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A BEST PROXIMITY POINT THEOREM FOR RELATIVELY NONEXPANSIVE MAPPINGS IN THE ABSENCE OF THE PROXIMAL NORMAL STRUCTURE PROPERTY

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Abstract. The well-known Kirk [8] fixed point theorem for nonexpansive mapping relies on the geometric notion called normal structure property. Göhde [5] provided sufficient conditions for the existence of a fixed point of a nonexpansive mapping without using normal structure property. In [4], Kirk et.al. introduced a notion called relatively nonexpansive mapping and provided sufficient conditions for the existence of best proximity points for such mappings using the proximal normal structure property. The main result of this manuscript provides the existence of best proximity points of a relatively nonexpansive mapping without using the proximal normal structure property. Also, our main result extends Göhde's fixed point theorem in best proximity point setting. An example is given to illustrate our main result.
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1. Introduction

Let X be a metric space and K be a nonempty subset of X. A mapping $T : K \to X$ is a *nonexpansive mapping* if $d(Tx, Ty) \leq d(x, y)$, for all $x, y \in K$. A point $x \in K$ is a *fixed point* of T if $T(x) = x$. It is easy to see examples of a nonexpansive self mapping on a closed bounded convex subset of a normed linear space having no fixed points. In 1965, Browder [3], Göhde [6] independently proved that if K is a nonempty closed bounded convex subset of a uniformly convex Banach space, then any nonexpansive self-mapping $T : K \to K$ has at least one fixed point in K. At the same time, in [8], Kirk furnished a generalized version of Browder and Göhde's result using a geometric notion called "*normal structure property*", which was introduced by Brodskii and Milman in [2]. Kirk's theorem states that

Theorem 1.1. [8] *Let K be a nonempty weakly compact convex subset of a normed linear space X and $f : K \to K$ be a nonexpansive mapping. Suppose that K has normal structure property. Then f has at least one fixed point in K.*

Since every nonempty closed bounded convex subset of a uniformly convex Banach space enjoys normal structure property, Kirk's theorem generalizes the results of Browder and Göhde and Theorem 1.1 is considered to be the fundamental fixed point theorem for nonexpansive mapping. One may find an interesting proof technique for Theorem 1.1 due to Jachymski [7], using partial ordering argument without invoking Zorn's lemma. Kirk's theorem drew attention of many researchers in the direction of finding conditions which relaxes normal structure property. Many authors attempt to approximate the fixed points of a nonexpansive mapping by iterative method. Let us consider a nonexpansive mapping $f : K \to K$ and $x_0 \in K$. Then the sequence $\{x_n\}$ of iteration of f starting at the point x_0 is defined as $x_n = f(x_{n-1})$, for all $n \in \mathbb{N}$. It is well-known fact that if a sequence of iteration of a continuous self-mapping f converges, then the limit of it must be a fixed point of f. This motivates many authors to construct an iterative type convergent sequence of a nonexpansive mapping $f : K \to K$ to obtain a fixed point of f. Ishikawa, Mann and Krasnoselskii iterations were developed in this direction. In [5], Göhde proved an interesting result (Corollary 3.1) which provides sufficient conditions on the sequence of iteration which assure the existence of at least one fixed point of a nonexpansive self-mapping, without using the normal structure property.

On the other hand, Kirk [4] et.al. introduced a notion called relatively nonexpansive mapping and a geometric concept called proximal normal structure which are proper generalization of nonexpansive mapping and normal structure property respectively. Using proximal normal structure property, the authors established sufficient conditions for the existence of generalized fixed points, called best proximity points for such mappings. In this manuscript, we provide sufficient conditions for the existence of best proximity points of a relatively nonexpansive mappings without using proximal normal structure property. The main result of this manuscript generalizes Göhde fixed point theorem in best proximity point setting.

2. Notations and Known Results

Let A, B be nonempty subsets of a normed linear space X and we fix the following notations for our further discussion.

$$\operatorname{dist}(A,B) := \inf\{\|a-b\| : a \in A,\ b \in B\}$$
$$A_0 := \{x \in A : \|x-y\| = \operatorname{dist}(A,B),\ \text{for some } y \in B\}$$
$$B_0 := \{y \in B : \|x-y\| = \operatorname{dist}(A,B),\ \text{for some } x \in A\}$$

In general A_0 and B_0 may be empty. In [9], Kirk et.al., provided sufficient conditions for the nonemptiness of A_0 and B_0. A mapping $T : A \cup B \to A \cup B$ is *cyclic* if $T(A) \subseteq B$ and $T(B) \subset A$. A point $x_0 \in A$ is a *best proximity point of* T if $\|x_0 - Tx_0\| = \operatorname{dist}(A,B)$. If $A = B$, then best proximity points are nothing but the fixed points of T. Some interesting best proximity point theorems in metric space setting can be found in [1] and reference therein. A cyclic mapping $T : A \cup B \to A \cup B$

is *relatively nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$, for all $x \in A$, $y \in B$. If $A = B$, then relatively nonexpansive mapping reduces to the usual nonexpansive mapping.

In [4], Kirk et.al. introduced a geometric notion called *proximal normal structure* and used it to prove the following theorem.

Theorem 2.1. *Let A and B be nonempty weakly compact convex subsets of a normed linear space X and $T : A \cup B \to A \cup B$ be a relatively nonexpansive mapping. Suppose that the pair (A, B) has proximal normal structure property. Then there is a point $x_0 \in A$ such that $\|x_0 - T(x_0)\| = \operatorname{dist}(A, B)$.*

Note that Theorem 1.1 is a particular case of Theorem 2.1 by considering $A = B = K$ in Theorem 2.1. A cyclic mapping $T : A \cup B \to A \cup B$ is *cyclic contraction* if there is a constant $q \in (0, 1)$ such that $\|Tx - Ty\| \leq q\|x - y\| + (1 - q)\operatorname{dist}(A, B)$, for all $x \in A$, $y \in B$. It is worth mentioning that relatively nonexpansive mappings and cyclic contraction mappings are need not be continuous. In this manuscript, we provide sufficient conditions for the existence of best proximity points of a relatively nonexpansive mapping without using proximal normal structure property.

3. Main Results

Now, let us state and prove a generalization of Göhde's fixed point theorem for a non-convex domain.

Theorem 3.1. *Let A, B be nonempty closed bounded convex subsets of a normed linear space X with $A_0 \neq \emptyset$ and $T : A \cup B \to A \cup B$ be a relatively nonexpansive mapping. Let M be a compact subset of A_0 such that for each $x \in A_0$ the iterated subsequence $\left\{T^{2n}(x)\right\}$ has a limit point in M. Suppose that T is continuous on M. Then there exists $x^* \in A$ such that $\|x^* - Tx^*\| = \operatorname{dist}(A, B)$.*

Proof. Choose $(u_0, v_0) \in A_0 \times B_0$ such that $\|u_0 - v_0\| = \operatorname{dist}(A, B)$. For each $q \in (0, 1)$, define a mapping $T_q : A \cup B \to A \cup B$ by

$$T_q(x) := \begin{cases} qTx + (1 - q)v_0 & \text{if } x \in A, \\ qTx + (1 - q)u_0 & \text{if } x \in B. \end{cases}$$

Since A, B are convex, the function T_q is well-defined. It is easy to verify that

$$\|T_q(x) - T_q(y)\| \leq q\|x - y\| + (1 - q)\operatorname{dist}(A, B),$$

for all $x \in A$, $y \in B$ and $T_q(A) \subseteq B$, $T_q(B) \subseteq A$. i.e., T_q is a cyclic contraction on $A \cup B$. Then for any $x \in A \cup B$, the iterated sequence $\{T^n x\}$ satisfy

$$\|T_q^n x - T_q^{n+1} x\| \leq q^n \|x - T_q x\| + (1 - q^n)\operatorname{dist}(A, B).$$

Thus, for any $x_0 \in A$,

$$\|T_q^{2n} x_0 - T_q^{2n+1} x_0\| \to \operatorname{dist}(A, B) \text{ as } n \to \infty.$$

We can find $n_0 \in \mathbb{N}$ such that $\|T_q^{2n_0} x_0 - T_q^{2n_0+1} x_0\| < (1 - q) + \operatorname{dist}(A, B)$.

Put $z_0 = T_q^{2n_0} x_0$. Then $\|z_0 - T_q(z_0)\| < (1-q) + \text{dist}(A,B)$. Also,

$$\begin{aligned}\|z_0 - Tz_0\| &\le \|z_0 - T_q z_0\| + \|T_q z_0 - Tz_0\| \\ &\le (1-q) + \text{dist}(A,B) + (1-q)[\|v_0\| + \|Tz_0\|] \\ &\le (1-q)R + \text{dist}(A,B)\end{aligned}$$

where $R > 2\text{diam}(B) + 1$. Since B is bounded such $R > 0$ exists. Since T is relatively nonexpansive and $z_0 \in A, Tz_0 \in B$, we conclude that, for any $n \in \mathbb{N}$,

$$\|T^{2n} z_0 - T^{2n+1} z_0\| \le \ldots \le \|Tz_0 - T^2 z_0\| \le \|z_0 - Tz_0\|.$$

Thus,

$$\|T^{2n} z_0 - T^{2n+1} z_0\| \le (1-q)R + \text{dist}(A,B), \text{ for all } n \in \mathbb{N}. \tag{3.1}$$

Let $\varepsilon > 0$. Since $\{T^{2n} z_0\}$ has a limit point in M, there exists $y_q \in M$ and $k \in \mathbb{N}$ such that $\|T^{2k} z_0 - y_q\| \le \varepsilon$. By the continuity of T at y_q, we can find $\delta > 0$ such that

$$\|x - y_q\| < \delta \Rightarrow \|Tx - Ty_q\| < \varepsilon.$$

Choose $\delta > 0$ small enough so that $\delta < \varepsilon$. Since y_q is a limit point of $\{T^{2n} z_0\}$, there exists $m \in \mathbb{N}$ such that $\|T^{2m} z_0 - y_q\| < \delta < \varepsilon$. Then $\|T^{2m+1} z_0 - Ty_q\| < \varepsilon$. Now,

$$\begin{aligned}\|Ty_q - y_q\| &\le \|Ty_q - T^{2m+1} z_0\| + \|T^{2m+1} z_0 - T^{2m} z_0\| + \|T^{2m} z_0 - y_q\| \\ &\le 2\varepsilon + (1-q)R + \text{dist}(A,B)\end{aligned}$$

Thus, for any $q \in (0,1)$, we obtain $y_q \in M$ such that

$$\|Ty_q - y_q\| \le 2\varepsilon + (1-q)R + \text{dist}(A,B),$$

where $\varepsilon > 0$ is arbitrary. Since M is compact, there is a convergent subsequence $\{y_{q_i}\}$ of $\{y_q\}$ converges to a point $x^* \in M$ as $q_i \to 1$ and hence $Ty_{q_i} \to Tx^*$ as $q_i \to 1$. Thus,

$$\begin{aligned}\text{dist}(A,B) &\le \|Tx^* - x^*\| \\ &\le \|Tx^* - Ty_{q_i}\| + \|Ty_{q_i} - y_{q_i}\| + \|y_{q_i} - x^*\| \\ &\le \|Tx^* - Ty_{q_i}\| + 2\varepsilon + (1-q_i)R + \text{dist}(A,B) + \|y_{q_i} - x^*\| \\ &\to 2\varepsilon + \text{dist}(A,B).\end{aligned}$$

i.e., $\text{dist}(A,B) \le \|Tx^* - x^*\| \le 2\varepsilon + \text{dist}(A,B)$. Since $\varepsilon > 0$ is arbitrary,

$$\|Tx^* - x^*\| = \text{dist}(A,B). \qquad \square$$

Let us illustrate the above theorem by the following example.

Example 3.1. Consider the Banach space ℓ^1 with $\|x\|_1 := \sum_n |x(n)|$, for all $x \in \ell^1$. Let $A := \{(0,x) : \|x\|_1 \le 1\}, B := \{(1,y) : \|y\|_1 \le 1\}$. Clearly, A and B are nonempty closed bounded convex subset of ℓ^1. Let $T : A \cup B \to A \cup B$ be a mapping defined as follows:

$$T(x,y) := \begin{cases} (1, \frac{y}{2}) & \text{if } x = 0, \\ (0, \frac{y}{2}) & \text{if } x = 1. \end{cases}$$

Clearly, $T(A) \subseteq B$ and $T(B) \subseteq A$. Also,

$$\|T(0,x) - T(1,y)\|_1 = 1 + \sum_n \left| \frac{x(n)}{2} - \frac{y(n)}{2} \right| \leq 1 + \sum_n |x(n) - y(n)| = \|(0,x) - (1,y)\|_1$$

i.e., T is a relatively nonexpansive mapping. Since

$$\|T(0,0) - T(1,0)\|_1 = \|(0,0) - (1,0)\|_1,$$

T is not a contractive type mapping. Let M be a compact subset of A containing $(0,0)$. It is easy to verify that for any $(0,x) \in A$, $T^{2n}(0,x) = \left(0, \frac{x}{2^{2n}}\right)$. i.e., $T^{2n}(0,x)$ has a limit point in $(0,0) \in M$. By Theorem 3.1, T has a best proximity point $(0,0)$ in A.

Since every nonexpansive self mapping can be considered as a relatively nonexpansive mapping, we can obtain the following Göhde's fixed point theorem for nonexpansive mapping as a corollary to the Theorem 3.1 by taking $A = B = K$.

Corollary 3.1. (Göhde's Fixed Point Theorem) *Let K be a nonempty closed bounded convex subset of a normed linear space X and $T : K \to K$ be a nonexpansive mapping. Suppose there is a compact subset M of K such that for each $x \in K$, the iterated sequence $\{T^n x\}$ has a limit point in M. Then T has at least one fixed point in M.*

References

[1] P. Ardsalee, S. Saejung, *Some common best proximity point theorems via a fixed point theorem in metric spaces*, Fixed Point Theory, **21**(2020), no. 2, 413-426.
[2] M.S. Brodskii, D.P. Milman, *On the center of a convex set*, (Russian), Doklady Akad. Nauk SSSR (N.S.), **59**(1948), 837-840.
[3] F.E. Browder, *Nonexpansive nonlinear operators in a Banach space*, Proc. Nat. Acad. Sci. U.S.A., **54**(1965), 1041-1044.
[4] A.A. Eldred, W.A. Kirk, P. Veeramani, *Proximal normal structure and relatively nonexpansive mappings*, Studia Math., **171**(2005), no. 3, 283-293.
[5] D. Göhde, *Uber Fixpunkte bei stetigen Selbstabbildungen mit kompakten Iterierten*, (German), Math. Nachr., **28**(1964), 45-55.
[6] D. Göhde, *Zum Prinzip der kontraktiven Abbildung*, (German), Math. Nachr., **30**(1965), 251-258.
[7] J. Jachymski, *Another proof of the Browder-Göhde-Kirk theorem via ordering argument*, Bull. Austral. Math. Soc., **65**(2002), no. 1, 105-107.
[8] W.A. Kirk, *A fixed point theorem for mappings which do not increase distances*, Amer. Math. Monthly, **72**(1965), 1004-1006.
[9] W.A. Kirk, S. Reich, P. Veeramani, *Proximinal retracts and best proximity pair theorems*, Numer. Funct. Anal. Optim., **24**(2003), 851-862.

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