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STRONG CONVERGENCE OF HALPERN'S TYPE ITERATION FOR α-NONEXPANSIVE SEMIGROUP IN BANACH SPACES AND CAT(0) SPACES

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Abstract. In this paper, we establish strong convergence of the Halpern's type iteration for a semigroup of α-nonexpansive mappings in Banach spaces. Using the concept of quasilinearization, we extend this result to $CAT(0)$ spaces, an important subclass of metric spaces. Our work generalizes and complements several comparable results existing in the literature.
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1. Introduction

Let (X,d) be a metric space. Let $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{R}^+$ be the set of positive integers, real numbers and nonnegative real numbers, respectively. Let C be a nonempty closed and convex subset of X. A mapping $T : C \to X$ is said to be nonexpansive if for all $x, y \in C$, we have

$$d(Tx, Ty) \le d(x, y).$$

In 2011, Aoyama and Kohsaka [3] introduced an important generalization of nonexpansive mappings called α-nonexpansive mappings. For $\alpha < 1$, they said $T : C \to X$ is α-nonexpansive mapping if for any $x, y \in C$,

$$d^2(Tx, Ty) \le \alpha d^2(Tx, y) + \alpha d^2(Ty, x) + (1 - 2\alpha)d^2(x, y) \text{ holds.} \tag{1.1}$$

Ariza et al. [4] noted that the case of $\alpha < 0$ is trivial. Particularly, they showed that $T = I$, the identity mapping, if $\alpha < 0$. Among various generalizations of nonexpansive mapping, this class is important because it contains several nonlinear mappings with application to minimization problem, variational inequality and zeros of maximal operators. Recently, Song et al. [27] defined T to be α-nonexpansive mapping if it satisfies

$$d(Tx,Ty) \leq \alpha d(Tx,y) + \alpha d(Ty,x) + (1-2\alpha)d(x,y) \quad \text{for all } x,y \in C. \tag{1.2}$$

Using convexity of the mapping $t \to t^2$, it is easy to see that (1.2) implies (1.1). We give an example to show that the class of mappings satisfying (1.2) is indeed a proper subclass of the class of mappings satisfying (1.1). Define a mapping $T:[0,3] \to [0,3]$ by

$$Tx = \begin{cases} 0\,, & x \neq 3 \\ 2\,, & x = 3\,. \end{cases}$$

It is easy to see that $2\|Tx-Ty\|^2 \leq \|Tx-y\|^2 + \|x-Ty\|^2$(See [11]). However, there is no $\alpha < 1$ such that for $x,y \in [0,3]$, $\|Tx-Ty\| \leq \alpha\|Tx-y\| + \alpha\|Ty-x)\| + (1-2\alpha)\|x-y\|$ holds. Indeed, for $x=2, y=3$

$$\|Tx-Ty\| = 2\,,\ \|x-y\| = 1\,,\ \|x-Ty\| = 0,\ \|y-Tx\| = 3$$

and $\|Tx-Ty\| \leq \alpha\|Tx-y\| + \alpha\|Ty-x\| + (1-2\alpha)\|x-y\|$ holds only if $\alpha \geq 1$.

In this paper, we consider α-nonexpansive mapping in the sense of Aoyama et al. [4] i.e. mapping for which (1.1) holds.

Definition 1.1. A one-parameter α-nonexpansive semigroup is a family $\mathcal{T} = \{T(t) : t \in \mathbb{R}^+\}$ of mappings on a closed and convex subset C of X satisfying :

(i) $T(0)x = x$ for all $x \in C$.
(ii) T(t+s) = T(t)T(s) for all $s,t \geq 0$.
(iii) for each $t > 0$, T(t) is α-nonexpansive i.e. for some $0 \leq \alpha < 1$ and $x,y \in C$,

$$d^2(Tx,Ty) \leq \alpha d^2(Tx,y) + \alpha d^2(Ty,x) + (1-2\alpha)d^2(x,y).$$

Finding common fixed point for classical nonexpansive semigroup has been extensively studied by several authors in different spaces under different conditions. Xu [30] studied the following implicit iteration for the nonexpansive semigroup in a Hilbert space,

$$x_n = \alpha_n u + (1-\alpha_n)\sigma_{t_n}(x_n)$$

for each $n \geq 1$, where $\sigma_t(x)$ is the average given by

$$\sigma_t(x) = \frac{1}{t}\int_0^t T(s)x\,ds \quad \text{for any } t > 0.$$

Later, Benavides et al. [5] and Aleyner et al. [2] studied strong convergence of the Halpern's iteration

$$x_{n+1} = \alpha_n u + (1-\alpha_n)T(t_n)x_n$$

under the assumption that the semigroup $\mathcal{T}$ satisfies uniform asymptotic regularity condition. Over the years, many researchers have extended the classical semigroup of nonexpansive mappings for a wider class of nonlinear mappings and spaces using various iterative schemes. For example, Cho and Kang [9] studied pseudo contraction semigroups in Banach spaces; Cho et al. [8] studied nonexpansive semigroup in $CAT(0)$ space and Zegeye et al. [32] studied asymptotically nonexpansive semigroup in Banach spaces.

Inspired by work of Naraghirad [22] and Song et al. [27], we consider Halpern's type iteration for an α-nonexpansive semigroup $\mathcal{T}$ and establish strong convergence of this iteration in Banach spaces as well as $CAT(0)$ spaces.

2. Preliminaries and Notation

Let X be a Banach space with dual X^*. For sequences, we denote strong convergence, weak convergence and weak star convergence by $\rightarrow$, $\rightharpoonup$, and $\rightharpoonup^*$, respectively. X is said to be uniformly convex if for each $r \in (0,2]$, the modulus of convexity of X, given by

$$\delta(r) = \inf\{1 - \frac{1}{2}\|x+y\| : \|x\| \leq 1,\ \|y\| \leq 1, \|x-y\| \geq r\}$$

satisfies the inequality $\delta(r) > 0$. Let $S_X = \{x \in X : \|x\| = 1\}$. X is said to be smooth if the norm of X is Gateaux differentiable i.e. for each $x, y \in S_X$, the limit $\lim_{t\to 0} \frac{\|x+ty\|-\|x\|}{t}$ exists.
The normalized duality mapping $J : X \to 2^{X^*}$ is defined by

$$J(x) = \{f \in X^* : \langle x, f\rangle = \|x\|^2,\ \|x\| = \|f\|\},\ \ \forall x \in X.$$

J is said to be weakly sequentially continuous if $\{x_n\} \subset X$ and $x_n \rightharpoonup x \in X$, then $J(x_n) \rightharpoonup^* J(x)$. If a Banach space X admits a sequentially continuous duality mapping J from weak topology to weak-star topology, then J is single-valued and X is smooth(See [15] for more details). If $X = H$, Hilbert space, then $J = I$, the identity mapping on H.

A Banach space X is said to satisfy the Opial condition [14] if for any sequence $\{x_n\}$, $x_n \rightharpoonup x$ implies that $\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\|$ for all $y \in X$ with $y \neq x$. It is well-known that all Hilbert spaces , all finite dimensional Banach spaces and the Banach spaces $l^p(1 \leq p < \infty)$ satisfy the Opial condition. Moreover, if X admits a weakly sequentially continuous duality mapping, then X is smooth and enjoys the Opial condition(See [15], [28] for more details).

The class of α-nonexpansive mappings contains the classes of mappings defined below:
Definition 2.1. Let C be a nonempty subset of a metric space (X, d). A mapping $T : C \to X$ is :

(i) mean nonexpansive if there exists $a, b \geq 0$ with $a + b \leq 1$ and $d(Tx, Ty) \leq ad(x, y) + bd(x, Ty)\ \ \forall x, y \in C$.

(ii) Nonspreading if $2d^2(Tx, Ty) \leq d^2(x, Ty) + d^2(y, Tx)\ \ \forall x, y \in C$.

(ii) Hybird if $3d^2(Tx, Ty) \leq d^2(x, Ty) + d^2(y, Tx) + d^2(x, y)\ \ \forall x, y \in C$.

The following lemmas would be instrumental for the development of our results.

Lemma 2.2. [13] *Let C be a nonempty subset of a metric space (X, d). Let $T : C \to X$ be an α-nonexpansive mapping for some $0 \le \alpha < 1$. Then*

$$d^2(x, Ty) \le \frac{1+\alpha}{1-\alpha} d^2(x, Tx) + \frac{2}{1-\alpha}[\alpha d(x, y) + d(Tx, Ty)]d(x, Tx) + d^2(x, y) \ \ \forall x, y \in C.$$

Lemma 2.3. [23] *Let C be a nonempty subset of a uniformly convex Banach space X with the Opial conditon. Let $T : C \to C$ be an α-nonexpansive mapping for some $0 \le \alpha < 1$.*

(i) *If $F(T) \neq \emptyset$. then T is quasi-nonexpansive and $F(T)$ is closed and convex.*

(ii) *If $\{x_n\}$ converges weakly to y and $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$, then $Ty = y$.*

Lemma 2.4. [29] *Let $r > 0$ be a fixed real number. If X is a uniformly convex Banach space, then there exists a continuous strictly increasing convex function $g : [0, \infty) \to [0, \infty)$ with $g(0) = 0$ such that*

$$\|\lambda x + (1-\lambda)y\|^2 \le \lambda \|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)g(\|x - y\|)$$

for any $\|x\|, \|y\| \le r$ and $\lambda \in [0, 1]$.

Definition 2.5. *Let C and D be nonempty subsets of a real Banach space X with $D \subset C$. A mapping $Q_D : C \to D$ is said to be :*

(a) *sunny if $Q_D(Q_D x + t(x - Q_D x)) = Q_D x$ for each $x \in X$ and $t \ge 0$.*
(b) *a retraction if $Q_D x = x$ for each $x \in D$.*

If $X = H$ (real Hilbert space), then $Q_D = P_D$, the metric projection of C onto D.

Lemma 2.6. [26] *Let C and D be nonempty subsets of a real Banach space X with $D \subset C$ and $Q_D : C \to D$ be a retraction. Then Q_D is sunny nonexpansive if and only if*

$$\langle z - Q_D(z), J(y - Q_D(z))\rangle \le 0$$

for all $z \in C$ and $y \in D$, where J is the normalized duality mapping of X.

Lemma 2.7. [26] *Let X be a real Banach space and J be the normalized duality mapping of X. Then,*

$$\|x + y\|^2 \le \|x\|^2 + 2\langle y, J(x + y)\rangle$$

for all $x, y \in X$.

Definition 2.8. A one-parameter semigroup $\mathcal{T} = \{T(t) : t \ge 0\}$ is said to be uniformly asymptotically regular (u.a.r.) if for any $s > 0$ and any bounded subset K of C, we have

$$\lim_{t\to\infty} \sup_{x\in K} d(T(s)T(t)x, T(t)x) = 0.$$

The following are examples of uniformly asymptotically regular semigroup

Examples 2.9. ([5]) Let C be a nonempty bounded, closed and convex subset of a Hilbert space H and $T : C \to C$ a contraction operator with Lipschitz constant $k < 1$. Then $\mathcal{T} = \{T^n : n \in \mathbb{N}\}$ is a u.a.r. contraction semigroup.

Example 2.10. ([1]) Let $X = l^2(\mathbb{N})$ be Hilbert space consisting of all functions x from $\mathbb{N}$ into $\mathbb{R}$ satisfying $\sum_{k\in\mathbb{N}} |x(k)|^2 < \infty$ with inner product

$$\langle x, y\rangle = \sum_{k\in\mathbb{N}} x(k)y(k).$$

Define a bounded, closed and convex subset C of X by

$$C = \{x \in X : 0 \le x(k) \le p_k\}$$

where $p_k = 2^{\frac{-k}{2}}$. Then $\{T(t) : t \ge 0\}$ defined by

$$(T(t)x)(k) = \max\left\{x(k) - tp_k^2, 0\right\}$$

is a u.a.r. nonexpansive semigroup.

The concept of uniform asymptotic regularity extends to a sequence of mappings as follows:

Definition 2.11. A family $\{T_n\}$ of self-mappings on a nonempty set C is said to be uniformly asymptotically regular (u.a.r.) if, for each positive integer m and any bounded subset $K \subset C$, we have

$$\lim_{n\to\infty} \sup_{x\in K} d(T_m(T_n x), T_n x) = 0.$$

Lemma 2.12. [31] *If $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \le (1 - b_n)a_n + b_n c_n \quad \forall n \ge 1$$

where $0 \le b_n \le 1$ and $c_n \in \mathbb{R}$ satisfy the following conditions:

$$1.\ \sum_{n=1}^{\infty} b_n = \infty \qquad 2.\ \limsup_{n\to\infty} c_n \le 0 \ \text{or} \ \sum_{n=1}^{\infty} |b_n c_n| < \infty,$$

then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.13. [21] *Let $\{a_n\}$ be a real sequence that has a subsequence $\{a_{n_k}\}$ which satisfies $a_{n_k} < a_{n_k+1}$ for all k. Then there exists an increasing sequence of integers $\{m_k\} \subset \mathbb{N}$ such that*

$$1.\ \lim_{k\to\infty} m_k = \infty \qquad 2.\ a_{m_k} \le a_{m_k+1} \qquad 3.\ a_k \le a_{m_k+1}$$

for all (sufficiently large) numbers $k \in \mathbb{N}$.

Throughout this paper, we denote the set of fixed point of $T(t)$, $t \ge 0$ by $F(T(t))$ and set $F = \cap_{t\ge 0} F(T(t))$.

3. Convergence Results in Banach space

Theorem 3.1. *Let X be a real uniformly convex Banach space which admits weakly sequentially continuous duality mapping J and C a nonempty closed and convex subset of X. Let $\mathcal{T} = \{T(t) : t \ge 0\}$ be the u.a.r. semigroup of α-nnexpansive mappings from C into itself with $F \ne \emptyset$. Let $\{x_n\}$ be the sequence defined by*

$$\begin{cases} u \in C, \quad x_1 \in C \quad \text{(chosen arbitrarily)} \\ x_{n+1} = \alpha_n u + (1 - \alpha_n)[\lambda_n x_n + (1 - \lambda_n)T(t_n)x_n]. \end{cases} \tag{3.1}$$

Let $\{\alpha_n\}, \{\lambda_n\}$ be sequences in $[0,1]$ and $t_n > 0$ satisfy the following assumptions:

(i) $\lim_{n\to\infty} \alpha_n = 0$

(ii) $\sum_{n=1}^{\infty} \alpha_n = \infty$

(iii) $0 < \liminf_{n\to\infty} \lambda_n \le \limsup_{n\to\infty} \lambda_n < 1$

(iv) $\lim_{n\to\infty} t_n = \infty$

Then, the sequence $\{x_n\}$ defined in (3.1) converges to $Q_F u$, where Q_F is a sunny nonexpansive retraction from X onto F.

Proof. By Lemma 2.3(i), it follows that $T(t_n)$ is a quasi nonexpansive mappping and the sunny nonexpansive retraction from X onto F is well defined. Define $z = Q_F u$.

Step 1: $\{x_n\}$ and $\{T(t_n)x_n\}$ are bounded.

Let $y_n = \lambda_n x_n + (1-\lambda_n)T(t_n)x_n$. Then $x_{n+1} = \alpha_n u + (1-\alpha_n)y_n$. Let $p \in F$. By Lemma 2.4, there exists a continuous strictly increasing convex function $g : [0,\infty) \to [0,\infty)$ with $g(0) = 0$ such that

$$\begin{aligned}\|y_n - p\|^2 &= \|\lambda_n(x_n - p) + (1-\lambda_n)(T(t_n)x_n - p)\|^2 \\ &\le \lambda_n\|x_n - p\|^2 + (1-\lambda_n)\|T(t_n)x_n - p\|^2 - \lambda_n(1-\lambda_n)g(\|x_n - T(t_n)x_n\|) \\ &\le \lambda_n\|x_n - p\|^2 + (1-\lambda_n)\|x_n - p\|^2 - \lambda_n(1-\lambda_n)g(\|x_n - T(t_n)x_n\|) \\ &= \|x_n - p\|^2 - \lambda_n(1-\lambda_n)g(\|x_n - T(t_n)x_n\|).\end{aligned}$$

Hence,

$$\|y_n - p\|^2 \le \|x_n - p\|^2 - \lambda_n(1-\lambda_n)g(\|x_n - T(t_n)x_n\|) \tag{3.2}$$

and

$$\|y_n - p\| \le \|x_n - p\|. \tag{3.3}$$

Now, (3.3) implies that

$$\begin{aligned}\|x_{n+1} - p\| &= \|\alpha_n u + (1-\alpha_n)y_n - p\| \\ &= \|\alpha_n(u-p) + (1-\alpha_n)(y_n - p)\| \\ &\le \alpha_n\|u - p\| + (1-\alpha_n)\|y_n - p\| \\ &\le \alpha_n\|u - p\| + (1-\alpha_n)\|x_n - p\|.\end{aligned}$$

By induction, we have

$$\|x_{n+1} - p\| \le \max\{\|u - p\|, \|x_1 - p\|\}$$

for all $n \in \mathbb{N}$. Thus $\{x_n\}$ is bounded. Since $\|T(t_n)x_n - p\| \le \|x_n - p\|$, we also have that $\{T(t_n)x_n\}$ is bounded.

Step 2: For any $n \in \mathbb{N}$, we prove that

$$\|x_{n+1} - z\|^2 \le (1-\alpha_n)\|x_n - z\|^2 + 2\alpha_n\langle x - z, J(x_{n+1} - z)\rangle. \tag{3.4}$$

By (3.1), we have

$$\begin{aligned}\|x_{n+1} - z\|^2 &= \|\alpha_n u + (1-\alpha_n)y_n - z\|^2 \\ &\le \alpha_n\|u - z\|^2 + (1-\alpha_n)\|y_n - z\|^2.\end{aligned}$$

Using (3.2), we get

$$\|x_{n+1}-z\|^2 \leq \alpha_n\|u-z\|^2+(1-\alpha_n)[\|x_n-p\|^2-\lambda_n(1-\lambda_n)g(\|x_n-T(t_n)x_n\|)]. \quad (3.5)$$

Setting $M=\sup\{\|u-z\|^2-\|x_n-z\|^2+\lambda_n(1-\lambda_n)g(\|x_n-T(t_n)x_n\|)\}$, we have that

$$\lambda_n(1-\lambda_n)g(\|x_n-T(t_n)x_n\|) \leq \|x_n-z\|^2-\|x_{n+1}-z\|^2+\alpha_n M\ . \quad (3.6)$$

By Lemma 2.7 and (3.1), we can deduce that

$$\begin{aligned}\|x_{n+1}-z\|^2 &= \|\alpha_n u+(1-\alpha_n)y_n-z\|^2\\ &\leq \|\alpha_n u+(1-\alpha_n)y_n-z-\alpha_n(u-z)\|^2+2\langle\alpha_n(u-z),J(x_{n+1}-z)\rangle\\ &= (1-\alpha_n)\|y_n-z\|^2+2\alpha_n\langle u-z,J(x_{n+1}-z)\rangle\\ &\leq (1-\alpha_n)\|x_n-z\|^2+2\alpha_n\langle u-z,J(x_{n+1}-z)\rangle.\end{aligned}$$

Step 3: If $\lim_{n\to\infty}\|x_n-T(t_n)x_n\|=0$, then $\lim_{n\to\infty}\|x_n-T(s)x_n\|=0$ for any $s>0$.
By(3.1), we have that

$$y_n-x_n=(1-\lambda_n)(x_n-T(t_n)x_n) \text{ and } x_{n+1}-y_n=\alpha_n(u-y_n).$$

With the aid of the assumption $\lim_{n\to\infty}\|x_n-T(t_n)x_n\|=0$ and condition (i), we can deduce that as $n\to\infty$,

$$\|y_n-x_n\|\to 0 \text{ and } \|x_{n+1}-y_n\|\to 0. \quad (3.7)$$

Since $\|x_{n+1}-x_n\|\leq\|x_{n+1}-y_n\|+\|y_n-x_n\|$, we have that

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=0. \quad (3.8)$$

Using $\mathcal{T}$ is u.a.r. and condition (iv), it follows that for any $s>0$

$$\lim_{n\to\infty}\|T(s)T(t_n)x_n-T(t_n)x_n\|\leq\lim_{n\to\infty}\sup_{x\in K}\|T(s)T(t_n)x-T(t_n)x\|=0 \quad (3.9)$$

where K is any bounded subset of C containing $\{x_n\}$ and $\{T(t_n)x_n\}$.
For any $s>0$, let $x=T(t_n)x_n$, $y=x_n$, $T=T(s)$ and

$$V=\sup\{\alpha\|T(t_n)x_n-x_n\|+\|T(s)T(t_n)x_n-T(s)T(t_n)x_n\|.$$

Now by Lemma 2.2, we have

$$\begin{aligned}\|T(t_n)x_n-T(s)x_n\|^2 &\leq \frac{1+\alpha}{1-\alpha}\|T(t_n)x_n-T(s)T(t_n)x_n\|^2+\|T(t_n)x_n-x_n\|^2\\ &+ \frac{2\alpha}{1-\alpha}\|T(t_n)x_n-x_n\|\|T(t_n)x_n-T(s)T(t_n)x_n\|\\ &+ \frac{2}{1-\alpha}\|T(s)x_n-T(s)T(t_n)x_n\|\|T(t_n)x_n-T(s)T(t_n)x_n\|\\ &\leq \frac{1+\alpha}{1-\alpha}\|T(t_n)x_n-T(s)T(t_n)x_n\|^2+\|T(t_n)x_n-x_n\|^2\\ &+ \frac{2}{1-\alpha}V\|T(t_n)x_n-T(s)T(t_n)x_n\|.\end{aligned}$$

Using the assumption $\lim_{n\to\infty}\|x_n-T(t_n)x_n\|=0$ of Step 3 above and (3.9), we have

$$\lim_{n\to\infty}\|T(t_n)x_n-T(s)x_n\|=0. \quad (3.10)$$

Also,

$$\begin{aligned}\|x_{n+1}-T(s)x_n\|^2 &= \|\alpha_n u+(1-\alpha_n)y_n-T(s)x_n\|^2\\ &= \|\alpha_n u+(1-\alpha_n)[\lambda_n x_n+(1-\lambda_n)T(t_n)x_n]-T(s)x_n\|^2\\ &= \|\alpha_n(u-T(t_n)x_n)+(1-\alpha_n)\lambda_n(x_n-T(t_n)x_n)\\ &\quad +T(t_n)x_n-T(s)x_n\|^2\\ &\leq 2\alpha_n\|u-T(t_n)x_n\|+4(1-\alpha_n)\lambda_n\|x_n-T(t_n)x_n\|^2\\ &+ 4\|T(t_n)x_n-T(s)x_n\|^2\end{aligned}$$

By $\lim_{n\to\infty}\|x_n-T(t_n)x_n\|=0$, condition (i) and (3.10), we get

$$\lim_{n\to\infty}\|x_{n+1}-T(s)x_n\|=0. \tag{3.11}$$

Hence for any $s>0$,

$$\begin{aligned}\|x_n-T(s)x_n\| &= \|x_n-x_{n+1}+x_{n+1}-T(s)x_n\|\\ &\leq \|x_n-x_{n+1}\|+\|x_{n+1}-T(s)x_n\|.\end{aligned}$$

From (3.8) and (3.11), we conclude that

$$\lim_{n\to\infty}\|x_n-T(s)x_n\|=0. \tag{3.12}$$

Step 4: We prove that $x_n\to z$ as $n\to\infty$.
Case 1: Suppose that there exists $n_0\in\mathbb{N}$ such that $\{\|x_n-z\|\}_{n=n_o}^{\infty}$ is nonincreasing. Then $\lim_{n\to\infty}\|x_n-z\|$ exists. Using conditions (i),(iii) and (3.6), we deduce that

$$\lim_{n\to\infty}g(\|x_n-T(t_n)x_n\|)=0$$

By the properties of g, we have that

$$\lim_{n\to\infty}\|x_n-T(t_n)x_n\|=0. \tag{3.13}$$

Thus By Step 3, (3.12) holds. Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{x_{n_k+1}\}$converges weakly to y. It follows from (3.12) and Lemma 2.3 that $y\in F$. We have that

$$\begin{aligned}\limsup_{n\to\infty}\langle u-z,J(x_{n+1}-z)\rangle &= \lim_{k\to\infty}\langle u-z,J(x_{n_k+1}-z)\rangle\\ &= \langle u-z,J(y-z)\rangle.\end{aligned}$$

Using Lemma 2.6, we conclude that

$$\limsup_{n\to\infty}\langle u-z,J(x_{n+1}-z)\rangle\leq 0. \tag{3.14}$$

Let $a_n=\|x_{n+1}-z\|^2$, $b_n=\alpha_n$ and $c_n=\langle u-z,J(x_{n+1}-z)\rangle$ in (3.4). Then, we deduce by conditions (i), (ii) and Lemma 2.12 that $x_n\to z$.
Case 2: Suppose $\{\|x_n-z\|\}_{n=n_o}^{\infty}$ is not nonincreasing, i.e. that there exists a subsequence $\{n_k\}$ of $\{n\}$ such that

$$\|x_{n_k}-z\|<\|x_{n_k+1}-z\| \qquad \forall k\in\mathbb{N}.$$

Then by Lemma 2.13, there exists an increasing sequence of integers $\{m_k\} \subset \mathbb{N}$ such that

1. $\lim_{k\to\infty} m_k = \infty$ 2. $\|x_{m_k} - z\| \le \|x_{m_k+1} - z\|$ 3. $\|x_k - z\| \le \|x_{m_k+1} - z\|$

for all (sufficiently large) numbers $k \in \mathbb{N}$. This, together with (3.6) implies that

$$\begin{aligned}\lambda_{m_k}(1-\lambda_{m_k})g(\|x_{m_k} - T(t_{m_k})x_{m_k}\|) &\le \|x_{m_k} - z\|^2 - \|x_{m_k+1} - z\|^2 + \alpha_{m_k} M \\ &\le \alpha_{m_k} M.\end{aligned}$$

Conditions (i),(iii) and properties of g imply

$$\lim_{k\to\infty} \|x_{m_k} - T(t_{m_k})x_{m_k}\| = 0.$$

As shown before, we have

$$\limsup_{k\to\infty} \langle u - z, J(x_{m_k+1} - z)\rangle \le 0\,. \tag{3.15}$$

From (3.4), we obtain

$$\|x_{m_k+1} - z\|^2 \le (1-\alpha_{m_k})\|x_{m_k} - z\|^2 + 2\alpha_{m_k}\langle x - z, J(x_{m_k+1} - z)\rangle. \tag{3.16}$$

This together with $\|x_{m_k} - z\| \le \|x_{m_k+1} - z\|$ implies that

$$\alpha_{m_k}\|x_{m_k} - z\|^2 \le 2\alpha_{m_k}\langle x - z, J(x_{m_k+1} - z)\rangle$$

and thus,

$$\|x_{m_k} - z\|^2 \le 2\langle x - z, J(x_{m_k+1} - z)\rangle\,.$$

It follows from (3.15) that

$$\lim_{k\to\infty} \|x_{m_k} - z\| = 0.$$

Consequently, (3.16) ensures that

$$\lim_{k\to\infty} \|x_{m_k+1} - z\| = 0\,.$$

Recalling that for sufficiently large $k \in \mathbb{N}$,

$$\|x_k - z\| \le \|x_{m_k+1} - z\|,$$

and so we conclude that $\lim_{n\to\infty} \|x_k - z\| = 0$. Hence, $x_n \to z$ as $n \to \infty$.

Corollary 3.2. *Let X be a real uniformly convex Banach space which admits weakly sequentially continuous duality mapping J and C a nonempty closed and convex subset of X. Let $\mathcal{T} = \{T(t) : t \ge 0\}$ be the u.a.r. semigroup of α-nnexpansive mappings from C into itself with $F \ne \emptyset$. Let $\{x_n\}$ be the sequence defined by*

$$\begin{cases} u \in C, \quad x_1 \in C \quad \textit{(chosen arbitrarily)} \\ \\ x_{n+1} = \alpha_n u + (1-\alpha_n)S_\lambda(t_n)x_n, \end{cases} \tag{3.17}$$

where $S_\lambda x = \lambda x + (1-\lambda)T(t)x$ for any $\lambda \in (0,1)$ and $t > 0$. Let $\{\alpha_n\}$ in $[0,1]$ and $t_n > 0$ satisfy the following assumptions:

(i) $\lim_{n\to\infty} \alpha_n = 0$

(ii) $\sum_{n=1}^{\infty} \alpha_n = \infty$

(iii) $\lim_{n\to\infty} t_n = \infty$

Then, the sequence $\{x_n\}$ *defined in (3.17) converges to* $Q_F u$*, where* Q_F *is a sunny nonexpansive retraction from* X *onto* F*.*

Proof. Put $\lambda_n = \lambda$ for all n in Theorem 3.1.

Corollary 3.3. *Let* X *be a real uniformly convex Banach space which admits weakly sequentially continuous duality mapping* J *and* C *a nonempty closed and convex subset of* X*. Let* $\mathcal{T} = \{T(t) : t \geq 0\}$ *be the u.a.r. semigroup of any of the following self mappings on* C *:*

(1) *Mean nonexpansive mappings.*
(2) *Nonspreading mappings.*
(3) *Hybird mappings.*

Suppose that $F \neq \emptyset$*. Let* $\{x_n\}$ *be the sequence defined by*

$$\begin{cases} u \in C, \quad x_1 \in C \quad \text{(chosen arbitrarily)} \\ x_{n+1} = \alpha_n u + (1-\alpha_n)[\lambda_n x_n + (1-\lambda_n)T(t_n)x_n]. \end{cases} \tag{3.18}$$

Let $\{\alpha_n\}, \{\lambda_n\}$ *in* $[0,1]$ *and* $t_n > 0$ *satisfy :*

(i) $\lim_{n\to\infty} \alpha_n = 0$
(ii) $\sum_{n=1}^{\infty} \alpha_n = \infty$
(iii) $0 < \liminf_{n\to\infty} \lambda_n \leq \limsup_{n\to\infty} \lambda_n < 1$
(iv) $\lim_{n\to\infty} t_n = \infty$

Then, the sequence $\{x_n\}$ *defined in (3.18) converges to* $Q_F u$*, where* Q_F *is a sunny nonexpansive retraction from* X *onto* F*.*

Proof. (1). Put $\alpha = \frac{b}{2}$ in Theorem 3.1 .
(2). Put $\alpha = \frac{1}{2}$ in Theorem 3.1 .
(3). Put $\alpha = \frac{1}{3}$ in Theorem 3.1 .

Remark 3.4. Theorem 3.1 and Corollaries 3.2- 3.3 extend and improve Theorem 3.3 of Song et al. [27], Theorem 3.2 of Benavides et al. [5] and Theorem 20 of Aleyner et al. [2].

Replacing the terms $T(t_n)$ and $T(s)$ of Theorem 3.1 with the terms T_n and T_m, respectively, we have the following result which improves and extends ([27], Theorem 3.4) and ([24], Theorem 4.1).

Theorem 3.5. *Let* X *be a real uniformly convex Banach space which admits weakly sequentially continuous duality mapping* J *and* C *a nonempty closed and convex subset of* X*. Let* $\{T_n\}$ *be the sequence of u.a.r.* α*-nonexpansive mappings from* C *into itself with* $F \neq \emptyset$*. Let* $\{x_n\}$ *be the sequence defined by*

$$\begin{cases} u \in C, \quad x_1 \in C \quad \text{(chosen arbitrarily)} \\ \\ x_{n+1} = \alpha_n u + (1-\alpha_n)[\lambda_n x_n + (1-\lambda_n)T_n x_n]. \end{cases} \tag{3.19}$$

Let $\{\alpha_n\}, \{\lambda_n\}$ be sequences in $[0,1]$ and satisfy the following conditions:

$$1.\ \lim_{n\to\infty} \alpha_n = 0, \qquad 2.\ \sum_{n=1}^{\infty} \alpha_n = \infty, \quad 3.\ 0 < \liminf_{n\to\infty} \lambda_n \le \limsup_{n\to\infty} \lambda_n < 1\,.$$

Then, the sequence $\{x_n\}$ defined in (3.19) converges to $Q_F u$, where Q_F is a sunny nonexpansive retraction from X onto F.

4. Preliminaries for $CAT(0)$ Spaces

Let (X,d) be a metric space and $I = [0,1] \subset \mathbb{R}$. A mapping $\varphi : I \to X$ with the property that $\varphi(0) = a$, $\varphi(1) = b$, and $d(\varphi(s), \varphi(t)) = |s-t|$ for all $0 \le t, s \le 1$ is known as a geodesic path from a to b. Geodesic segment connecting a and b is the image of φ. If a geodesic segment connects any two points of X, then (X,d) is called a geodesic space. Moreover, X is uniquely geodesic if such a segment is unique for every pair of elements of X. The unique segment joining a to b is represented by $[a,b]$. A geodesic triangle $\triangle(y_1,y_2,y_3)$ comprises of three points y_1,y_2,y_3 in X as well as geodesic segments connecting any two of the points y_1,y_2,y_3 . For any geodesic triangle $\triangle(y_1,y_2,y_3)$ in (X,d), a comparison triangle is defined as $\overline{\triangle}(y_1,y_2,y_3) := \triangle(\overline{y_1},\overline{y_2},\overline{y_3})$ in $\mathbb{R}^2$ with $d_{\mathbb{R}^2}(\overline{y_i},\overline{y_j}) = d(y_i,y_j)$ for $i,j = 1,2,3$. A $CAT(0)$ space is a metric space in which every two points are connected by a geodesic segment and for every $x, y \in \triangle(y_1,y_2,y_3)$ in X and $\overline{x},\overline{y} \in \overline{\triangle} := \triangle(\overline{y_1},\overline{y_2},\overline{y_3})$ in $\mathbb{R}^2$,

$$d(x,y) \le d_{\mathbb{R}^2}(\overline{x},\overline{y}) \ \textit{holds.}$$

By $u = (1-t)x_1 \oplus tx_2$, we mean u is a point on the geodesic segment joining x_1 to x_2 where $d(x_2,u) = (1-t)d(x_1,x_2)$ and $d(x_1,u) = td(x_1,x_2)$.

Some well-known examples of $CAT(0)$ space are complete,simply connected Riemannian manifold having nonpositive sectional curvature, Pre-Hilbert spaces, R-trees, Euclidean spaces, the complex Hilbert ball with a hyperbolic metric (See [18] for more details). Complete $CAT(0)$ spaces are often called Hadamard spaces.

In 2008, Berg and Nikolaev[6] introduced the concept of quasilinearization as follows: Denote a pair $(a,b) \in X \times X$ by $\overrightarrow{ab}$ and call it a vector. The quasilinearization is the map $\langle .,. \rangle : (X\times X) \times (X \times X) \to \mathbb{R}$ defined by

$$\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle = \frac{1}{2}\left[d^2(a,b) + d^2(b,c) - d^2(a,c) - d^2(b,d)\right], \quad (a,b,c,c,d \in X). \tag{4.1}$$

It is easy to verify that for any $a,b,c,d,x \in X$,

$$\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle = \langle \overrightarrow{cd}, \overrightarrow{ab} \rangle,\ \langle \overrightarrow{ab}, \overrightarrow{cd} \rangle = -\langle \overrightarrow{ba}, \overrightarrow{cd} \rangle \ \text{and}\ \langle \overrightarrow{ax}, \overrightarrow{cd} \rangle + \langle \overrightarrow{xb}, \overrightarrow{cd} \rangle = \langle \overrightarrow{ab}, \overrightarrow{cd} \rangle \tag{4.2}$$

Moreover, we say that X satisfies the Cauchy-Schwartz inequality if $\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle \le d(a,b)d(c,d)$ for all $a,b,c,d \in X$. It is known ([6]) that a geodesically connected metric space is $CAT(0)$ space if and only if it satisfies the Cauchy-Schwartz inequality.

Recently, Dehghan and Rooin [10] introduced the duality mapping in $CAT(0)$ spaces and studied its relation with subdifferential, by using the concept of quasilinearization. Then they presented a characterization of metric projection in $CAT(0)$ spaces as follows:

Let (X, d) be a complete CAT(0) space and C a nonempty closed and convex subset of X. Let $u \in C$ and $x \in X$. Then $u = P_C x$ if and only if

$$\langle \overrightarrow{yu}, \overrightarrow{ux} \rangle \geq 0 \text{ for all } y \in C.$$

The concept of $\triangle$-convergence was introduced by Lim [20]. Kirk and Panyanak[19] showed that $\triangle$-convergence in $CAT(0)$ space is similar to weak convergence in Banach space setting.

Next, we explain the concept of $\triangle$-convergence and collect some of its basic properties.

Definition 4.1. Let $\{x_n\}$ be a bounded sequence in a $CAT(0)$ space (X, d). For $x \in X$, we set

$$r(x, \{x_n\}) = \limsup_{n \to \infty} d(x, x_n).$$

The asymptotic radius of $\{x_n\}$ is given by

$$r(\{x_n\}) = \inf\{r(x, \{x_n\}) : x \in X)\}$$

and asymptotic center of $\{x_n\}$ is the set

$$A(\{x_n\}) = \{x \in X : r(x, \{x_n\}) = r(\{x_n\})\}.$$

Definition 4.2. A sequence $\{x_n\}$ in X is said to $\triangle-$ converge to $x \in X$ if x is the unique asymptotic center of every subsequence $\{u_n\}$ of $\{x_n\}$. In this case, we write $\triangle - \lim_{n \to \infty} x_n = x$ and call x, the $\triangle-$limit of $\{x_n\}$.

The following known lemmas are needed to prove our result in $CAT(0)$ space.

Lemma 4.3. [19] *Every bounded sequence in a complete $CAT(0)$ space X has a $\triangle$-convergent subsequence.*

Lemma 4.4. [17] *Let X be a complete $CAT(0)$ space, $\{x_n\}$ a sequence in X and $x \in X$. Then $\{x_n\}$ $\triangle$-converges to x if and only if* $\limsup_{n\to\infty}\langle xx_n, xy\rangle \leq 0$ *for all $y \in X$.*

Lemma 4.5. [25] *Let X be a complete $CAT(0)$ space, $\{x_n\}$ a sequence in X and $x \in X$. Suppose there exists a nonempty subset K of X satisfying :*

(i) *For every $z \in K$, $\lim_{n \to \infty} d(x_n, z)$ exists,*
(ii) *If a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ is $\triangle$-convergent to $x \in X$, then $x \in K$.*

Then $\{x_n\}$ $\triangle$-converges to $x \in K$.

Lemma 4.6. [12] *Let X be a $CAT(0)$ space. For any $x, y, z, w \in X$ and $\lambda \in [0, 1]$, we have the following:*

(a) $d(\lambda x \oplus (1 - \lambda)y, z) \leq \lambda d(x, z) + (1 - \lambda)d(y, z)$.
(b) $d^2(\lambda x \oplus (1 - \lambda)y, z) \leq \lambda d^2(x, z) + (1 - \lambda)d^2(y, z) - \lambda(1 - \lambda)d^2(x, y)$.

The above inequality (b) is called the (CN) inequality of Bruhat and Tits(For more details, see [7]*).*

Lemma 4.7. (see [16], [12]) *Let X be a $CAT(0)$ space. For any $x, y, z, w \in X$ and $t \in [0, 1]$ with $u_t = \lambda z \oplus (1 - t)w$, we have the following:*

(i) $\langle \overrightarrow{u_t x}, \overrightarrow{u_t y} \rangle \leq t\langle \overrightarrow{zx}, \overrightarrow{u_t y} \rangle + (1 - t)\langle \overrightarrow{wx}, \overrightarrow{u_t y} \rangle$.
(ii) $\langle \overrightarrow{u_t x}, \overrightarrow{zy} \rangle \leq t\langle \overrightarrow{zx}, \overrightarrow{zy} \rangle + (1 - t)\langle \overrightarrow{wx}, \overrightarrow{zy} \rangle$.
(iii) $\langle \overrightarrow{u_t x}, \overrightarrow{wy} \rangle \leq t\langle \overrightarrow{zx}, \overrightarrow{wy} \rangle + (1 - t)\langle \overrightarrow{wx}, \overrightarrow{wy} \rangle$.

Lemma 4.8. [10] *Let X be a $CAT(0)$ space and $x, y, z, \in X$. Then for each $\lambda \in [0,1]$, we have*

$$d^2(\lambda x \oplus (1-\lambda)y, z) \leq \lambda^2 d^2(x,z) + (1-\lambda)^2 d^2(y,z) + 2\lambda(1-\lambda)\langle \overrightarrow{xz}, \overrightarrow{yz}\rangle.$$

Lemma 4.9. [23] *Let C be a nonempty subset of a $CAT(0)$ space X. Let $T : C \to X$ be an α-nonexpansive mapping for some $0 \leq \alpha < 1$.*

(i) *If $F(T) \neq \emptyset$, then T is quasi-nonexpansive. Moreover, $F(T)$ is closed and convex.*
(ii) *If $\{x_n\}$ is a sequence in C such that $\lim_{n\to\infty} d(x_n, Tx_n) = 0$ and $\triangle - \lim_{n\to\infty} x_n = z$ for some $z \in X$, then $z \in C$ and $Tz = z$.*

5. Convergence Results in $CAT(0)$ space

Theorem 5.1. *Let X be a complete $CAT(0)$ space and C a nonempty closed and convex subset of X. Let $\mathcal{T} = \{T(t) : t \geq 0\}$ be the u.a.r semigroup of α-nonexpansive mappings from C into itself with $F \neq \emptyset$. Let $\{x_n\}$ be the sequence defined by*

$$\begin{cases} u \in C, \quad x_1 \in C \quad \textit{(chosen arbitrary)} \\ \\ x_{n+1} = \alpha_n u \oplus (1-\alpha_n)[\lambda_n x_n \oplus (1-\lambda_n)T(t_n)x_n]. \end{cases} \tag{5.1}$$

Let $\{\alpha_n\}, \{\lambda_n\}$ in $[0,1]$ and $t_n > 0$ satisfy the following conditions :

(i) $\lim_{n\to\infty} \alpha_n = 0$
(ii) $\sum_{n=1}^{\infty} \alpha_n = \infty$
(iii) $0 < \liminf_{n\to\infty} \lambda_n \leq \limsup_{n\to\infty} \lambda_n < 1$
(iv) $\lim_{n\to\infty} t_n = \infty$

Then, the sequence $\{x_n\}$ defined in (5.1) converges to $\overline{x} = P_F u$, where P_F is the metric projection of X onto F.

Proof. Let $z \in F$ and $y_n = \lambda_n x_n \oplus (1-\lambda_n)T(t_n)x_n$.
Step 1: $\{x_n\}$ and $\{T(t_n)x_n\}$ are bounded.
By Lemma 4.6 and Lemma 4.9(i), we obtain

$$\begin{aligned} d^2(y_n, z) &\leq \lambda_n d^2(x_n, z) + (1-\lambda_n)d^2(T(t_n)x_n, z) - \lambda_n(1-\lambda_n)d^2(x_n, T(t_n)x_n) \\ &\leq \lambda_n d^2(x_n, z) + (1-\lambda_n)d^2(x_n, z) - \lambda_n(1-\lambda_n)d^2(x, T(t_n)x_n) \\ &= d^2(x_n, z) - \lambda_n(1-\lambda_n)d^2(x, T(t_n)x_n) \\ &\leq d^2(x_n, z). \end{aligned}$$

By replacing $\|.-.\|$ with $d(.,.)$ in Step 1 of the proof of Theorem 3.1, we have

$$d(x_{n+1}, z) \leq \max\{d(u,z), d(x_1, z)\}$$

Hence, $\{x_n\}$ and $\{T(t_n)x_n\}$ are bounded.
Step 2: If $\lim_{n\to\infty} d(x_n, T(t_n)x_n) = 0$, then $\lim_{n\to\infty} d(x_n, T(s)x_n) = 0$ for any $s > 0$.

By replacing $\|. - .\|$ with $d(.,.)$ in Step 3 of the proof of Theorem 3.1 and using properties of d in a $CAT(0)$ space, it is easy to verify that

$$\lim_{n\to\infty} d(x_n, y_n) = 0 \text{ and } \lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{5.2}$$

Moreover, using arguments similar to those in Step 3 of the proof of Theorem 3.1 and replacing $\|. - .\|$ with $d(.,.)$, we get

$$\lim_{n\to\infty} d(T(t_n)x_n, T(s)x_n) = 0 \text{ and } \lim_{n\to\infty} d(T(s)T(t_n)x_n, T(t_n)x_n) = 0. \tag{5.3}$$

Now,

$$\begin{aligned} d(y_n, T(s)x_n) &\leq d(y_n, T(t_n)x_n) + d(T(t_n)x_n, T(s)x_n) \\ &= \lambda_n d(x_n, T(t_n)x_n) + d(T(t_n)x_n, T(s)x_n). \end{aligned}$$

Taking limit as $n \to \infty$, we have

$$\lim_{n\to\infty} d(y_n, T(s)x_n) = 0 \tag{5.4}$$

By Lemma 4.6, we have

$$\begin{aligned} d^2(x_{n+1}, T(s)x_n) &= d^2(\alpha_n u \oplus (1-\alpha_n)y_n, T(s)x_n) \\ &\leq \alpha_n d^2(u, T(s)x_n) + (1-\alpha_n)d^2(y_n, T(s)x_n) \\ &\quad -\alpha_n(1-\alpha_n)d^2(u, y_n) \\ &\leq \alpha_n d^2(u, T(s)x_n) + (1-\alpha_n)d^2(y_n, T(s)x_n). \end{aligned}$$

By condition (i) and (5.4), it follows that

$$\lim_{n\to\infty} d(x_{n+1}, T(s)x_n) = 0. \tag{5.5}$$

Hence, for any $s > 0$,

$$d(x_n, T(s)x_n) \leq d(x_n, x_{n+1}) + d(x_{n+1}, T(s)x_n)$$

and

$$\lim_{n\to\infty} d(x_n, T(s)x_n) = 0. \tag{5.6}$$

Step 3: $x_n \to \overline{x} \in F$ as $n \to \infty$.

Case 1: Suppose that there exists $n_0 \in \mathbb{N}$ such that $\{d(x_n, z)\}_{n=n_o}^{\infty}$ is nonincreasing. Then $\lim_{n\to\infty} d(x_n, z)$ exists. With the aid of Lemma 4.6 and reasoning as in Step 2 of the proof of Theorem 3.1, we can show that

$$\lambda_n(1-\lambda_n)d^2(x_n, T(t_n)x_n) \leq d^2(x_n - z) - d^2(x_{n+1}, z) + \alpha_n M.$$

Since $\lim_{n\to\infty} d(x_n, z)$ exists, it follows from conditions (i) and (iii) that

$$\lim_{n\to\infty} d(x_n, T(t_n)x_n) = 0. \tag{5.7}$$

Consequently, we deduce from Step 2 above that

$$\lim_{n\to\infty} d(x_n, T(s)x_n) = 0 \text{ and } \lim_{n\to\infty} d(x_n, y_n) = 0. \tag{5.8}$$

Since $\{x_n\}$ is bounded, it follows by Lemma 4.3 that there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ that $\triangle$-converges to $\overline{x} \in X$. Using (5.6) and Lemma 4.9, we have that $\overline{x} \in F$. Now, Lemma 4.5 ensures that $\{x_n\}$ $\triangle$-converges to $\overline{x} \in F$.
Exploiting properties of quasilinearization in (4.2), Cauchy-Schwartz inequality and Lemma 4.8, we have

$$\begin{aligned} d^2(x_{n+1},\overline{x}) &= d^2(\alpha_n u \oplus (1-\alpha_n)y_n, \overline{x}) \\ &\leq \alpha_n^2 d^2(u,\overline{x}) + (1-\alpha_n)^2 d^2(y_n,\overline{x}) + 2(\alpha_n(1-\alpha_n)\langle \overrightarrow{u\overline{x}}, \overrightarrow{y_n\overline{x}}\rangle \\ &\leq \alpha_n^2 d^2(u,\overline{x}) + (1-\alpha_n)^2 d^2(x_n,\overline{x}) + 2\alpha_n(1-\alpha_n)[\langle \overrightarrow{u\overline{x}}, \overrightarrow{y_n x_n}\rangle + \langle \overrightarrow{u\overline{x}}, \overrightarrow{x_n\overline{x}}\rangle] \\ &\leq (1-\alpha_n)d^2(x_n,\overline{x}) + \alpha_n\left[\alpha_n d^2(u,\overline{x}) + 2(1-\alpha_n)[d(u,\overline{x})d(y_n x_n) + \langle \overrightarrow{u\overline{x}}, \overrightarrow{x_n\overline{x}}\rangle]\right]. \end{aligned}$$

Since $\{x_n\}$ $\triangle$-converges to $\overline{x}$, it follows by Lemma 4.4 that $\limsup_{n\to\infty}\langle u\overline{x}, x_n\overline{x}\rangle \leq 0$. This inequality, (5.8) and condition(i) imply

$$\limsup_{n\to\infty}\left[\alpha_n d^2(u,x) + 2(1-\alpha_n)[d(u,\overline{x})d(y_n x_n) + \langle \overrightarrow{u\overline{x}}, \overrightarrow{x_n\overline{x}}\rangle]\right] = 0.$$

By Lemma 2.12, we have that $\lim_{n\to\infty} d(x_n,\overline{x}) = 0$. Hence $x_n \to \overline{x} \in F$ as $n \to \infty$.
Case 2: Suppose $\{d(x_n,z)\}_{n=n_o}^{\infty}$ is not nonincreasing. So, there exists a subsequence $\{n_k\}$ of $\{n\}$ such that

$$d(x_{n_k},z) < d(x_{n_k+1},z) \quad \forall k \in \mathbb{N}.$$

Then by Lemma 2.13, there exists an increasing sequence of integers $\{m_k\} \subset \mathbb{N}$ such that

$$1.\ \lim_{k\to\infty} m_k = \infty \qquad 2.\ d(x_{m_k},z) \leq d(x_{m_k+1},z) \qquad 3.\ d(x_k,z) \leq d(x_{m_k+1},z)$$

for all (sufficiently large) numbers $k \in \mathbb{N}$. As shown before, we have

$$\lambda_{m_k}(1-\lambda_{m_k})d^2(x_{m_k}, T(t_{m_k})x_{m_k}) \leq d^2(x_{m_k},z) - d^2(x_{m_k+1},z) + \alpha_n M.$$

This inequality, condition (i) and $d(x_{m_k},z) \leq d(x_{m_k+1},z)$ imply

$$\lim_{k\to\infty} d(x_{m_k}, T(t_{m_k})x_{m_k}) = 0.$$

Moreover, we have from Step 2 that for any $s > 0$,

$$\lim_{k\to\infty} d(x_{m_k}, y_{m_k}) = 0 \ \text{ and } \ \lim_{k\to\infty} d(x_{m_k}, T(s)x_{m_k}) = 0.$$

As in Case 1 above, $\{x_{m_k}\}$ $\triangle$-converges to $\overline{x} \in F$ and

$$\begin{aligned} d^2(x_{m_k+1},\overline{x}) &\leq (1-\alpha_{m_k})d^2(x_{m_k},\overline{x}) \\ &\quad + \alpha_{m_k}\left[\alpha_{m_k} d^2(u,\overline{x}) + 2(1-\alpha_{m_k})[d(u,\overline{x})d(y_{m_k},x_{m_k}) + \langle \overrightarrow{u\overline{x}}, \overrightarrow{x_{m_k}\overline{x}}\rangle]\right]. \end{aligned}$$

Since $d(x_{m_k},z) \leq d(x_{m_k+1},z)$, we have that

$$d^2(x_{m_k},\overline{x}) \leq \alpha_{m_k} d^2(u,\overline{x}) + 2(1-\alpha_{m_k})[d(u,\overline{x})d(y_{m_k},x_{m_k}) + \langle \overrightarrow{u\overline{x}}, \overrightarrow{x_{m_k}\overline{x}}\rangle.$$

Condition (i) and the fact that $\limsup_{n\to\infty}\langle u\overline{x}, x_{m_k}\overline{x}\rangle \leq 0$ imply that $\lim_{k\to\infty} d(x_{m_k},\overline{x}) = 0$ and consequently, $\lim_{k\to\infty} d(x_{m_k+1},\overline{x}) = 0$. Finally,

$$d(x_k,\overline{x}) \leq d(x_{m_k+1},\overline{x}) \ \text{ implies that } \ \lim_{k\to\infty} d(x_k,\overline{x}) = 0$$

Hence the desired result follows.
Step 4: Now we show that $\overline{x} \in F$ is the nearest common fixed point of $\mathcal{T}$ to u i.e. $\overline{x} = P_F u$.
For any $z \in F$, it follows by Lemma 4.6(ii) that

$$\begin{aligned} d^2(x_n, z) &= d^2(\alpha_n u \oplus (1-\alpha_n) y_n, z) \\ &\leq \alpha_n d^2(u, z) + (1-\alpha_n) d^2(y_n, z) - \alpha_n(1-\alpha_n) d^2(u, y_n) \qquad (5.9) \\ &\leq d^2(u, z) + d^2(x_n, z) - \alpha_n(1-\alpha_n) d^2(u, y_n). \end{aligned}$$

Since $d(y_n, \overline{x}) \leq d(x_n, \overline{x})$, we have that $y_n \to \overline{x}$ as $n \to \infty$. Taking limit as $n \to \infty$ in (5.9), we have

$$d^2(\overline{x}, z) \leq d^2(u, z) + d^2(x_n, z) - \alpha_n(1-\alpha_n) d^2(u, y_n)\,.$$

This implies $d(u, \overline{x}) \leq d(u, z) \ \ \forall z \in F$. Hence $\overline{x} = P_F u$.
Remark 5.2. An analogue of Theorem 3.3 for a sequence of u.a.r. α-nonexpansive mappings on a $CAT(0)$ space can be easily established on the lines of its proof.

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