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# EIGENVALUE INTERVALS FOR ITERATIVE SYSTEMS OF SECOND-ORDER NONLINEAR EQUATIONS WITH IMPULSES AND M-POINT BOUNDARY CONDITIONS

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**Abstract.** This paper is devoted to determining the eigenvalue intervals of the parameters $\lambda_1, \lambda_2, ..., \lambda_n$ for which there exist positive solutions of the iterative systems of second-order with $m$-point impulsive boundary value problem. We use the Guo-Krasnosel'skii fixed point theorem on the cones in order to achieve our results. An example is also presented to demonstrate the applicability of the main results obtained.
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## 1. Introduction

It is widely agreed that the theory and applications of differential equations with impulsive effects are an important area of research, because it is significantly richer than the corresponding theory of differential equations without impulsive effects. Several models such as population, ecology, biological system, pharmacokinetics, biotechnology, and optimum control can be stated using impulsive differential equations. In addition, impulsive differential equations provide for a more realistic approach to modeling many real-world issues in areas including control theory, electronics, chemistry, mechanics, economics, medicine, electrical circuits, and population dynamics. We recommend the reader to references [1, 2, 3, 15, 24, 25] for an introduction to the general theory of impulsive differential equations, and [7, 18] for applications of impulsive differential equations.

Many authors have investigated second-order impulsive boundary value problems in the literature; for a list of such, see [5, 6, 9, 12, 13, 17, 16, 27, 28, 29, 30] in references. See [12, 27] in the references for some recent studies on second-order with m-point impulsive boundary value problems. In addition, some authors have been interested in systems of second-order impulsive boundary value problems, for these, we refer to reader to [6, 9, 17, 16]. On the other hand, because of the importance of both theory and applications, achieving optimal eigenvalue intervals for the existence of positive solutions of iterative systems with nonlinear boundary value problems has gained a lot of interest by an application of Guo–Krasnosel'skii fixed point theorem. [4, 10, 14, 11, 19, 21, 20, 22, 23, 26] are a few papers in this line. However, there is no work concerning the eigenvalues for iterative system of nonlinear second-order with m-point impulsive boundary value problem.

Motivated by the mentioned above result, in this study, we consider the following iterative system of nonlinear second-order with m-point impulsive boundary value problem (IBVP):

$$\begin{cases} z_i''(t) + \lambda_i p_i(t) g_i(z_{i+1}(t)) = 0, \ t \in J = [0,1], \ 1 \leq i \leq n, \\ z_{n+1}(t) = z_1(t), \\ \triangle z_i|_{t=t_k} = \lambda_i I_{ik}(z_{i+1}(t_k)), \ t \neq t_k, k = 1, 2, ..., p, \\ \triangle z_i'|_{t=t_k} = -\lambda_i J_{ik}(z_{i+1}(t_k)), \\ a z_i(0) - b z_i'(0) = \sum_{j=1}^{m-2} \alpha_j z_i(\xi_j), \\ c z_i(1) + d z_i'(1) = \sum_{j=1}^{m-2} \beta_j z_i(\xi_j) \end{cases} \tag{1.1}$$

where $J = [0,1], \ t \neq t_k, \ k = 1, 2, ..., p$ with $0 < t_1 < t_2 < ... < t_p < 1$. For $1 \leq i \leq n$, $\triangle z_i|_{t=t_k}$ and $\triangle z_i'|_{t=t_k}$ represent the jump of $z_i(t)$ and $z_i'(t)$ at $t = t_k$, i.e.,

$$\triangle z_i|_{t=t_k} = z_i(t_k^+) - z_i(t_k^-), \quad \triangle z_i'|_{t=t_k} = z_i'(t_k^+) - z_i'(t_k^-),$$

where $z_i(t_k^+)$, $z_i'(t_k^+)$ and $z_i(t_k^-)$, $z_i'(t_k^-)$ symbolize the right-hand limit and left-hand limit of $z_i(t)$ and $z_i'(t)$ at $t = t_k$, $k = 1, 2, ..., p$, respectively.

Throughout this paper, we suppose that the following conditions are provided.

- $(H1)$ $a, b, c, d \in [0, \infty)$ with $ac + ad + bc > 0$; $\alpha_j, \beta_j \in [0, \infty)$, $\xi_j \in (0,1)$, for $j \in \{1, ..., m-2\}$,
- $(H2)$ $g_i : \mathbb{R}^+ \to \mathbb{R}^+$ is continuous, for $1 \leq i \leq n$,
- $(H3)$ $p_i \in C([0,1], \mathbb{R}^+)$. On any closed subinterval of $[0,1]$, for $1 \leq i \leq n$, $p_i$ does not vanish identically.
- $(H4)$ $I_{ik} \in C(\mathbb{R}, \mathbb{R}^+)$ and $J_{ik} \in C(\mathbb{R}, \mathbb{R}^+)$ are bounded functions such that $[d + c(1 - t_k)] J_{ik}(\tau) > c I_{ik}(\tau)$, $t < t_k$, $k = 1, 2, ..., p$, for $1 \leq i \leq n$, where $\tau$ be any nonnegative number.
- $(H5)$ Each of

$$g_i^0 = \lim_{z \to 0^+} \frac{g_i(z)}{z}, \quad I_{ik}^0 = \lim_{z \to 0^+} \frac{I_{ik}(z)}{z}, \quad J_{ik}^0 = \lim_{z \to 0^+} \frac{J_{ik}(z)}{z},$$

$$g_i^\infty = \lim_{z \to \infty} \frac{g_i(z)}{z}, \quad I_{ik}^\infty = \lim_{z \to \infty} \frac{I_{ik}(z)}{z}, \quad J_{ik}^\infty = \lim_{z \to \infty} \frac{J_{ik}(z)}{z}, \ 1 \leq i \leq n,$$

exists as positive real number.

The goal of this study is to determine the eigenvalue intervals of $\lambda_i, 1 \leq i \leq n$, for which the iterative system of nonlinear second-order with m-point IBVP (1.1) has positive solutions. For this, the main tool relied upon is the Guo-Krasnosel'skii fixed point theorem.

This paper's main structure is as follows. We present several definitions and basic lemmas in Section 2, which are important tools for our main result. In Section 3, we find the eigenvalue intervals for which the iterative system of the IBVP (1.1) has positive solutions. We provide an example in Section 4 to show the applicability of our main results.

## 2. Preliminaries

In this section, we first introduce some background definitions in Banach spaces, and then present auxiliary lemmas that will be useful later.

Let $J' = J \backslash \{t_1, t_2, ..., t_p\}$. $C(J)$ indicate the Banach space of all continuous mapping $z : J \to \mathbb{R}$ with the norm $\|z\| = \sup_{t \in J} |z(t)|$, $PC(J) = \{z : J \to \mathbb{R} : z \in C(J'),\ z(t_k^+)$ and $z(t_k^-)$ exist and $z(t_k^-) = z(t_k),\ k = 1, 2, ..., p\}$ is also a Banach space with norm $\|z\|_{PC} = \sup_{t \in J} |z(t)|$. Let $\mathbb{B} = PC(J) \cap C^2(J')$. A function $(z_1, ..., z_n) \in \mathbb{B}^n$ is referred a solution of the iterative system of the IBVP (1.1) provided that it yields the iterative system of the IBVP (1.1).

We will first consider the case of $i = 1$ in the iterative system of the IBVP (1.1). So, we will give the solution $z_1$ of the IBVP (2.1). Then, we can find $z_n$, since $z_1$ is known. If this argument continues, we can obtain $z_{n-1}$, then $z_{n-2}$ etc. and finally $z_2$. As a result, the solution $(z_1, ..., z_n)$ for the iterative system of the IBVP (1.1) is obtained.

Let $h \in C[0, 1]$, then we consider the following IBVP:

$$\begin{cases} -z_1''(t) = h(t),\ t \in J = [0, 1], t \neq t_k, k = 1, 2, ..., p, \\ \triangle z_1|_{t=t_k} = \lambda_1 I_{1k}(z_2(t_k)), \\ \triangle z_1'|_{t=t_k} = -\lambda_1 J_{1k}(z_2(t_k)), \\ az_1(0) - bz_1'(0) = \displaystyle\sum_{j=1}^{m-2} \alpha_j z_1(\xi_j), \\ cz_1(1) + dz_1'(1) = \displaystyle\sum_{j=1}^{m-2} \beta_j z_1(\xi_j). \end{cases} \tag{2.1}$$

The solutions of the corresponding homogeneous equation are denoted by $\theta$ and $\phi$.

$$-z_1''(t) = 0, t \in [0, 1], \tag{2.2}$$

under the initial conditions

$$\begin{cases} \theta(0) = b, \quad \theta'(0) = a, \\ \phi(1) = d, \quad \phi'(1) = -c. \end{cases} \tag{2.3}$$

Using the initial conditions (2.3), we can deduce from equation (2.2) for $\theta$ and $\phi$ the following equations:

$$\theta(t) = b + at, \quad \phi(t) = d + c(1-t). \tag{2.4}$$

Set

$$\rho := ad + ac + bc, \tag{2.5}$$

and

$$\triangle = \begin{vmatrix} -\sum_{j=1}^{m-2}\alpha_j(b+a\xi_j) & \rho - \sum_{j=1}^{m-2}\alpha_j[d+c(1-\xi_j)] \\ \rho - \sum_{j=1}^{m-2}\beta_j(b+a\xi_j) & -\sum_{j=1}^{m-2}\beta_j[d+c(1-\xi_j)] \end{vmatrix}. \tag{2.6}$$

**Lemma 2.1.** *Let (H1)-(H5) hold. Suppose that*
*(H6)* $\triangle \neq 0$.
*If* $z_1 \in \mathbb{B}$ *is a solution of the equation*

$$z_1(t) = \int_0^1 G(t,s)h(s)ds + \sum_{k=1}^{p} W_{1k}(t,t_k) + (b+at)A_1(h) + (d+c(1-t))B_1(h), \tag{2.7}$$

*where*

$$G(t,s) = \frac{1}{\rho}\begin{cases} (b+as)[d+c(1-t)], & s \le t, \\ (b+at)[d+c(1-s)], & t \le s, \end{cases} \tag{2.8}$$

$$W_{1k}(t,t_k) = \frac{1}{\rho}\begin{cases} (b+at)[-c\lambda_1 I_{1k}(z_2(t_k)) + (d+c(1-t_k))\lambda_1 J_{1k}(z_2(t_k))], & t < t_k, \\ (d+c(1-t))[a\lambda_1 I_{1k}(z_2(t_k)) + (b+at_k)J_{1k}(z_2(t_k))], & t_k < t, \end{cases} \tag{2.9}$$

$$A_1(h) = \frac{1}{\triangle}\begin{vmatrix} \sum_{j=1}^{m-2}\alpha_j K_{1j} & \rho - \sum_{j=1}^{m-2}\alpha_j[d+c(1-\xi_j)] \\ \sum_{j=1}^{m-2}\beta_j K_{1j} & -\sum_{j=1}^{m-2}\beta_j[d+c(1-\xi_j)] \end{vmatrix}, \tag{2.10}$$

$$B_1(h) = \frac{1}{\triangle}\begin{vmatrix} -\sum_{j=1}^{m-2}\alpha_j(b+a\xi_j) & \sum_{j=1}^{m-2}\alpha_j K_{1j} \\ \rho - \sum_{j=1}^{m-2}\beta_j(b+a\xi_j) & \sum_{j=1}^{m-2}\beta_j K_{1j} \end{vmatrix}, \tag{2.11}$$

*and*

$$K_{1j} = \int_0^1 G(\xi_j,s)h(s)ds + \sum_{k=1}^{p} W_{1k}(\xi_j,t_k), \tag{2.12}$$

*then* $z_1$ *is a solution of the IBVP* (2.1).

*Proof.* Let $z_1$ satisfies the integral equation (2.7), then we get

$$z_1(t) = \int_0^1 G(t,s)h(s)ds + \sum_{k=1}^{p} W_{1k}(t,t_k) + (b+at)A_1(h) + (d+c(1-t))B_1(h),$$

i.e.,

$$\begin{aligned} z_1(t) \quad = \quad & \frac{1}{\rho}\int_0^t (b+as)[d+c(1-t)]h(s)ds + \frac{1}{\rho}\int_t^1 (b+at)[d+c(1-s)]h(s)ds \\ & + \frac{1}{\rho}\sum_{0<t_k<t}(d+c(1-t))[a\lambda_1 I_{1k}(z_2(t_k)) + (b+at_k)J_{1k}(z_2(t_k))] \\ & + \frac{1}{\rho}\sum_{t<t_k<1}(b+at)[-c\lambda_1 I_{1k}(z_2(t_k)) + (d+c(1-t_k))\lambda_1 J_{1k}(z_2(t_k))] \\ & + (b+at)A_1(h) + (d+c(1-t))B_1(h), \end{aligned}$$

$$\begin{aligned} z_1'(t) \quad = \quad & \frac{1}{\rho}\int_0^t (-c)(b+as)h(s)ds + \frac{1}{\rho}\int_t^1 (a)[d+c(1-s)]h(s)ds \\ & + \frac{1}{\rho}\sum_{0<t_k<t}(-c)[a\lambda_1 I_{1k}(z_2(t_k)) + (b+at_k)J_{1k}(z_2(t_k))] \\ & + \frac{1}{\rho}\sum_{t<t_k<1}(a)[-c\lambda_1 I_{1k}(z_2(t_k)) + (d+c(1-t_k))\lambda_1 J_{1k}(z_2(t_k))] \\ & + aA_1(h) + (-c)B_1(h). \end{aligned}$$

Thus

$$z_1''(t) \quad = \quad \frac{1}{\rho}(-ct - (d+c(1-t)))h(t) = -h(t),$$

i.e.,

$$z_1''(t) + h(t) = 0.$$

Since

$$\begin{aligned} z_1(0) \quad = \quad & \frac{1}{\rho}\int_0^1 b[d+c(1-s)]h(s)ds \\ & + \frac{1}{\rho}\sum_{k=1}^{p} b[-c\lambda_1 I_{1k}(z_2(t_k)) + (d+c(1-t_k))\lambda_1 J_{1k}(z_2(t_k))] \\ & + bA_1(h) + (c+d)B_1(h) \end{aligned}$$

and

$$\begin{aligned} z_1'(0) &= \frac{1}{\rho}\int_0^1 (a)[d+c(1-s)]h(s)ds \\ &+\frac{1}{\rho}\sum_{k=1}^{p}(a)[-c\lambda_1 I_{1k}(z_2(t_k))+(d+c(1-t_k))\lambda_1 J_{1k}(z_2(t_k))] \\ &+aA_1(h)+(-c)B_1(h), \end{aligned}$$

we get

$$\begin{aligned} az_1(0)-bz_1'(0)=\rho B_1(h)=\sum_{j=1}^{m-2}\alpha_j\bigg[\int_0^1 G(\xi_j,s)h(s)ds+\sum_{k=1}^{p}W_{1k}(\xi_j,t_k) \\ +(b+a\xi_j)A_1(h)+(d+c(1-\xi_j))B_1(h)\bigg]. \end{aligned} \tag{2.13}$$

Since

$$\begin{aligned} z_1(1) &= \frac{1}{\rho}\int_0^1 (b+as)(c+d)h(s)ds \\ &+\frac{1}{\rho}\sum_{k=1}^{p}(c+d)[a\lambda_1 I_{1k}(z_2(t_k))+(b+at_k)J_{1k}(z_2(t_k))] \\ &+(a+b)A_1(h)+dB_1(h) \end{aligned}$$

and

$$\begin{aligned} z_1'(1) &= \frac{1}{\rho}\int_0^1 (-c)(b+as)h(s)ds \\ &+\frac{1}{\rho}\sum_{k=1}^{p}(-c)[a\lambda_1 I_{1k}(z_2(t_k))+(b+at_k)J_{1k}(z_2(t_k))] \\ &+aA_1(h)+(-c)B_1(h), \end{aligned}$$

we get

$$\begin{aligned} cz_1(1)+dz'(1)=\rho A_1(h)=\sum_{j=1}^{m-2}\beta_j\bigg[\int_0^1 G(\xi_j,s)h(s)ds+\sum_{k=1}^{p}W_{1k}(\xi_j,t_k) \\ +(b+a\xi_j)A_1(h)+(d+c(1-\xi_j))B_1(h)\bigg]. \end{aligned} \tag{2.14}$$

From equations (2.13) and (2.14), we have the following equations:

$$\begin{cases} -\Big[\sum_{j=1}^{m-2}\alpha_j(b+a\xi_j)\Big]A_1(h)+\Big[\rho-\sum_{j=1}^{m-2}\alpha_j(d+c(1-\xi_j))\Big]B_1(h)=\sum_{j=1}^{m-2}\alpha_j K_{1j}, \\ \Big[\rho-\sum_{j=1}^{m-2}\beta_j(b+a\xi_j)\Big]A_1(h)+\Big[-\sum_{i=1}^{m-2}\beta_j(d+c(1-\xi_j))\Big]B_1(h)=\sum_{j=1}^{m-2}\beta_j K_{1j} \end{cases}$$

which yields that $A_1(h)$ and $B_1(h)$ satisfy (2.10) and (2.11), respectively. □

**Lemma 2.2.** *Let (H1)-(H6) hold. Suppose that*

$$(H7)\quad \triangle<0,\ \rho-\sum_{j=1}^{m-2}\beta_j(b+a\xi_j)>0,\ \rho-\sum_{j=1}^{m-2}\alpha_j(d+c(1-\xi_j))>0.$$

*Then for $z_1\in\mathbb{B}$ with $h\geq 0$, the solution $z_1$ of the IBVP (2.1) satisfies $z_1(t)\geq 0$ for $t\in[0,1]$.*

*Proof.* Firstly, it is clear that the Green's function $G_1$ is positive for $t,s\in[0,1]\times[0,1]$. In addition, with the condition $(H7)$, $A_1(h)$ and $B_1(h)$ are positive. Lastly, since $I_{1k}$ and $J_{1k}$ are positive, we obtain the positivity of $W_{1k}$. As a result, $z_1(t)$ is positive for $t\in[0,1]$. □

**Lemma 2.3.** *Let (H1)-(H7) hold. Suppose that*

$$(H8)\quad c-\sum_{j=1}^{m-2}\beta_j<0.$$

*Then the solution $z_1\in\mathbb{B}$, of the IBVP (2.1) satisfies $z_1'(t)\geq 0$ for $t\in[0,1]$.*

*Proof.* The proof of this lemma is presented in [11]. □

**Lemma 2.4.** *Assume that (H1)-(H8) hold, then for any $t,s\in J$, we have*

$$0\leq G(t,s)\leq G(s,s). \tag{2.15}$$

*Proof.* It is easily obtained from equation (2.8). □

**Lemma 2.5.** *Let (H1)-(H6) hold. Let $\sigma\in(0,\frac{1}{2})$. Then for any $t,s\in J$, we have*

$$G(t,s)\geq\gamma G(s,s) \tag{2.16}$$

*where* $\gamma:=\min\left\{\dfrac{b+a\sigma}{b+a},\dfrac{d+c\sigma}{d+c}\right\}$.

*Proof.* [11] provides the proof for this lemma. □

Let $\mathcal{P}=\{z_1\in PC(J): z_1(t)$ is nonnegative, nondecreasing and concave on $J\}$. So, $\mathcal{P}$ is a cone of $PC(J)$.

**Lemma 2.6.** *Let (H1)-(H8) hold and $z_1(t)\in\mathcal{P}$, $\sigma\in(0,\frac{1}{2})$. Then,*

$$\min_{t\in[\sigma,1-\sigma]} z_1(t)\geq\sigma\|z_1\|_{PC} \tag{2.17}$$

*where* $\|z_1\|_{PC}=\sup_{t\in J}|z_1(t)|$.

*Proof.* We know that $z_1(t)$ is concave on $J$ because of $z_1\in\mathcal{P}$. As a result,

$$\min_{t\in[\sigma,1-\sigma]} z_1(t)=z_1(\sigma)\text{ and }\|z_1\|_{PC}=\sup_{t\in J}|z_1(t)|=z_1(1).$$

Because the graph of $z_1$ is concave down on $J$, we obtain

$$\frac{z_1(1)-z_1(0)}{1-0} \leq \frac{z_1(\sigma)-z_1(0)}{\sigma-0},$$

i.e., $z_1(\sigma) \geq \sigma z_1(1) + (1-\sigma) z_1(0)$. So, $z_1(\sigma) \geq \sigma z_1(1)$. The proof is completed. □

We note that an $n$- tuple $(z_1(t), z_2(t), ..., z_n(t))$ is a solution of the iterative system of the IBVP (1.1) if and only if

$$\begin{aligned}
z_1(t) =& \lambda_1 \int_0^1 G(t,s_1)p_1(s_1)g_1\Bigg(\lambda_2 \int_0^1 G(s_1,s_2)p_2(s_2)g_2\Bigg(\lambda_3 \int_0^1 G(s_2,s_3)p_3(s_3)g_3 ...\\
& g_{n-1}\Bigg(\lambda_n \int_0^1 G(s_{n-1},s_n)p_n(s_n)g_n(z_1(s_n))ds_n + \sum_{k=1}^{p} W_{nk}(s_{n-1},t_k)\\
& + A_n(b+as_{n-1}) + B_n(d+c(1-s_{n-1}))\Bigg)ds_{n-1} + \sum_{k=1}^{p} W_{n-1,k}(s_{n-2},t_k)\\
& + A_{n-1}(b+as_{n-2}) + B_{n-1}(d+c(1-s_{n-2}))\Bigg)ds_{n-2} + ...\\
& + \sum_{k=1}^{p} W_{3k}(s_2,t_k) + A_3(b+as_2) + B_3(d+c(1-s_2))\Bigg)ds_2\\
& + \sum_{k=1}^{p} W_{2k}(s_1,t_k) + A_2(b+as_1) + B_2(d+c(1-s_1))\Bigg)ds_1\\
& + \sum_{k=1}^{p} W_{1k}(t,t_k) + A_1(b+at) + B_1(d+c(1-t)).
\end{aligned}$$

$$\begin{aligned}
z_i(t) =& \lambda_i \int_0^1 G(t,s)p_i(s)g_i(z_{i+1}(s))ds + \sum_{k=1}^{p} W_{ik}(t,t_k) + A_i(b+at)\\
& + B_i(d+c(1-t)),\ t \in J,\\
z_{n+1}(t) =& z_1(t),
\end{aligned}$$

and

$$A_i := A(\lambda_i p_i(.)g_i(z_{i+1}(.))), \quad B_i := B(\lambda_i p_i(.)g_i(z_{i+1}(.))),$$

where

$$A(\lambda_i p_i(.)g_i(z_{i+1}(.)))$$

$$= \frac{1}{\triangle} \begin{vmatrix} \sum_{j=1}^{m-2} \alpha_j \Big[\int_0^1 G(\xi_j,s)\lambda_i p_i(s)g_i(z_{i+1}(s))ds + \sum_{k=1}^{p} W_{ik}(\xi_j,t_k)\Big] & \rho - \sum_{j=1}^{m-2} \alpha_j[d+c(1-\xi_j)] \\ \sum_{j=1}^{m-2} \beta_j \Big[\int_0^1 G(\xi_j,s)\lambda_i p_i(s)g_i(z_{i+1}(s))ds + \sum_{k=1}^{p} W_{ik}(\xi_j,t_k)\Big] & -\sum_{j=1}^{m-2} \beta_j[d+c(1-\xi_j)] \end{vmatrix},$$

$$B(\lambda_i p_i(.)g_i(z_{i+1}(.)))$$

$$= \frac{1}{\triangle}\begin{vmatrix} -\sum_{j=1}^{m-2}\alpha_j(b+a\xi_j)+ & \sum_{j=1}^{m-2}\alpha_j\Big[\int_0^1 G(\xi_j,s)\lambda_i p_i(s)g_i(z_{i+1}(s))ds+\sum_{k=1}^{p}W_{ik}(\xi_j,t_k)\Big] \\ \rho-\sum_{j=1}^{m-2}\beta_j(b+a\xi_j) & \sum_{j=1}^{m-2}\beta_j\Big[\int_0^1 G(\xi_j,s)\lambda_i p_i(s)g_i(z_{i+1}(s))ds+\sum_{k=1}^{p}W_{ik}(\xi_j,t_k)\Big] \end{vmatrix},$$

$$W_{ik}(t,t_k) = \frac{1}{\rho}\begin{cases}(b+at)[-c\lambda_i I_{ik}(z_{i+1}(t_k)) + (d+c(1-t_k))\lambda_i J_{ik}(z_{i+1}(t_k))], & t < t_k,\\ (d+c(1-t))[a\lambda_i I_{ik}(z_{i+1}(t_k)) + (b+at_k)J_{ik}(z_{i+1}(t_k))], & t_k < t.\end{cases}$$

To identify the eigenvalue intervals for which the iterative system of the IBVP (1.1) has at least one positive solution in a cone, we will apply the following Guo-Krasnosel'skii's fixed point theorem [8].

**Theorem 2.1.** [8] *Let $X$ be a Banach space, and $P \subset X$ be cone in $X$. Assume that $\Omega_1$ and $\Omega_2$ are two bounded open subsets of $X$ with $0 \in \Omega_1, \bar{\Omega}_1 \subset \Omega_2$. Let $A : P \cap (\bar{\Omega}_2 \backslash \Omega_1) \to P$ be a completely continuous operator, satisfying either*

$$(\mathbf{i})\|Ax\| \le \|x\|,\ x \in P \cap \partial\Omega_1, \quad \|Ax\| \ge \|x\|,\ x \in P \cap \partial\Omega_2,$$

*or*

$$(\mathbf{ii})\|Ax\| \ge \|x\|,\ x \in P \cap \partial\Omega_1, \quad \|Ax\| \le \|x\|,\ x \in P \cap \partial\Omega_2.$$

*Then $A$ has at least one fixed point in $P \cap (\bar{\Omega}_2 \backslash \Omega_1)$.*

## 3. Main results

In this section, we establish criteria to determine the eigenvalues for which the iterative system of the IBVP (1.1) has at least one positive solution in a cone.

Now, we define an integral operator $\mathcal{P} \to \mathbb{B}$, for $z_1 \in \mathcal{P}$, by

$$\begin{aligned} Tz_1(t) = \lambda_1 \int_0^1 & G(t,s_1)p_1(s_1)g_1\bigg(\lambda_2\int_0^1 G(s_1,s_2)p_2(s_2)g_2\bigg(\lambda_3\int_0^1 G(s_2,s_3)p_3(s_3)g_3 \ldots \\ & g_{n-1}\bigg(\lambda_n\int_0^1 G(s_{n-1},s_n)p_n(s_n)g_n(z_1(s_n))ds_n + \sum_{k=1}^{p} W_{nk}(s_{n-1},t_k) \\ & + A_n(b+as_{n-1}) + B_n(d+c(1-s_{n-1}))\bigg)ds_{n-1} + \sum_{k=1}^{p} W_{n-1,k}(s_{n-2},t_k) \\ & + A_{n-1}(b+as_{n-2}) + B_{n-1}(d+c(1-s_{n-2}))\bigg)ds_{n-2} + \ldots \\ & + \sum_{k=1}^{p} W_{3k}(s_2,t_k) + A_3(b+as_2) + B_3(d+c(1-s_2))\bigg)ds_2 \end{aligned} \tag{3.1}$$

$$+\sum_{k=1}^{p} W_{2k}(s_1,t_k) + A_2(b+as_1) + B_2(d+c(1-s_1))\Big)ds_1$$
$$+\sum_{k=1}^{p} W_{1k}(t,t_k) + A_1(b+at) + B_1(d+c(1-t)).$$

Notice from (H1)-(H8) and Lemmas 2.2, 2.3 and the definition of $T$ that, for $z_1 \in \mathcal{P}$, $Tz_1(t) \geq 0,\ (Tz_1)'(t) \geq 0$ *and* $(Tz_1)'(t)$ is concave on $J$. Therefore, $T(\mathcal{P}) \subset \mathcal{P}$. In addition, the Arzela-Ascoli theorem shows that the operator $T$ is completely continuous.

Now, we investigate the appropriate fixed points of $T$ which belong to the cone $\mathcal{P}$.

The following notations are presented for the convenience. Let

$$N_1 := \max_{1\leq i\leq n}\left\{\left[\gamma\mu\int_{\mu}^{1-\mu} G(s,s)p_i(s)ds g_i^{\infty}\right]^{-1}\right\}$$

and

$$\begin{aligned} N_2 \quad &:= \quad \min_{1\leq i\leq n}\left\{\left[\left(\int_0^1 G(s,s)p_i(s)ds + \frac{p}{\rho}(2a+b)(c+d) + \bar{A}_i(a+b) + \bar{B}_i(c+d)\right)\right.\right. \\ &\qquad \left.\left.\cdot\left(\max\{g_i^0, I_{ik}^0, J_{ik}^0\}\right)\right]^{-1}\right\}, \end{aligned}$$

where

$$\bar{A}_i := \frac{1}{\triangle}\begin{vmatrix} \sum_{j=1}^{m-2}\alpha_j\left[\int_0^1 G(\xi_j,s)p_i(s)ds + \frac{p}{\rho}(2a+b)(c+d)\right] & \rho - \sum_{j=1}^{m-2}\alpha_j[d+c(1-\xi_j)] \\ \sum_{j=1}^{m-2}\beta_j\left[\int_0^1 G(\xi_j,s)p_i(s)ds + \frac{p}{\rho}(2a+b)(c+d)\right] & -\sum_{j=1}^{m-2}\beta_j[d+c(1-\xi_j)] \end{vmatrix},$$

$$\bar{B}_i := \frac{1}{\triangle}\begin{vmatrix} -\sum_{j=1}^{m-2}\alpha_j(b+a\xi_j) + & \sum_{j=1}^{m-2}\alpha_j\left[\int_0^1 G(\xi_j,s)p_i(s)ds + \frac{p}{\rho}(2a+b)(c+d)\right] \\ \rho - \sum_{j=1}^{m-2}\beta_j(b+a\xi_j) & \sum_{j=1}^{m-2}\beta_j\left[\int_0^1 G(\xi_j,s)p_i(s)ds + \frac{p}{\rho}(2a+b)(c+d)\right] \end{vmatrix}.$$

It also appears that

$$A_i := A(\lambda_i p_i(s) g_i(z_{i+1}(s))) \leq \lambda_i \bar{A}_i \max\{g_i(z_{i+1}), I_{ik}(z_{i+1}), J_{ik}(z_{i+1})\}$$

and

$$B_i := B(\lambda_i p_i(s) g_i(z_{i+1}(s))) \leq \lambda_i \bar{B}_i \max\{g_i(z_{i+1}), I_{ik}(z_{i+1}), J_{ik}(z_{i+1})\}.$$

**Theorem 3.1.** *Assume that conditions (H1)-(H8) are satisfied. Then, for each* $\lambda_1, \lambda_2, ..., \lambda_n$ *satisfying*

$$N_1 < \lambda_i < N_2, \quad 1 \leq i \leq n, \tag{3.2}$$

*there exists an n-tuple* $(z_1, z_2, ..., z_n)$ *satisfying* (1.1) *such that* $z_i(t) > 0,\ 1 \leq i \leq n,$ *on J.*

*Proof.* Let $\lambda_r,\ 1 \le r \le n$, be as in (3.2). Now, let $\varepsilon > 0$ be chosen such that

$$\max_{1\le i\le n}\left\{\left[\gamma\mu\int_\mu^{1-\mu} G(s,s)p_i(s)ds(g_i^\infty - \varepsilon)\right]^{-1}\right\} \le \min_{1\le r\le n}\lambda_r$$

and

$$\max_{1\le r\le n}\lambda_r \le \min_{1\le i\le n}\left\{\left[\left(\int_0^1 G(s,s)p_i(s)ds + \frac{p}{\rho}(2a+b)(c+d) + \bar{A}_i(a+b) + \bar{B}_i(c+d)\right)\right.\right.$$
$$\left.\left.\cdot\left(\max\{g_i^0+\varepsilon, I_{ik}^0+\varepsilon, J_{ik}^0+\varepsilon\}\right)\right]^{-1}\right\}.$$

The fixed points of the completely continuous operator $T : \mathcal{P} \to \mathcal{P}$ defined by (3.1) are investigated. Based on the definitions of $g_i^0, I_{ik}^0, J_{ik}^0,\ 1 \le i \le n$, there is a $K_1 > 0$ such that, for each $1 \le i \le n$,

$$g_i(z) \le (g_i^0+\varepsilon)z,\ I_{ik}(z) \le (I_{ik}^0+\varepsilon)z,\ J_{ik}(z) \le (J_{ik}^0+\varepsilon)z,\ 0 < z < K_1.$$

Let $z_1 \in \mathcal{P}$ with $\|z_1\| = K_1$. We obtain from Lemma 2.4 and the choice of $\varepsilon$, for $0 \le s_{n-1} \le 1$,

$$\begin{aligned}
&\lambda_n\int_0^1 G(s_{n-1},s_n)p_n(s_n)g_n(z_1(s_n))ds_n + \sum_{k=1}^p W_{nk}(s_{n-1},t_k)\\
&\qquad + A_n(b+as_{n-1}) + B_n(d+c(1-s_{n-1}))\\
&\quad\le \lambda_n\left[\left(\int_0^1 G(s_n,s_n)p_n(s_n)ds_n + \frac{p}{\rho}(2a+b)(c+d) + \bar{A}_n(a+b) + \bar{B}_n(c+d)\right)\right.\\
&\qquad\left.\cdot\left(\max\{g_n^0+\varepsilon, I_{nk}^0+\varepsilon, J_{nk}^0+\varepsilon\}\right)\right]\|z_1\|\\
&\quad\le K_1.
\end{aligned}$$

It continues in a similar manner from Lemma 2.4, for $0 \le s_{n-2} \le 1$, that

$$\begin{aligned}
&\lambda_{n-1}\int_0^1 G(s_{n-2},s_{n-1})p_{n-1}(s_{n-1})g_{n-1}\left(\lambda_n\int_0^1 G(s_{n-1},s_n)p_n(s_n)g_n(z_1(s_n))ds_n\right.\\
&\qquad\left.+\sum_{k=1}^p W_{nk}(s_{n-1},t_k) + A_n(b+as_{n-1}) + B_n(d+c(1-s_{n-1}))\right)ds_{n-1}\\
&\qquad+\sum_{k=1}^p W_{n-1,k}(s_{n-2},t_k) + A_{n-1}(b+as_{n-2}) + B_{n-1}(d+c(1-s_{n-2}))\\
&\qquad\le \lambda_{n-1}\left[\left(\int_0^1 G(s_{n-1},s_{n-1})p_{n-1}(s_{n-1})ds_{n-1} + \frac{p}{\rho}(2a+b)(c+d)\right.\right.\\
&\qquad\left.\left.+\bar{A}_{n-1}(a+b) + \bar{B}_{n-1}(c+d)\right)\cdot\left(\max\{g_{n-1}^0+\varepsilon, I_{n-1,k}^0+\varepsilon, J_{n-1k}^0+\varepsilon\}\right)\right]\|z_1\|\\
&\qquad\le \|z_1\| = K_1.
\end{aligned}$$

If we continue this bootstrapping argument, we get, for $0 \le t \le 1$,

$$\begin{aligned}
&\lambda_1 \int_0^1 G(t,s_1)p_1(s_1)g_1(\lambda_2 ...)ds_1 + \sum_{k=1}^{p} W_{1k}(t,t_k) + A_1(b+at) + B_1(d+c(1-t)) \\
&\quad \le \lambda_1 \Bigg[ \Bigg( \int_0^1 G(s_1,s_1)p_1(s_1)ds_1 + \frac{p}{\rho}(2a+b)(c+d) + \bar{A}_1(a+b) + \bar{B}_1(c+d) \Bigg) \\
&\qquad\qquad \cdot \Bigg( \max\{g_1^0+\varepsilon, I_{1k}^0+\varepsilon, J_{1k}^0+\varepsilon\} \Bigg) \Bigg] K_1 \\
&\quad \le K_1 = \|z_1\|.
\end{aligned}$$

Thus, $\|Tz_1\| \le K_1 = \|z_1\|$. If we established $\Omega_1 = \{z \in \mathbb{B} : \|z\| < K_1\}$, then

$$\|Tz_1\| \le \|z_1\| \ \textit{for} \ z_1 \in \mathcal{P} \cap \partial\Omega_1. \tag{3.3}$$

Next, from the definitions of $g_i^\infty$, $1 \le i \le n$, there is a $\bar{K}_2 > 0$ such that, for each $1 \le i \le n$,

$$g_i(z) \ge (g_i^\infty - \varepsilon)z, \quad z \ge \bar{K}_2.$$

Let

$$K_2 = \max\left\{2K_1, \frac{\bar{K}_2}{\mu}\right\}.$$

Let $z_1 \in \mathcal{P}$ and $\|z_1\| = K_2$. Therefore, from Lemmas 2.5 and 2.6,

$$\min_{t\in[\mu,1-\mu]} z_1(t) \ge \mu\|z_1\| \ge \bar{K}_2$$

is obtained.
As a consequence, with the help of Lemmas 2.5, 2.6 and the choice of $\varepsilon$,
for $0 \le s_{n-1} \le 1$, we get

$$\begin{aligned}
&\lambda_n \int_0^1 G(s_{n-1},s_n)p_n(s_n)g_n(z_1(s_n))ds_n + \sum_{k=1}^{p} W_{nk}(s_{n-1},t_k) \\
&\qquad + A_n(b+as_{n-1}) + B_n(d+c(1-s_{n-1})) \\
&\quad \ge \lambda_n\gamma \int_\mu^{1-\mu} G(s_n,s_n)p_n(s_n)g_n(z_1(s_n))ds_n \\
&\quad \ge \lambda_n\gamma \int_\mu^{1-\mu} G(s_n,s_n)p_n(s_n)(g_n^\infty-\varepsilon)z_1(s_n)ds_n \\
&\quad \ge \lambda_n\gamma\mu \int_\mu^{1-\mu} G(s_n,s_n)p_n(s_n)ds_n(g_n^\infty-\varepsilon)\|z_1\| \\
&\quad \ge \|z_1\| = K_2.
\end{aligned}$$

It continues in a similar manner from Lemmas 2.5, 2.6 and the choice of $\varepsilon$, for $0 \leq s_{n-2} \leq 1$,

$$\begin{aligned}
&\lambda_{n-1} \int_0^1 G(s_{n-2}, s_{n-1}) p_{n-1}(s_{n-1}) g_{n-1}\bigg(\lambda_n \int_0^1 G(s_{n-1}, s_n) p_n(s_n) g_n(z_1(s_n)) ds_n \\
&\qquad + \sum_{k=1}^{p} W_{nk}(s_{n-1}, t_k) + A_n(b + a s_{n-1}) + B_n(d + c(1 - s_{n-1}))\bigg) ds_{n-1} \\
&\qquad + \sum_{k=1}^{p} W_{n-1,k}(s_{n-2}, t_k) + A_{n-1}(b + a s_{n-2}) + B_{n-1}(d + c(1 - s_{n-2})) \\
&\quad \geq \lambda_{n-1} \gamma \int_\mu^{1-\mu} G(s_{n-1}, s_{n-1}) p_{n-1}(s_{n-1}) ds_{n-1} (g_{n-1}^{\infty} - \varepsilon) K_2 \\
&\quad \geq \lambda_{n-1} \gamma \mu \int_\mu^{1-\mu} G(s_{n-1}, s_{n-1}) p_{n-1}(s_{n-1}) ds_{n-1} (g_{n-1}^{\infty} - \varepsilon) K_2 \\
&\quad \geq K_2.
\end{aligned}$$

Again, if we use a bootstrapping argument, we obtain

$$\begin{aligned}
&\lambda_1 \int_0^1 G(t, s_1) p_1(s_1) g_1(\lambda_2 ...) ds_1 + \sum_{k=1}^{p} W_{1k}(t, t_k) + A_1(b + at) + B_1(d + c(1 - t)) \\
&\quad \geq K_2,
\end{aligned}$$

thus,

$$T z_1(t) \geq K_2 = \|z_1\|.$$

Therefore, $\|T z_1\| \geq \|z_1\|$. If we put $\Omega_2 = \{z \in \mathbb{B} : \|z\| < K_2\}$, then

$$\|T z_1\| \geq \|z_1\| \textit{ for } z_1 \in \mathcal{P} \cap \partial \Omega_2. \tag{3.4}$$

We can see that $T$ has a fixed point $z_1 \in \mathcal{P} \cap (\bar{\Omega}_2 \backslash \Omega_1)$ by applying Theorem 2.1 to (3.3) and (3.4). As a result, by setting $z_{n+1} = z_1$, we get a positive solution $(z_1, z_2, ..., z_n)$ of the iterative system of the IBVP (1.1) given iteratively by

$$\begin{aligned}
z_r(t) \quad = \quad & \lambda_r \int_0^1 G(t, s) p_r(s) g_r(z_{r+1}(s)) ds + \sum_{k=1}^{p} W_{rk}(t, t_k) + A_r(b + at) \\
& + B_r(d + c(1 - t)), \quad r = n, n-1, ..., 1.
\end{aligned}$$

The proof is completed.

The positive numbers $N_3$ and $N_4$ are defined as follows for our next result:

$$N_3 := \max_{1 \leq i \leq n} \left\{ \left[ \gamma \mu \int_\mu^{1-\mu} G(s, s) p_i(s) ds g_i^0 \right]^{-1} \right\}$$

and

$$N_4 := \min_{1\le i\le n}\Big\{\Big[\Big(\int_0^1 G(s,s)p_i(s)ds+\frac{p}{\rho}(2a+b)(c+d)+\bar{A}_i(a+b)+\bar{B}_i(c+d)\Big)\cdot\Big(\max\{g_i^\infty, I_{ik}^\infty, J_{ik}^\infty\}\Big)\Big]^{-1}\Big\}.$$

**Theorem 3.2.** *Assume that conditions (H1)-(H8) are satisfied. Then, for each* $\lambda_1, \lambda_2, ..., \lambda_n$ *satisfying*

$$N_3 < \lambda_i < N_4, \quad 1 \le i \le n, \tag{3.5}$$

*there exists an n-tuple* $(z_1, z_2, ..., z_n)$ *satisfying* (1.1) *such that* $z_i(t) > 0,\ 1 \le i \le n,$ *on J.*

*Proof.* Let $\lambda_k,\ 1 \le k \le n$, be as in (3.5). Now, let $\varepsilon > 0$ be chosen such that

$$\max_{1\le i\le n}\Big\{\Big[\gamma\mu\int_\mu^{1-\mu} G(s,s)p_i(s)ds(g_i^0-\varepsilon)\Big]^{-1}\Big\} \le \min_{1\le r\le n}\lambda_r$$

and

$$\max_{1\le r\le n}\lambda_r \le \min_{1\le i\le n}\Big\{\Big[\Big(\int_0^1 G(s,s)p_i(s)ds+\frac{p}{\rho}(2a+b)(c+d)+\bar{A}_i(a+b)+\bar{B}_i(c+d)\Big)\cdot\Big(\max\{g_i^\infty+\varepsilon, I_{ik}^\infty+\varepsilon, J_{ik}^\infty+\varepsilon\}\Big)\Big]^{-1}\Big\}.$$

Let $T$ be completely continuous, cone-preserving operator defined by (3.1). From the definitions of $g_i^0, I_{ik}^0, J_{ik}^0,\ 1 \le i \le n$, there exists an $\bar{K}_3 > 0$ such that, for each $1 \le i \le n$,

$$g_i(z) \ge (g_i^0-\varepsilon)z,\ I_{ik}(z) \ge (I_{ik}^0-\varepsilon)z,\ J_{ik}(z) \ge (J_{ik}^0-\varepsilon)z,\ 0 < z \le \bar{K}_3.$$

Besides, from the definitions of $g_i^0, I_{ik}^0, J_{ik}^0$, it follows that

$$g_i(0) = I_{ik}(0) = J_{ik}(0) = 0,\ 1 \le i \le n,$$

and so there exist $0 < M_n < M_{n-1} < ... < M_2 < \bar{K}_3$ such that

$$\lambda_i \max\{g_i(t), I_{ik}(z_{i+1}(t_k)), J_{ik}(z_{i+1}(t_k))\} \le \frac{M_{i-1}}{\int_0^1 G(s,s)p_i(s)ds+\frac{p}{\rho}(2a+b)(c+d)+\bar{A}_i(a+b)+\bar{B}_i(c+d)},$$
$$for \quad t\in[0,M_i],\ 3\le i\le n$$

and

$$\lambda_2 \max\{g_2(t), I_{2k}(z_3(t_k)), J_{2k}(z_3(t_k))\} \le \frac{\bar{K}_3}{\int_0^1 G(s,s)p_2(s)ds+\frac{p}{\rho}(2a+b)(c+d)+\bar{A}_2(a+b)+\bar{B}_2(c+d)},$$
$$for \quad t\in[0,M_2].$$

Let $z_1 \in \mathcal{P}$ with $\|z_1\| = M_n$. We obtain from Lemma 2.4, for $0 \le s_{n-1} \le 1$,

$$\begin{aligned}
&\lambda_n \int_0^1 G(s_{n-1}, s_n) p_n(s_n) g_n(z_1(s_n)) ds_n + \sum_{k=1}^{p} W_{nk}(s_{n-1}, t_k) \\
&\qquad + A_n(b + as_{n-1}) + B_n(d + c(1 - s_{n-1})) \\
&\le \lambda_n \Bigg[ \bigg( \int_0^1 G(s_n, s_n) p_n(s_n) ds_n + \frac{p}{\rho}(2a + b)(c + d) + \bar{A}_n(a + b) + \bar{B}_n(c + d) \bigg) \\
&\qquad \cdot \max\{\|g_n(z_1)\|, \|I_{nk}(z_1)\|, \|J_{nk}(z_1)\|\} \Bigg] \\
&\le \frac{\bigg( \int_0^1 G(s_n, s_n) p_n(s_n) ds_n + \frac{p}{\rho}(2a + b)(c + d) + \bar{A}_n(a + b) + \bar{B}_n(c + d) \bigg) M_{n-1}}{\bigg( \int_0^1 G(s_n, s_n) p_n(s_n) ds_n + \frac{p}{\rho}(2a + b)(c + d) + \bar{A}_n(a + b) + \bar{B}_n(c + d) \bigg)} \\
&= M_{n-1}.
\end{aligned}$$

If we continue with this bootstrapping argument, we have

$$\begin{aligned}
&\lambda_2 \int_0^1 G(s_1, s_2) p_2(s_2) g_2 \bigg( \lambda_3 \int_0^1 G(s_2, s_3) p_3(s_3) ... g_n(z_1(s_n)) ds_n ... ds_3 \\
&\qquad + \sum_{k=1}^{p} W_{3k}(s_2, t_k) + A_3(b + as_2) + B_3(d + c(1 - s_2)) \bigg) ds_2 \\
&\qquad + \sum_{k=1}^{p} W_{2k}(s_1, t_k) + A_2(b + as_1) + B_2(d + c(1 - s_1)) \\
&\le \bar{K}_3.
\end{aligned}$$

Then

$$\begin{aligned}
Tz_1(t) &\ge \lambda_1 \int_0^1 G(t, s_1) p_1(s_1) g_1 \bigg( \lambda_2 \int_0^1 G(s_1, s_2) p_2(s_2) ... g_n(z_1(s_n)) ds_n ... ds_2 \\
&\qquad + \sum_{k=1}^{p} W_{2k}(s_1, t_k) + A_2(b + as_1) + B_2(d + c(1 - s_1)) \bigg) ds_1 \\
&\qquad + \sum_{k=1}^{p} W_{1k}(t, t_k) + A_1(b + at) + B_1(d + c(1 - t)) \\
&\ge \lambda_1 \gamma \mu \int_\mu^{1-\mu} G(s, s) p_1(s_1)(g_1^0 - \varepsilon) \|z_1\| ds_1 \\
&\ge \|z_1\|.
\end{aligned}$$

Thus, $\|Tz_1\| \ge \|z_1\|$. If we set $\Omega_3 = \{z \in \mathbb{B} | \|z\| < K_n\}$, then

$$\|Tz_1\| \ge \|z_1\| \textit{ for } z_1 \in \mathcal{P} \cap \partial\Omega_3. \tag{3.6}$$

Because each $g_i^\infty, I_{ik}^\infty, J_{ik}^\infty$ are assumed to be a positive real number, it follows that $g_i, I_{ik}, J_{ik}, 1 \le i \le n$, is unbounded at $\infty$.

For each $1 \le i \le n$, set

$$g_i^*(z) = \sup_{0\le s\le z} g_i(s), \quad I_{ik}^*(z) = \sup_{0\le s\le z} I_{ik}(s), \quad J_{ik}^*(z) = \sup_{0\le s\le z} J_{ik}(s).$$

Then, for each $1 \le i \le n$, $g_i^*,\ I_{ik}^*,\ J_{ik}^*$ are nondecreasing real-valued functions, $g_i \le g_i^*,\ I_{ik} \le I_{ik}^*,\ J_{ik} \le J_{ik}^*$, and

$$\lim_{z\to\infty} \frac{g_i^*(z)}{z} = g_i^\infty, \quad \lim_{z\to\infty} \frac{I_{ik}^*(z)}{z} = I_{ik}^\infty, \quad \lim_{z\to\infty} \frac{J_{ik}^*(z)}{z} = J_{ik}^\infty.$$

Then, according to the definitions of $g_i^\infty, I_{ik}^\infty, J_{ik}^\infty, 1 \le i \le n$, there exists $\bar{K}_4$ such that, for each $1 \le i \le n$,

$$g_i^*(z) \le (g_i^\infty + \varepsilon)z,\ I_{ik}^*(z) \le (I_{ik}^\infty + \varepsilon)z,\ J_{ik}^*(z) \le (J_{ik}^\infty + \varepsilon)z,\ z \ge \bar{K}_4.$$

As a result, there exists $K_4 > \max\{2\bar{K}_3, \bar{K}_4\}$ such that, for each $1 \le i \le n$,

$$g_i^*(z) \le g_i^*(K_4),\ I_{ik}^*(z) \le I_{ik}^*(K_4),\ J_{ik}^*(z) \le J_{ik}^*(K_4),\ 0 < x \le K_4.$$

Let $z_1 \in \mathcal{P}$ with $\|z_1\| = K_4$. Then, with the help of the bootstrapping argument, we get

$$\begin{aligned}
Tz_1(t) \le{}& \lambda_1 \int_0^1 G(t,s_1)p_1(s_1)g_1(\lambda_2 ...)ds_1 + \sum_{k=1}^{p} W_{1k}(t,t_k) \\
&+ A_1(b+at) + B_1(d + c(1-t)) \\
\le{}& \lambda_1 \bigg( \int_0^1 G(s_1,s_1)p_1(s_1)ds_1 + \frac{p}{\rho}(2a+b)(c+d) + \bar{A}_1(a+b) + \bar{B}_1(c+d) \bigg) \\
&\cdot \max\{g_1^*(z_2), I_{1k}^*(z_2), J_{1k}^*(z_2)\} \\
\le{}& \lambda_1 \bigg( \int_0^1 G(s_1,s_1)p_1(s_1)ds_1 + \frac{p}{\rho}(2a+b)(c+d) + \bar{A}_1(a+b) + \bar{B}_1(c+d) \bigg) \\
&\cdot \max\{g_1^*(K_4), I_{1k}^*(K_4), J_{1k}^*(K_4)\} \\
\le{}& \lambda_1 \bigg( \int_0^1 G(s_1,s_1)p_1(s_1)ds_1 + \frac{p}{\rho}(2a+b)(c+d) + \bar{A}_1(a+b) + \bar{B}_1(c+d) \bigg) \\
&\cdot \max\{(g_1^\infty + \varepsilon)K_4, (I_{1k}^\infty + \varepsilon)K_4, (J_{1k}^\infty + \varepsilon)K_4\} \\
\le{}& K_4 = \|z_1\|.
\end{aligned}$$

Thus, $\|Tz_1\| \le \|z_1\|$. So, if we put $\Omega_4 = \{z \in \mathbb{B} | \|z\| < K_4\}$, then

$$\|Tz_1\| \le \|z_1\| \ for\ z_1 \in \mathcal{P} \cap \partial\Omega_4. \tag{3.7}$$

By applying Theorem 2.1 to (3.6) and (3.7), we can see that $T$ includes a fixed point $z_1 \in \mathcal{P} \cap (\bar{\Omega}_4 \backslash \Omega_3)$, whereby we can get an n-tuple $(z_1, z_2, ..., z_n)$ satisfying the iterative system of the IBVP (1.1) for the chosen values of $\lambda_i, 1 \le i \le n$, by using $z_{n+1} = z_1$. The proof is completed.

## 4. An Example

**Example 4.1.** In the iterative system of the IBVP (1.1), suppose that $n = m = p = 3$, $p_i(t) = 2$ for $1 \le i \le 3,\ a = c = 4, b = d = 2, \xi_1 = \dfrac{1}{3}, \mu = \dfrac{1}{3}, \alpha_1 = 1$ and $\beta_1 = 6$ i.e.,

$$\begin{cases} z_i''(t) + 2\lambda_i g_i(z_{i+1}(t)) = 0,\ t \in J = [0,1], t \neq t_k, 1 \le i, k \le 3, \\ \triangle z_i|_{t=t_k} = \lambda_i I_{ik}(z_{i+1}(t_k)), \\ \triangle z_i'|_{t=t_k} = -\lambda_i J_{ik}(z_{i+1}(t_k)), \\ 4z_i(0) - 2z_i'(0) = z_i(\frac{1}{3}), \\ 4z_i(1) + 2z_i'(1) = 6z_i(\frac{1}{3}). \end{cases} \tag{4.1}$$

where

$g_1(z_2) = z_2(10^4 - 9999e^{-z_2}),\ g_2(z_3) = z_3(2.10^4 - 19999e^{-2z_3}),$
$g_3(z_1) = z_1(10^4 - (9999,5)e^{-z_1}),$

$$I_{1k}(z_2) = \frac{2z_2^2 + 3z_2}{6 + z_2},\ I_{2k}(z_3) = \frac{z_3^3 + 2z_3}{4 + z_3^2},\ I_{3k}(z_1) = \frac{3z_1^2 + z_1}{4 + 3z_1},$$
$$J_{1k}(z_2) = \frac{4z_2^2 + 6z_2}{1 + z_2},\ J_{2k}(z_3) = \frac{2z_3^3 + 4z_3}{1 + z_3^2},\ J_{3k}(z_1) = \frac{6z_1^2 + 2z_1}{2 + 3z_1}.$$

It is clear that (H1)-(H8) has been satisfied. By simple calculation, we get

$$\rho = 32,\ \theta(t) = 2 + 4t,\ \phi(t) = 6 - 4t,\ \triangle = -\frac{704}{3},\ \gamma = \frac{5}{9},\ \bar{A}_i = \frac{3656}{704},\ \bar{B}_i = \frac{1828}{2112}$$

for $1 \le i \le 3$ and

$$G(t,s) = \frac{1}{32}\begin{cases} (2+4s)(6-4t), & s \le t, \\ (2+4t)(6-4s), & t \le s. \end{cases}$$

We obtain

$g_1^0 = 1,\ g_2^0 = 1,\ g_3^0 = \dfrac{1}{2},\ g_1^\infty = 10^4,\ g_2^\infty = 2.10^4,\ g_3^\infty = 10^4,$

$I_{1k}^0 = \dfrac{1}{2},\ I_{2k}^0 = \dfrac{1}{2},\ I_{3k}^0 = \dfrac{1}{4},\ I_{1k}^\infty = 2,\ I_{2k}^\infty = 1,\ I_{3k}^\infty = 1,$

$J_{1k}^0 = 6,\ J_{2k}^0 = 4,\ J_{3k}^0 = 1,\ J_{1k}^\infty = 4,\ J_{2k}^\infty = 2,\ J_{3k}^\infty = 2,$

$N_1 = \max\{0,0016351401869159, 0,00081757009345794 39\}$

and

$N_2 = \min\{0,003885552808195, 0,0233133168491699, 0,0058283292122925\}.$

Using Theorem 2.1, we obtain the optimal eigenvalue interval of

$$0,0016351401869159 < \lambda_i < 0,003885552808195,\ i = 1,2,3,$$

which has a positive solution to the impulsive boundary value problem (4.1).

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