}$$

On the other hand, we have

$$\begin{aligned}\frac{1}{s}\langle\overrightarrow{x_s z},\overrightarrow{yx_s}\rangle &= \frac{1}{s}\langle\overrightarrow{x_s z},\overrightarrow{yz}\rangle+\frac{1}{s}\langle\overrightarrow{x_s z},\overrightarrow{zx_s}\rangle\\ &\leq \frac{1}{s}d(x_s,z)d(y,z)-\frac{1}{s}d^2(x_s,z)\\ &= -\frac{1}{s}d(x_s,z)(d(x_s,z)-d(y,z))\\ &< 0,\end{aligned}$$

which contradicts (4.1). Therefore, $d(x_n,z)\leq\delta,\quad$ for all $n\in\mathbb{N}$. Hence, (x_n) is bounded. Hence,

$$\lim_{n\to\infty} D([\frac{1}{n}\overrightarrow{x_n z}],\mathbf{0})=\lim_{n\to\infty}\frac{1}{n}d(x_n,z)=0. \tag{4.2}$$

Moreover, by Lemma 2.4, there exists a subsequence (x_{n_k}) of (x_n) such that (x_{n_k}) is Δ-convergent to an element p in X and for all $k\in\mathbb{N}$, we have $(x_{n_k},[\frac{1}{n_k}\overrightarrow{x_{n_k}z}])\in gra(A)$. Hence, by (4.2) and Lemma 4.1, $p\in A^{-1}(\mathbf{0})$.
Now assume that $A^{-1}(\mathbf{0})\neq\emptyset$. Let $p\in A^{-1}(\mathbf{0})$. By monotonicity of A, we get

$$\langle x^*,\overrightarrow{px}\rangle\geq 0,\quad \text{for all }\ x\in\mathbb{D}(A),\ x^*\in A(x) \tag{4.3}$$

Let $z\in X$ be fixed. Set $\delta=d(p,z)+1$ and $y=p$. Then for every $x\in\mathbb{D}(A)-B_\delta[z]$ and $x^*\in A(x)$, we have

$$d(y,z)=d(p,z)<d(p,z)+1=\delta<d(x,z),$$

which implies $y\in\mathbb{D}(A)\cap B_{d(x,z)}(z)$ and by (4.3), $\langle x^*,\overrightarrow{yx}\rangle\geq 0,\ \forall\ x^*\in A(x)$. Thus, $\inf_{x^*\in Ax}\langle x^*,\overrightarrow{yx}\rangle\geq 0$. Hence, there exists $z\in X$ and $0<\delta=d(p,z)+1$ such that for every $x\in\mathbb{D}(A)-B_\delta[z]$, there exists $y=p\in\mathbb{D}(A)\cap B_{d(x,z)}(z)$ satisfying $\inf_{x^*\in Ax}\langle x^*,\overrightarrow{yx}\rangle\geq 0$. This indicates **R1** holds and completes the proof. □

Remark 4.1. By Remark 3.1, Theorem 4.1 improves the conditions in [3, Theorem1] and generalizes [3, Theorem1] from Hadamrad manifolds to Hadamard spaces. Moreover, Theorem 4.1 improves and extends [33, Theorem 3.1] from Hilbert spaces to Hadamard spaces.

5. Boundedness of the solution set

In the following theorem, the equivalence between coercivity condition **R2** and boundedness of the solution set of inclusion problem (1.1) is established without the convexity condition on the domain of the monotone operator.

Theorem 5.1. *Let X be a Hadamard space with dual X^* and $A: X\to 2^{X^*}$ be a monotone operator that satisfies the range condition. Then the coercivity condition* **R2** *holds if and only if the set $A^{-1}(\mathbf{0})=\{x\in\mathbb{D}(A): 0\in Ax\}$ is nonempty and bounded.*

Proof. Suppose the coercivity condition **R2** holds. Then, by Remark 3.1 and Theorem 4.1, $A^{-1}(\mathbf{0})$ is nonempty. Suppose $A^{-1}(\mathbf{0})$ is not bounded. Then, there exists $p\in$

$A^{-1}(\mathbf{0})$ such that $d(p,z) > \delta$. Therefore, $p \in \mathbb{D}(A) - B_\delta[z]$. Hence, by **R2**, for $x = p$, there exists $y \in \mathbb{D}(A) \cap B_{d(p,z)}(z)$ satisfying

$$\sup_{y^* \in Ay} \langle y^*, \overrightarrow{yp} \rangle > 0. \tag{5.1}$$

On the other hand, by monotonicity of A, for all $y \in \mathbb{D}(A)$ and $y^* \in Ay$, we have $\langle y^*, \overrightarrow{yp} \rangle \leq 0$, which implies $\sup_{y^* \in Ay} \langle y^*, \overrightarrow{yp} \rangle \leq 0$. This contradicts (5.1). Hence $A^{-1}(\mathbf{0})$ is bounded.

Now assume that $A^{-1}(\mathbf{0})$ is nonempty and bounded and **R2** does not hold. Let $p \in A^{-1}(\mathbf{0})$ and $\delta > 0$ be fixed and arbitrary. There exists $x \in \mathbb{D}(A) - B_\delta[p]$ such that $\sup_{y^* \in Ay} \langle y^*, \overrightarrow{yx} \rangle \leq 0$, for all $y \in \mathbb{D}(A) \cap B_{d(x,p)}(p)$. Set $\lambda > 0$ and $u_t = tp \oplus (1-t)x$ where $t \in (0,1)$. By Theorem 2.4, we get

$$d(J_\lambda^A u_t, p) \leq d(u_t, p) = (1-t)d(x,p) < d(x,p),$$

which, by part **(ii)** of Remark 2.2, implies

$$\langle \overrightarrow{J_\lambda^A u_t u_t}, \overrightarrow{J_\lambda^A u_t x} \rangle \leq \lambda \sup_{y^* \in A(J_\lambda^A u_t)} \langle y^*, \overrightarrow{J_\lambda^A u_t x} \rangle \leq 0. \tag{5.2}$$

By firmly nonexpansiveness of J_λ^A and (5.2), we obtain

$$\begin{aligned}
d^2(J_\lambda^A u_t, J_\lambda^A x) &\leq \langle \overrightarrow{J_\lambda^A u_t J_\lambda^A x}, \overrightarrow{u_t x} \rangle \\
&= \langle \overrightarrow{J_\lambda^A u_t u_t}, \overrightarrow{u_t x} \rangle + \langle \overrightarrow{u_t J_\lambda^A x}, \overrightarrow{u_t x} \rangle \\
&= \langle \overrightarrow{J_\lambda^A u_t u_t}, \overrightarrow{u_t J_\lambda^A u_t} \rangle + \langle \overrightarrow{J_\lambda^A u_t u_t}, \overrightarrow{J_\lambda^A u_t x} \rangle + \langle \overrightarrow{u_t J_\lambda^A x}, \overrightarrow{u_t x} \rangle \\
&= -d^2(u_t, J_\lambda^A u_t) + \langle \overrightarrow{J_\lambda^A u_t u_t}, \overrightarrow{J_\lambda^A u_t x} \rangle + \langle \overrightarrow{u_t J_\lambda^A x}, \overrightarrow{u_t x} \rangle \\
&\leq -d^2(u_t, J_\lambda^A u_t) + \langle \overrightarrow{u_t J_\lambda^A x}, \overrightarrow{u_t x} \rangle,
\end{aligned}$$

which, by Lemma 2.1, implies

$$\begin{aligned}
d^2(u_t, J_\lambda^A u_t) &\leq \langle \overrightarrow{u_t J_\lambda^A x}, \overrightarrow{u_t x} \rangle - d^2(J_\lambda^A u_t, J_\lambda^A x) \\
&\leq t\langle \overrightarrow{p J_\lambda^A x}, \overrightarrow{u_t x} \rangle + (1-t)\langle \overrightarrow{x J_\lambda^A x}, \overrightarrow{u_t x} \rangle - d^2(J_\lambda^A u_t, J_\lambda^A x) \\
&\leq t(t\langle \overrightarrow{p J_\lambda^A x}, \overrightarrow{px} \rangle + (1-t)\langle \overrightarrow{p J_\lambda^A x}, \overrightarrow{xx} \rangle) \\
&\quad + (1-t)(t\langle \overrightarrow{x J_\lambda^A x}, \overrightarrow{px} \rangle + (1-t)\langle \overrightarrow{x J_\lambda^A x}, \overrightarrow{xx} \rangle) - d^2(J_\lambda^A u_t, J_\lambda^A x) \\
&= t^2(\langle \overrightarrow{p J_\lambda^A x}, \overrightarrow{px} \rangle + \langle \overrightarrow{J_\lambda^A x x}, \overrightarrow{px} \rangle) + t\langle \overrightarrow{J_\lambda^A x x}, \overrightarrow{xp} \rangle - d^2(J_\lambda^A u_t, J_\lambda^A x) \\
&= t^2 d^2(p,x) - td^2(J_\lambda^A x, x) + t\langle \overrightarrow{J_\lambda^A x x}, \overrightarrow{J_\lambda^A x p} \rangle - d^2(J_\lambda^A u_t, J_\lambda^A x).
\end{aligned}$$

This, together with monotonicity of A, yields that

$$d^2(u_t, J_\lambda^A u_t) \leq t^2 d^2(p,x) - td^2(J_\lambda^A x, x) - d^2(J_\lambda^A u_t, J_\lambda^A x). \tag{5.3}$$

Let (t_n) be a sequence in $(0,1)$ such that $t_n \to 0$, as $n \to \infty$. Then (5.3) implies

$$\lim_{n\to\infty} d(u_{t_n}, J_\lambda^A u_{t_n}) = 0. \tag{5.4}$$

On the other hand $\lim_{n\to\infty} d(u_{t_n}, x) = \lim_{n\to\infty} t_n d(p, x) = 0$ which implies (u_{t_n}) Δ-converges to x, as $n \to \infty$. Therefore, by (5.4) and part **(i)** of Remark 2.2, we get $x \in A^{-1}(\mathbf{0})$. This, together with $x \in \mathbb{D}(A) - B_\delta[p]$, contradicts the boundedness of $A^{-1}(\mathbf{0})$. Hence, the coercivity condition **R2** holds. □

Remark 5.1. By Remark 3.1, Theorem 5.1 generalizes Theorem 2 of [3] and Theorem 4.1 of [33] to Hadamard spaces and improves the conditions in [3, Theorem 2] and [33, Theorem 4.1]. In particular, this theorem shows that the convexity condition on the maximal monotone operator's domain in [32, Theorem 4.1] is removable.

6. Application to convex minimization

One of the most widely used examples of monotone operators that satisfies the range condition, is subdifferential of a convex, proper and lower semicontinuous function. In [2], the subdifferential of a proper function on a Hadamard space X was defined, as follows.

Definition 6.1. [2] Let X be a Hadamard space with dual X^* and $f : X \to (-\infty, +\infty]$ be a proper function with efficient domain $D(f) := \{x : f(x) < +\infty\}$, then the subdifferential of f is the multi-valued function $\partial f : X \to 2^{X^*}$ defined by

$$\partial f(x) = \{x^* \in X^* : f(z) - f(x) \geq \langle x^*, \overrightarrow{xz}\rangle \quad (z \in X)\},$$

when $x \in D(f)$ and $\partial f(x) = \emptyset$, otherwise.

The following theorem shows that subdifferential of a convex, proper and lower semicontinuous function is a monotone operator satisfies the range condition in Hadamard spaces.

Theorem 6.1. [2, Theorem 4.2] [21, Proposition 5.2] *Let $f : X \to (-\infty, +\infty]$ be a proper, lower semicontinuous and convex function on a Hadamard space X with dual X^*, then*
(i) f attains its minimum at $x \in X$ if and only if $\mathbf{0} \in \partial f(x)$.
(ii) $\partial f : X \to 2^{X^*}$ is a monotone operator.
(iii) for any $y \in X$ and $\alpha > 0$, there exists a unique point $x \in X$ such that $[\alpha\overrightarrow{xy}] \in \partial f(x)$. (i.e. $D(J_\lambda^{\partial f}) = X$, for all $\lambda > 0$).

Khatibzadeh and the author in [21, Proposition 5.3.] proved that if $f : X \to (-\infty, +\infty]$ is a proper, lower semicontinuous and convex function on a Hadamard space X with dual X^*, then

$$J_\lambda^{\partial f} x = Argmin_{z\in X}\{f(z) + \frac{1}{2\lambda} d^2(z, x)\}, \qquad \text{for all } \lambda > 0,\ x \in X.$$

Suppose $f : X \to (-\infty, +\infty]$ is a proper, lower semicontinuous and convex function on a Hadamard space X with dual X^*. Using this together with Theorem 6.1, the inclusion problem (1.1) in the case of $A = \partial f$ turns into

$$\text{finding } x \in D(f), \text{ such that } f(x) = \min_{y\in X} f(y). \tag{6.1}$$

The solution set of the problem (6.1) is

$$(\partial f)^{-1}(\mathbf{0}) = \{x \in X : f(x) \leq f(y), \text{ for all } y \in X\}.$$

Using Theorems 4.1 and 5.1, the coercivity condition **R1**, in the case of $A = \partial f$, is equivalent to the existence of a solution to the minimization problem (6.1), and the coercivity condition **R2**, in the case of $A = \partial f$, is equivalent to boundedness of a solution set of the minimization problem (6.1).

Corollary 6.2. *Let X be a Hadamard space with dual X^* and $f : X \to (-\infty, +\infty]$ be a proper, lower semicontinuous and convex function. Then the coercivity condition* **R1***, in the case of $A = \partial f$, holds if and only if the solution set of the problem* (6.1) *is nonempty, (i.e. $(\partial f)^{-1}(\mathbf{0}) \neq \emptyset$).*

Proof. Proof is deducted from Theorem 4.1 and Theorem 6.1. □

Corollary 6.3. *Let X be a Hadamard space with dual X^* and $f : X \to (-\infty, +\infty]$ be a proper, lower semicontinuous and convex function. Then the coercivity condition* **R2***, in the case of $A = \partial f$, holds if and only if the solution set of the problem* (6.1) *(i.e. $(\partial f)^{-1}(\mathbf{0})$) is nonempty and bounded.*

Proof. Proof is deducted from Theorem 5.1 and Theorem 6.1. □

Example 6.1. Let (X, d) be a Hadamard space with dual X^* and $x_0 \in X$. Define $f : X \longrightarrow (-\infty, \infty]$ by $f(x) = \frac{1}{2}d^2(x, x_0)$. Then f is proper, convex and continuous. By Definition 6.1, for all $x \in X$, we have

$$\partial f(x) = \{x^* \in X^* : \tfrac{1}{2}d^2(y, x_0) - \tfrac{1}{2}d^2(x, x_0) \geq \langle x^*, \overrightarrow{xy}\rangle \quad (y \in X)\}.$$

For all $x \in X$, $[\overrightarrow{x_0x}] \in \partial f(x)$. Thus, $\mathbb{D}(\partial f) = X$. By Theorem 6.1, the operator ∂f is monotone and satisfies the range condition. We show that the operator ∂f satisfies the coercivity condition **R1**. For this, in **R1**, put $z = x_0$ and $\delta > 0$ is fixed. Suppose $x \in X - B_\delta[x_0]$ is arbitrary. Choose $0 < t < 1$ and set $y = tx \oplus (1-t)x_0$. Then $d(y, x_0) = td(x, x_0) < d(x, x_0)$, which implies $y \in X \cap B_{d(x,x_0)}(x_0)$. Therefore, for all $x^* \in \partial f(x)$, we get

$$\langle x^*, \overrightarrow{xy}\rangle \leq \frac{1}{2}d^2(y, x_0) - \frac{1}{2}d^2(x, x_0) < 0,$$

which implies $\inf_{x^* \in \partial f(x)} \langle x^*, \overrightarrow{yx}\rangle \geq 0$.

Remark 6.1. By Remark 3.1, Corollaries 6.2 and 6.3 improve and extend, respectively, Theorem 3 and Theorem 4 of [3] to Hadamard spaces.

7. Application to fixed point theory

Let (X, d) be a Hadamard space with dual X^* and $T : X \to X$ be a nonexpansive mapping. Define $A : X \longrightarrow 2^{X^*}$ with $Az = [\overrightarrow{Tzz}]$, then $F(T) = A^{-1}(0)$ and Proposition 4.2 of [20] shows the operator $Az = [\overrightarrow{Tzz}]$ is a monotone operator. The range condition for this operator was studied in [21].

In the case of $Az = [\overrightarrow{Tzz}]$, the inclusion problem (1.1) turns into

$$\text{finding } \ x \in X, \ \text{ such that } \ x \in F(T). \tag{7.1}$$

The solution set of the problem (6.1) is $F(T)$.

In the sequel, we give the consequences of Theorems 4.1 and 5.1 which state that the coercivity condition **R1**, in the case of $Az = [\overrightarrow{Tzz}]$, is a sufficient and necessary condition for the nonemptiness of the solution set of the fixed point problem (7.1), (i.e. $F(T)$), and the coercivity condition **R2**, in the case of $Az = [\overrightarrow{Tzz}]$, is equivalent to boundedness of a solution set of the fixed point problem (7.1), (i.e. $F(T)$). The following corollaries are easily derived from Theorems 4.1 and 5.1.

Corollary 7.1. *Let X be a Hadamard space with dual X^* and $T : X \to X$ be a nonexpansive mapping. Suppose the operator $A : X \to 2^{X^*}$ with $Az = [\overrightarrow{Tzz}]$ satisfies the range condition. Then the coercivity condition* **R1** *holds if and only if $F(T) \neq \emptyset$.*

Corollary 7.2. *Let X be a Hadamard space with dual X^* and $T : X \to X$ be a nonexpansive mapping. Suppose the operator $A : X \to 2^{X^*}$ with $Az = [\overrightarrow{Tzz}]$ satisfies the range condition. Then the coercivity condition* **R2** *holds if and only if the set $F(T)$ is nonempty and bounded.*

Example 7.1. Let (X, d) be a flat Hadamard space with dual X^*, $x_0 \in X$ and $0 < t < 1$. Define $T : X \longrightarrow X$ by $Tx = tx \oplus (1-t)x_0$. Then, for all $x, y \in X$, we obtain

$$d(Tx, Ty) = d(tx \oplus (1-t)x_0, ty \oplus (1-t)x_0) = td(x, y) < d(x, y),$$

which implies T is a nonexpansive mapping. Define $A : X \longrightarrow 2^{X^*}$ by $A(x) = [\overrightarrow{Txx}]$. By [21, Section 6], the operator A is monotone and satisfies the range condition. We show that the operator A satisfies the coercivity condition **R2**. For this, in **R2**, put $z = x_0$ and $\delta > 0$ is fixed. Suppose $x \in X - B_\delta[x_0]$ is arbitrary. Set $y = tx \oplus (1-t)x_0$. Then $d(y, x_0) = td(x, x_0) < d(x, x_0)$, which implies $y \in X \cap B_{d(x,x_0)}(x_0)$. On the other hand,

$$\begin{aligned}
2\langle[\overrightarrow{Tyy}], \overrightarrow{yx}\rangle &= 2\langle\overrightarrow{(t(tx \oplus (1-t)x_0) \oplus (1-t)x_0)(tx \oplus (1-t)x_0)}, \overrightarrow{(tx \oplus (1-t)x_0)x}\rangle \\
&= d^2(t(tx \oplus (1-t)x_0) \oplus (1-t)x_0, x) \\
&\quad - d^2(t(tx \oplus (1-t)x_0) \oplus (1-t)x_0, tx \oplus (1-t)x_0) \\
&\quad - d^2(tx \oplus (1-t)x_0, x) \\
&= t(1-t)^2 d^2(x, x_0) + (1-t)d^2(x, x_0) - t^3(1-t)d^2(x, x_0) \\
&\quad - (1-t)^2 t^2 d^2(x, x_0) - (1-t)^2 d^2(x, x_0) \\
&= 2t(1-t)^2 d^2(x, x_0) > 0,
\end{aligned}$$

which implies $\sup_{y^* \in A(y)} \langle y^*, \overrightarrow{yx}\rangle > 0$.

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