Fixed Point Theory, 25(2024), No. 1, 249-270
DOI: 10.24193/fpt-ro.2024.1.16
http://www.math.ubbcluj.ro/~nodeacj/sfptcj.html

CONVERGENCE ANALYSIS OF A NEW RELAXED ALGORITHM WITH INERTIAL FOR SOLVING SPLIT FEASIBILITY PROBLEMS

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Abstract. In this paper, we introduce a new algorithm for solving split feasibility problems in Hilbert spaces. The algorithm stems from the relaxed CQ method and the alternated inertial step method. The proposed algorithm uses variable step-sizes which are updated at each iteration by a cheap computation without linesearch. This step-size rule allows the resulting algorithm to work more easily without the prior knowledge of the operator norm. Numerical experiments are implemented to illustrate the theoretical results and also to compare with existing algorithms.
Key Words and Phrases: Split feasibility problems, relaxed CQ-algorithms, alternated inertial method, fixed point, bounded linear operator, Hilbert space.
2020 Mathematics Subject Classification: 49J53, 65K10, 49M37, 90C25, 47H10.

1. Introduction

The split feasibility problem (SFP) is:

$$\text{Find } x \in H_1 \text{ such that } x \in C \text{ and } Ax \in Q, \tag{1.1}$$

where C and Q are nonempty closed and convex subsets of real Hilbert spaces H_1 and H_2 respectively and $A : H_1 \to H_2$ is a bounded linear operator. We shall denote the solution set of SFP (1.1) by Ω. The split-feasibility problem (1.1) was first studied by Censor and Elfving [16]. The SFP has applications in many important problems such as signal processing, image reconstruction, intensity modulated radiation therapy (IMRT) treatment planning, and many well-known iterative algorithms for solving the SFP (1.1) has been established [6, 14, 16, 15, 17, 18].

In [16], the authors used their multidistance idea to obtain iterative algorithms for solving the SFP (1.1). Their algorithms as well as some other algorithms obtained later (see [12]) involve matrix inverses at each iteration. Byrne [14, 13] was among the first to propose the so-called CQ algorithm which generates a sequence $\{x_n\}$ by the recursive procedure,

$$x_{n+1} = P_C(x_n - \tau_n A^*(I - P_Q)Ax_n), \tag{1.2}$$

where the stepsize τ_n is chosen in the interval $(0, \frac{2}{\|A\|^2})$, P_C and P_Q are the orthogonal projections onto C and Q which are closed and convex subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively and A^* is the transpose of the $m \times n$ matrix A. Compared with Censor and Elfving's algorithm [16] where the matrix inverse A^{-1} is involved, the CQ algorithm (1.2) seems more easily executed since it only deals with orthogonal projections with no need to compute matrix inverses.

Now consider the nonempty closed convex subsets in (1.1) having the following form:

$$C = \{x \in \mathbb{R}^n : c(x) \leq 0\}, Q = \{y \in \mathbb{R}^m : q(y) \leq 0\},$$

where $c : \mathbb{R}^n \to \mathbb{R}$ and $q : \mathbb{R}^m \to \mathbb{R}$ are both convex functions. In this situation, the efficiency of the CQ method is extremely affected because in general the computation of projections onto such subsets is very difficult. Motivated by Fukushima's relaxed projection method in [21], Yang [44] suggested calculating the projection onto a half space containing the original subset instead of the latter set itself. More precisely, Yang [44] introduced the following relaxed CQ algorithm:

$$x_{n+1} = P_{C_n}(x_n - \tau_n A^*(I - P_{Q_n})Ax_n), \tag{1.3}$$

where C_n and Q_n are constructed as follows

$$C_n = \{x \in \mathbb{R}^n : c(x_n) \leq \langle \xi_n, x_n - x\rangle\}, \tag{1.4}$$

with $\xi_n \in \partial c(x_n)$ and

$$Q_n = \{x \in \mathbb{R}^m : q(Ax_n) \leq \langle \zeta_n, Ax_n - x\rangle\}, \tag{1.5}$$

with $\zeta_n \in \partial q(Ax_n)$. We note that both the CQ algorithm and the relaxed CQ algorithm use a fixed stepsize related to the matrix norm, which sometimes affects convergence of the algorithms especially if A is a dense matrix and has a high dimension.

In order to remove this setback, López et al. [27] introduced a new method for selecting the variable stepsize. They defined the stepsize λ_n as follows:

$$\lambda_n = \frac{\frac{1}{2}\rho_n\|(I - P_{Q_n})Ax_n\|^2}{\|A^*(I - P_{Q_n})Ax_n\|^2} \tag{1.6}$$

where $\{\rho_n\}$ is a sequence in $(0, 4)$ such that $\inf\limits_{n\in\mathbb{N}} \rho_n(4 - \rho_n) > 0$ and proved that the sequence $\{x_n\}$ generated by (1.3) converges weakly to a solution of SFP (1.1).

Recently, Kesornprom et al. [26] proposed the following algorithm which can be seen as an improvement over the algorithm of Gibali et al. [22] in the sense that it only involves one projection per iteration.

$$\begin{cases} y_n = x_n - \lambda_n A^*(I - P_{Q_n})Ax_n, \\ x_{n+1} = P_{C_n}(y_n - \varphi_n(A^*(I - P_{Q_n})Ay_n), \end{cases} \tag{1.7}$$

where $\lambda_n = \frac{\frac{1}{2}\rho_n\|(I-P_{Q_n})Ax_n\|^2}{\|A^*(I-P_{Q_n})Ax_n\|^2+\theta_n}$ and $\varphi_n = \frac{\frac{1}{2}\rho_n\|(I-P_{Q_n})Ax_n\|^2}{\|A^*(I-P_{Q_n})Ay_n\|^2+\theta_n}$, $0 < \rho_n < 4$, $0 < \theta_n < 1$.

In iterative methods, it is always desirable to develop more efficient and faster iterative algorithms. In order to obtain iterative algorithms with better rate of convergence than the previous ones in the literature, many authors have studied inertial type algorithms (see, e.g., [1, 2, 4, 3, 5, 7, 10, 8, 9, 19, 28, 29, 32, 34]). These results analysed the convergence properties of inertial extrapolation type algorithms and demonstrated their improved performance numerically on some imaging and data analysis problems.

In particular, many authors have proposed a variety of inertial-type methods for solving SFPs, see [20, 36, 38, 37] and some of the references therein. One thing that is very evident with respect to the inertial-type methods in [20, 36, 38, 37], is that the sequence $\{x_n\}$ generated does not yield Fejér monotonicity of $\{\|x_n - x^*\|\}$, $x^* \in \Omega$ and can move or swing back and forth around Ω, see, for example, [7, 28]. This is the reason such inertial extrapolation step sometimes does not converge faster than its counterpart non-inertial methods, see, e.g., [30]. The alternated inertial method introduced recently in [31] has partially resolved this problem as it has shown some improvements over the vanilla inertial methods, see [24, 25, 35], for details.

The purpose of this paper is to propose an alternated inertial relaxed CQ algorithm for solving SFPs in real Hilbert spaces. Our contribution in this paper includes:

(1) Our proposed inertial method regains the Fejér monotonicity of $\{\|x_n - x^*\|\}$, $x^* \in \Omega$, in some sense.
(2) The algorithm can be considered as an alternated inertial relaxed version of the splitting-relaxed projection method studied in [42] for solving SFPs.
(3) We give some numerical examples to show the comparative advantage of our proposed method over some recent related methods.

2. Preliminaries

Let H_1 and H_2 be real Hilbert spaces, $\omega_w(x_n)$ the set of cluster points of $\{x_n\}$ in the weak topology, " $\to$ " strong convergence and " $\rightharpoonup$ " weak convergence. Let T be an operator on H. Then T is called
(i) κ-inverse strongly monotone (κ-ism) if there is $\kappa > 0$ such that

$$\langle Tx - Ty, x - y\rangle \geq \kappa\|Tx - Ty\|^2, x, y \in H;$$

(ii) L-Lipschitz continuous if there is $L > 0$ such that

$$\|Tx - Ty\| \le L\|x - y\|, x, y \in H;$$

(iii) firmly nonexpansive if it is 1-ism;
(iv) nonexpansive if it is 1-Lipschitz continuous.

Lemma 2.1. (Gobel-Kirk [23]). *Let T be an operator on H. Then the following are equivalent.*
(i) T is firmly nonexpansive;
(ii) $I - T$ is firmly nonexpansive;
(iii) $\|Tx - Ty\|^2 \le \langle x - y, Tx - Ty\rangle, x, y \in H$.

Let C be a nonempty closed and convex subset of a Hilbert space H. The projection mapping from H onto C is denoted by P_C and defined by

$$P_C(x) = \arg\min_{y \in C} \|x - y\|,\ x \in H.$$

It is well known that the projection mapping is firmly nonexpansive and is characterized by the following variational inequality:

$$\langle P_C(x) - x, P_C(x) - y\rangle \le 0, \forall y \in C. \tag{2.1}$$

Lemma 2.2. (Byrne [14]). *Let $A : H_1 \to H_2$ be a bounded linear operator and Q a nonempty closed and convex subset of H_2, then the operator $A^*(I - P_Q)A$ is $\frac{1}{\|A\|^2}$-ism and hence $\|A\|^2$-Lipschitz continuous.*

Lemma 2.3. (See [39]) *Assume that $\{a_n\}$ and $\{b_n\}$ are two sequences of non-negative numbers such that*

$$a_{n+1} \le a_n + b_n \ \forall n \in \mathbb{N}.$$

If $\sum_{n=1}^{\infty} b_n < +\infty$, then $\lim_{n\to\infty} a_n$ exists.

We recall that a function $f : H \to \mathbb{R}$ is called convex if

$$f(\alpha x + (1 - \alpha)y) \le \alpha f(x) + (1 - \alpha)f(y), \forall \alpha \in [0, 1] \text{ and } \forall x, y \in H.$$

A function $f : H \to \mathbb{R}$ is called subdifferentiable at x if there exists at least one subgradient at x. The set of subgradients of f at the point x is called the subdifferential of f at x, and is denoted by $\partial f(x)$. A function f is called subdifferentiable if it is subdifferentiable at all points in H. A convex function $f : H \to \mathbb{R}$ is called weakly lower semicontinuious at x if $x_n \rightharpoonup x$ implies

$$f(x) \le \liminf_{n\to\infty} f(x_n);$$

weakly lower semicontinuous if it is weakly lower semicontinuous at all points in H.

Definition 2.4. A sequence $\{x_n\}$ in H is said to converge weakly to $\bar{x} \in H$ if $\forall z \in H$,

$$\lim_{n\to\infty} \langle x_n, z\rangle = \langle \bar{x}, z\rangle.$$

Lemma 2.5. ([33]) *Let C be a non empty subset of H and $\{x_n\}$ be a sequence in H such that the following conditions hold:*
(i) for every $x \in C$, $\lim_{n\to\infty} \|x_n - x\|$ exists;
(ii) every sequential weak cluster point of $\{x_n\}$ is in C.
Then $\{x_n\}$ converges weakly to a point in C.

Lemma 2.6. ([17]) *Let $\{C_i\}_{i=1}^t$ and $\{Q_j\}_{j=1}^r$ be nonempty closed and convex subsets of H_1 and H_2 respectively and $A : H_1 \to H_2$ a bounded linear operator. Let $P(x) := \sum_{i=1}^t l_i(x - P_{C_i}(x)) + \sum_{j=1}^r \lambda_j A^*(I - P_{Q_j})Ax$ where $l_i(i = 1, \cdots, t)$ and $\lambda_j(j = 1, \cdots, r)$ are all positive constants. Then $P(x)$ is Lipschitz continuous with $L := \sum_{i=1}^t l_i + ||A||^2 \sum_{j=1}^r \lambda_j$.*

3. Main results

In this section we consider the nonempty closed convex subsets in the SFP (1.1) with the following form:

$$C = \{x \in H_1 : c(x) \le 0\}, Q = \{y \in H_2 : q(y) \le 0\},$$

where $c : H_1 \to \mathbb{R}$ and $q : H_2 \to \mathbb{R}$ are both weakly lower semicontinuous convex functions. Assume that both ∂c and ∂q are nonempty and bounded on bounded sets. We note that in finite dimensional Hilbert spaces, every convex function is subdifferentiable everywhere and its subdifferentials are uniformly bounded on bounded sets (see [6]). So our assumptions are automatically satisfied in finite dimensional Hilbert spaces setting. Define C_n and Q_n as follows:

$$C_n = \{x \in H_1 : c(w_n) \le \langle \xi_n, w_n - x\rangle\}, \tag{3.1}$$

where $\xi_n \in \partial c(w_n)$, and

$$Q_n = \{y \in H_2 : q(Aw_n) \le \langle \zeta_n, Aw_n - y\rangle\} \tag{3.2}$$

where $\zeta_n \in \partial q(Aw_n)$. Observe that for every $n \ge 0$, $C \subseteq C_n$ and $Q \subseteq Q_n$. We also set $f_n(x) = A^*(I - P_{Q_n})Ax$.

Algorithm 3.1. Initialization : Choose four parameters $\tau_1 > 0, \theta \in (0, 1], \mu \in (0, 1)$ and α_n such that $0 \le \alpha_n \le \left(\frac{1-\mu}{1+\mu}\right)^2$. Select initial $x_0, x_1 \in H_1$ and set $n := 1$.
Step 1: Given x_{n-1} and x_n $(n \ge 1)$, compute

$$w_n = \begin{cases} x_n, \ n = \text{even} \\ x_n + \alpha_n(x_n - x_{n-1}), \ n = \text{odd}, \end{cases} \tag{3.3}$$

Step 2:

$$y_n = P_{C_n}(w_n - \tau_n f_n(w_n)), \tag{3.4}$$

if $y_n = w_n$, then stop and w_n is the solution of the SFP. Otherwise,
Step 3: Compute

$$x_{n+1} = (1 - \theta)w_n + \theta y_n + \theta\tau_n(f_n(w_n) - f_n(y_n)) \tag{3.5}$$

and update

$$\tau_{n+1} = \begin{cases} \min\left\{\tau_n, \frac{\mu\|w_n - y_n\|}{\|f_n(w_n) - f_n(y_n)\|}\right\}, \text{ if } \|f_n(w_n) - f_n(y_n)\| \neq 0, \\ \tau_n, \text{ Otherwise.} \end{cases} \tag{3.6}$$

Set $n := n + 1$ and return to **Step 1**.

Remark 3.2. The stepsize τ_{n+1} is found at each iteration by a cheap computation and does not depend on the operator norm. From the definition, we see that if $f_n(w_n) = f_n(y_n)$, then $\tau_{n+1} = \tau_n$. Otherwise since f_n is Lipschitz continuous with Lipschitz constant $\|A\|^2$, then

$$\frac{\mu\|w_n - y_n\|}{\|f_n(w_n) - f_n(y_n)\|} \geq \frac{\mu\|w_n - y_n\|}{\|A\|^2\|w_n - y_n\|} = \frac{\mu}{\|A\|^2}.$$

Thus, we have that $\{\tau_n\}$ is bounded below by $\min\{\tau_1, \frac{\mu}{\|A\|^2}\}$. Moreover, from definition, the sequence $\{\tau_n\}$ is monotonically non-increasing. Thus there exists $\tau > 0$ such that $\lim_{n\to\infty} \tau_n = \tau$.

Lemma 3.3. *If $y_n = w_n$ for some $n \geq 1$, then w_n is a solution of the SFP* (1.1), *i.e.* $w_n \in \Omega$.

Proof. Let $y_n = w_n$, and pick $x^* \in \Omega$. Substituting $w_n = x$ in (3.1) yeilds $c(w_n) \leq 0, i.e.\ w_n \in C$. Also, by (3.5) and (2.1), we have

$$\langle w_n - \lambda_n f_n(w_n) - w_n, y - w_n\rangle \leq 0, \ \forall y \in C.$$

Therefore,

$$\langle f_n(w_n), w_n - x^*\rangle \leq 0. \tag{3.7}$$

Since $I - P_{Q_n}$ is firmly nonexpansive, then

$$\begin{aligned} \|(I - P_{Q_n})Aw_n\|^2 &= \|(I - P_{Q_n})Aw_n - (I - P_{Q_n})Ax^*\|^2 \\ &\leq \langle (I - P_{Q_n})Aw_n, Aw_n - Ax^*\rangle \\ &= \langle f_n(w_n), w_n - x^*\rangle \leq 0. \end{aligned} \tag{3.8}$$

Thus $Aw_n \in Q_n$, by the definition of Q_n, we have $q(Aw_n) \leq 0, i.e., Aw_n \in Q$ and therefore w_n is a solution of the SFP (1.1). □

Theorem 3.4. *Let $\{x_n\}$ be the sequence generated by Algorithm 3.1 and $\Omega \neq \emptyset$. Then $\{x_n\}$ converges weakly to a point in Ω.*

Proof. Let $x^* \in \Omega$. From the definition of y_n and (2.1), we obtain

$$\langle y_{2n+1} - (w_{2n+1} - \tau_{2n+1} f_{2n+1}(w_{2n+1})), y_{2n+1} - x^*\rangle \leq 0,$$

which gives

$$\begin{aligned} &2\langle w_{2n+1} - y_{2n+1}, y_{2n+1} - x^*\rangle - 2\tau_{2n+1}\langle f_{2n+1}(w_{2n+1}) - f_{2n+1}(y_{2n+1}), y_{2n+1} - x^*\rangle \\ &- 2\tau_{2n+1}\langle f_{2n+1}(y_{2n+1}), y_{2n+1} - x^*\rangle \geq 0. \end{aligned} \tag{3.9}$$

Using the identity $2\langle a,b\rangle = \|a+b\|^2 - \|a\|^2 - \|b\|^2$ with $a = w_{2n+1} - y_{2n+1}$ and $b = y_{2n+1} - x^*$, we get

$$2\langle w_{2n+1} - y_{2n+1}, y_{2n+1} - x^*\rangle = \|w_{2n+1} - x^*\|^2 - \|w_{2n+1} - y_{2n+1}\|^2 - \|y_{2n+1} - x^*\|^2. \tag{3.10}$$

Moreover, since $x^* \in \Omega$, then $f_{2n+1}(x^*) = 0$. Therefore, it follows from f_{2n+1} being monotone that

$$\langle f_{2n+1}(y_{2n+1}), y_{2n+1} - x^*\rangle \geq \langle f_{2n+1}(x^*), y_{2n+1} - x^*\rangle = 0. \tag{3.11}$$

Combining the relations (3.9), (3.10) and (3.11), we obtain

$$\begin{aligned}\|y_{2n+1} - x^*\|^2 \quad \leq \quad & \|w_{2n+1} - x^*\|^2 - \|w_{2n+1} - y_{2n+1}\|^2 \\ & -2\tau_{2n+1}\langle f_{2n+1}(w_{2n+1}) - f_{2n+1}(y_{2n+1}), y_{2n+1} - x^*\rangle. \end{aligned} \tag{3.12}$$

But

$$\begin{aligned} & \|x_{2n+2} - x^*\|^2 \\ = \; & \|(1-\theta)w_{2n+1} + \theta y_{2n+1} + \theta\tau_{2n+1}(f_{2n+1}(w_{2n+1}) - f_{2n+1}(y_{2n+1})) - x^*\|^2 \\ = \; & (1-\theta)^2\|w_{2n+1} - x^*\|^2 + \theta^2\|y_{2n+1} - x^*\|^2 \\ & +\theta^2\tau_{2n+1}^2\|f_{2n+1}(w_{2n+1}) - f_{2n+1}(y_{2n+1})\|^2 \\ & +2\theta(1-\theta)\langle w_{2n+1} - x^*, y_{2n+1} - x^*\rangle \\ & +2\tau_{2n+1}\theta(1-\theta)\langle w_{2n+1} - x^*, f_{2n+1}(w_{2n+1}) - f_{2n+1}(y_{2n+1})\rangle \\ & +2\tau_{2n+1}\theta^2\langle y_{2n+1} - x^*, f_{2n+1}(w_{2n+1}) - f_{2n+1}(y_{2n+1})\rangle. \end{aligned} \tag{3.13}$$

From the identity $2\langle a,b\rangle = \|a\|^2 + \|b\|^2 - \|a-b\|^2$ for $a = w_{2n+1} - x^*$ and $b = y_{2n+1} - x^*$, we have

$$\begin{aligned} 2\langle w_{2n+1} - x^*, y_{2n+1} - x^*\rangle \quad = \quad & \|w_{2n+1} - x^*\|^2 + \|y_{2n+1} - x^*\|^2 \\ & -\|w_{2n+1} - y_{2n+1}\|^2. \end{aligned} \tag{3.14}$$

Substituting (3.14) into (3.13), we get

$$\begin{aligned} \|x_{2n+2} - x^*\|^2 = \; & (1-\theta)^2\|w_{2n+1} - x^*\|^2 + \theta^2\|y_{2n+1} - x^*\|^2 \\ & + \theta^2\tau_{2n+1}^2\|f_{2n+1}(w_{2n+1}) - f_{2n+1}(y_{2n+1})\|^2 \\ & + \theta(1-\theta)[\|w_{2n+1} - x^*\|^2 + \|y_{2n+1} - x^*\|^2 - \|w_{2n+1} - y_{2n+1}\|^2] \\ & + 2\tau_{2n+1}\theta(1-\theta)\langle w_{2n+1} - x^*, f_{2n+1}(w_{2n+1}) - f_{2n+1}(y_{2n+1})\rangle \\ & + 2\tau_{2n+1}\theta^2\langle y_{2n+1} - x^*, f_{2n+1}(w_{2n+1}) - f_{2n+1}(y_{2n+1})\rangle \\ = \; & (1-\theta)\|w_{2n+1} - x^*\|^2 + \theta\|y_{2n+1} - x^*\|^2 \\ & - \theta(1-\theta)\|w_{2n+1} - y_{2n+1}\|^2 \\ & + \theta^2\tau_{2n+1}^2\|f_{2n+1}(w_{2n+1}) - f_{2n+1}(y_{2n+1})\|^2 \\ & + 2\tau_{2n+1}\theta(1-\theta)\langle w_{2n+1} - x^*, f_{2n+1}(w_{2n+1}) - f_{2n+1}(y_{2n+1})\rangle \\ & + 2\tau_{2n+1}\theta^2\langle y_{2n+1} - x^*, f_{2n+1}(w_{2n+1}) - f_{2n+1}(y_{2n+1})\rangle. \end{aligned} \tag{3.15}$$

Thus from (3.12) and (3.15), we have

$$\begin{aligned}
\|x_{2n+2}-x^*\|^2 &\leq (1-\theta)\|w_{2n+1}-x^*\|^2+\theta[\|w_{2n+1}-x^*\|^2-\|w_{2n+1}-y_{2n+1}\|^2\\
&\quad -2\tau_{2n+1}\langle f_{2n+1}(w_{2n+1})-f_{2n+1}(y_{2n+1}),y_{2n+1}-x^*\rangle]\\
&\quad -\theta(1-\theta)\|w_{2n+1}-y_{2n+1}\|^2\\
&\quad +\theta^2\tau_{2n+1}^2\|f_{2n+1}(w_{2n+1})-f_{2n+1}(y_{2n+1})\|^2\\
&\quad +2\tau_{2n+1}\theta(1-\theta)\langle w_{2n+1}-x^*,f_{2n+1}(w_{2n+1})-f_{2n+1}(y_{2n+1})\rangle\\
&\quad +2\tau_{2n+1}\theta^2\langle y_{2n+1}-x^*,f_{2n+1}(w_{2n+1})-f_{2n+1}(y_{2n+1})\rangle.\\
&= \|w_{2n+1}-x^*\|^2-\theta(2-\theta)\|w_{2n+1}-y_{2n+1}\|^2\\
&\quad +\theta^2\tau_{2n+1}^2\|f_{2n+1}(w_{2n+1})-f_{2n+1}(y_{2n+1})\|^2\\
&\quad +2\tau_{2n+1}\theta(1-\theta)\langle w_{2n+1}-y_{2n+1},f_{2n+1}(w_{2n+1})-f_{2n+1}(y_{2n+1})\rangle\\
&\leq \|w_{2n+1}-x^*\|^2-\theta(2-\theta)\|w_{2n+1}-y_{2n+1}\|^2\\
&\quad +\theta^2\tau_{2n+1}^2\frac{\mu^2}{\tau_{2n+2}^2}\|w_{2n+1}-y_{2n+1}\|^2\\
&\quad +2\tau_{2n+1}\theta(1-\theta)\frac{\mu}{\tau_{2n+2}}\|w_{2n+1}-y_{2n+1}\|^2.\\
&= \|w_{2n+1}-x^*\|^2-\theta\Big[2-\theta-2\mu(1-\theta)\frac{\tau_{2n+1}}{\tau_{2n+2}}\\
&\quad -\theta\mu^2\frac{\tau_{2n+1}^2}{\tau_{2n+2}^2}\Big]\|w_{2n+1}-y_{2n+1}\|^2. \qquad (3.16)
\end{aligned}$$

Moreover,

$$\begin{aligned}
\|w_{2n+1}-x^*\|^2 &= \|x_{2n+1}+\alpha_{2n+1}(x_{2n+1}-x_{2n})-x^*\|^2\\
&= \|(1+\alpha_{2n+1})(x_{2n+1}-x^*)-\alpha_{2n+1}(x_{2n}-x^*)\|^2\\
&= (1+\alpha_{2n+1})\|x_{2n+1}-x^*\|^2-\alpha_{2n+1}\|x_{2n}-x^*\|^2\\
&\quad +\alpha_{2n+1}(1+\alpha_{2n+1})\|x_{2n+1}-x_{2n}\|^2. \qquad (3.17)
\end{aligned}$$

Using similar arguments in showing (3.16), we have

$$\begin{aligned}
\|x_{2n+1}-x^*\|^2 &\leq \|w_{2n}-x^*\|^2-\theta\Big[2-\theta-2\mu(1-\theta)\frac{\tau_{2n}}{\tau_{2n+1}}\\
&\quad -\theta\mu^2\frac{\tau_{2n}^2}{\tau_{2n+1}^2}\Big]\|w_{2n}-y_{2n}\|^2.\\
&= \|x_{2n}-x^*\|^2-\theta\Big[2-\theta-2\mu(1-\theta)\frac{\tau_{2n}}{\tau_{2n+1}}\\
&\quad -\theta\mu^2\frac{\tau_{2n}^2}{\tau_{2n+1}^2}\Big]\|x_{2n}-y_{2n}\|^2. \qquad (3.18)
\end{aligned}$$

Substituting (3.17) and (3.18) into (3.16), gives

$$\begin{aligned}\|x_{2n+2}-x^*\|^2 \le \|x_{2n}-x^*\|^2 &- (1+\alpha_{2n+1})\theta\Big[2-\theta-2\mu(1-\theta)\frac{\tau_{2n}}{\tau_{2n+1}}\\ &- \theta\mu^2\frac{\tau_{2n}^2}{\tau_{2n+1}^2}\Big]\|x_{2n}-y_{2n}\|^2+\alpha_{2n+1}(1+\alpha_{2n+1})\|x_{2n+1}-x_{2n}\|^2\\ &- \theta\Big[2-\theta-2\mu(1-\theta)\frac{\tau_{2n+1}}{\tau_{2n+2}}-\theta\mu^2\frac{\tau_{2n+1}^2}{\tau_{2n+2}^2}\Big]\|w_{2n+1}-y_{2n+1}\|^2. \quad (3.19)\end{aligned}$$

Also,

$$\begin{aligned}\|x_{2n+1}-x_{2n}\| &= \|(1-\theta)w_{2n}+\theta y_{2n}+\theta\tau_{2n}(f_{2n}(w_{2n})-f_{2n}(y_{2n}))-x_{2n}\|\\ &\le \theta\|w_{2n}-y_{2n}\|+\theta\tau_{2n}\|f_{2n}(w_{2n})-f_{2n}(y_{2n})\|\\ &\le \theta\|w_{2n}-y_{2n}\|+\theta\tau_{2n}\frac{\mu}{\tau_{2n+1}}\|w_{2n}-y_{2n}\|\\ &= \theta\Big(1+\tau_{2n}\frac{\mu}{\tau_{2n+1}}\Big)\|x_{2n}-y_{2n}\|. \quad (3.20)\end{aligned}$$

Therefore, from (3.19) and (3.20), we have

$$\begin{aligned}\|x_{2n+2}-x^*\|^2 &\le \|x_{2n}-x^*\|^2-(1+\alpha_{2n+1})\theta\Big[2-\theta-2\mu(1-\theta)\frac{\tau_{2n}}{\tau_{2n+1}}\\ &\quad -\theta\mu^2\frac{\tau_{2n}^2}{\tau_{2n+1}^2}-\alpha_{2n+1}\theta\Big(1+\tau_{2n}\frac{\mu}{\tau_{2n+1}}\Big)^2\Big]\|x_{2n}-y_{2n}\|^2\\ &\quad -\theta\Big[2-\theta-2\mu(1-\theta)\frac{\tau_{2n+1}}{\tau_{2n+2}}-\theta\mu^2\frac{\tau_{2n+1}^2}{\tau_{2n+2}^2}\Big]\|w_{2n+1}-y_{2n+1}\|^2\\ &\le \|x_{2n}-x^*\|^2-(1+\alpha_{2n+1})\theta\Big[2-\theta-2\mu(1-\theta)\frac{\tau_{2n}}{\tau_{2n+1}}\\ &\quad -\theta\mu^2\frac{\tau_{2n}^2}{\tau_{2n+1}^2}-\theta\Big(\frac{1-\mu}{1+\mu}\Big)^2\Big(1+\tau_{2n}\frac{\mu}{\tau_{2n+1}}\Big)^2\Big]\|x_{2n}-y_{2n}\|^2\\ &\quad -\theta\Big[2-\theta-2\mu(1-\theta)\frac{\tau_{2n+1}}{\tau_{2n+2}}-\theta\mu^2\frac{\tau_{2n+1}^2}{\tau_{2n+2}^2}\Big]\|w_{2n+1}-y_{2n+1}\|^2.\end{aligned}$$

(3.21)

Since $\tau_n \to \tau > 0$, we have

$$\begin{aligned}&\lim_{n\to\infty}\Big[2-\theta-2\mu(1-\theta)\frac{\tau_{2n}}{\tau_{2n+1}}-\theta\mu^2\frac{\tau_{2n}^2}{\tau_{2n+1}^2}-\theta\Big(\frac{1-\mu}{1+\mu}\Big)^2\Big(1+\tau_{2n}\frac{\mu}{\tau_{2n+1}}\Big)^2\Big]\\ &=\Big[2-\theta-2\mu(1-\theta)-\theta\mu^2-\theta\Big(\frac{1-\mu}{1+\mu}\Big)^2(1+\mu)^2\Big]\\ &=[2-\theta-2\mu(1-\theta)-\theta\mu^2-\theta(1-\mu)^2]\\ &=(1-\mu)[2-2\theta+2\theta\mu]\\ &=2(1-\mu)[1-\theta(1-\mu)]>0. \quad (3.22)\end{aligned}$$

Also,

$$\begin{aligned}&\lim_{n\to\infty}\Big[2-\theta-2\mu(1-\theta)\frac{\tau_{2n+1}}{\tau_{2n+2}}-\theta\mu^2\frac{\tau_{2n+1}^2}{\tau_{2n+2}^2}\Big]\\&=2-\theta-2\mu(1-\theta)-\theta\mu^2\\&=(1-\mu)(2-\theta+\theta\mu)>0.\end{aligned}\tag{3.23}$$

Let ϵ be fixed such that

$$0<\epsilon<2-\theta-2\mu(1-\theta)-\theta\mu^2-\theta(1-\mu)^2\leq 2-\theta-2\mu(1-\theta)-\theta\mu^2.$$

Then, it follows from the relations (3.22) and (3.23) that there exists $n_0>1$ such that

$$\begin{aligned}&2-\theta-2\mu(1-\theta)\frac{\tau_{2n+1}}{\tau_{2n+2}}-\theta\mu^2\frac{\tau_{2n+1}^2}{\tau_{2n+2}^2}\\&>2-\theta-2\mu(1-\theta)\frac{\tau_{2n}}{\tau_{2n+1}}-\theta\mu^2\frac{\tau_{2n}^2}{\tau_{2n+1}^2}-\theta\Big(\frac{1-\mu}{1+\mu}\Big)^2\Big(1+\tau_{2n}\frac{\mu}{\tau_{2n+1}}\Big)^2\\&\geq\epsilon>0,\forall n\geq n_0.\end{aligned}\tag{3.24}$$

This together with (3.21), implies that

$$\begin{aligned}\|x_{2n+2}-x^*\|^2 \quad\leq\quad &\|x_{2n}-x^*\|^2-(1+\alpha_{2n+1})\theta\epsilon\|x_{2n}-y_{2n}\|\\&-\theta\epsilon\|w_{2n+1}-y_{2n+1}\|^2,\ \forall n\geq n_0,\end{aligned}\tag{3.25}$$

which implies

$$\|x_{2n+2}-x^*\|^2 \quad\leq\quad \|x_{2n}-x^*\|^2,\ \forall n\geq n_0.\tag{3.26}$$

Therefore, we have that $\lim_{n\to\infty}\|x_{2n}-x^*\|$ exists and that $\{x_{2n}\}$ is bounded. Furthermore, we get from (3.25) that

$$\lim_{n\to\infty}\|x_{2n}-y_{2n}\|=0\tag{3.27}$$

and

$$\lim_{n\to\infty}\|w_{2n+1}-y_{2n+1}\|=0.\tag{3.28}$$

Again, from (2.1) and the definition of y_n, we have

$$\langle w_{2n}-\tau_{2n}f_{2n}(w_{2n})-y_{2n},x^*-y_{2n}\rangle\leq 0,$$

which implies

$$\begin{aligned}\langle f_{2n}(w_{2n}),y_{2n}-x^*\rangle \quad&\leq\quad \frac{1}{\tau_{2n}}\langle w_{2n}-y_{2n},y_{2n}-x^*\rangle\\&\leq\quad \frac{1}{\tau_{2n}}\|w_{2n}-y_{2n}\|\|y_{2n}-x^*\|.\end{aligned}\tag{3.29}$$

Furthermore, since f_{2n} is $\|A\|^2$-Lipschitz, then

$$\begin{aligned}\langle f_{2n}(w_{2n}),w_{2n}-y_{2n}\rangle \quad&\leq\quad \|f_{2n}(w_{2n})-f_{2n}(x^*)\|\|y_{2n}-w_{2n}\|\\&\leq\quad \|A\|^2\|w_{2n}-x^*\|\|y_{2n}-w_{2n}\|.\end{aligned}\tag{3.30}$$

Combining (3.29) and (3.30), we have

$$\langle f_{2n}(w_{2n}), w_{2n} - x^* \rangle \leq M \|y_{2n} - w_{2n}\|, \tag{3.31}$$

where $M = \sup\{\|A\|^2 \|w_{2n} - x^*\| + \frac{1}{\tau_{2n}} \|y_{2n} - x^*\|\}$. However, since f_{2n} is $\frac{1}{\|A\|^2}$-ism, we have

$$\begin{aligned} \langle f_{2n}(w_{2n}), w_{2n} - x^* \rangle &= \langle f_{2n}(w_{2n}) - f_{2n}(x^*), w_{2n} - x^* \rangle \\ &\geq \frac{1}{\|A\|^2} \|f_{2n}(w_{2n}) - f_{2n}(x^*)\|^2 \\ &= \frac{1}{\|A\|^2} \|f_{2n}(w_{2n})\|^2. \end{aligned} \tag{3.32}$$

From (3.31) and (3.32), we get

$$\begin{aligned} \|f_{2n}(w_{2n})\|^2 &\leq \|A\|^2 M \|y_{2n} - w_{2n}\| \\ &= \|A\|^2 M \|y_{2n} - x_{2n}\| \to 0, n \to \infty. \end{aligned} \tag{3.33}$$

Now, since $I - P_{Q_{2n}}$ is firmly nonexpansive, it follows that

$$\begin{aligned} \|(I - P_{Q_{2n}})Aw_{2n}\|^2 &= \|(I - P_{Q_{2n}})Aw_{2n} - (I - P_{Q_{2n}})Ax^*\|^2 \\ &\leq \langle (I - P_{Q_{2n}})Aw_{2n}, Aw_{2n} - Ax^* \rangle \\ &= \langle A^*(I - P_{Q_{2n}})Aw_{2n}, w_{2n} - x^* \rangle \\ &\leq \|f_{2n}(w_{2n})\| \|w_{2n} - x^*\| \to 0, n \to \infty. \end{aligned} \tag{3.34}$$

Similarly, one can show that

$$\lim_{n\to\infty} \|(I - P_{Q_{2n+1}})Aw_{2n+1}\| = 0. \tag{3.35}$$

Since, ∂q is bounded on bounded sets, we have that there exists $\zeta > 0$ such that $\|\zeta_{2n}\| \leq \zeta$. Since $P_{Q_{2n}}Aw_{2n} \in Q_{2n}$, we obtain from (3.2) and (3.34) that,

$$\begin{aligned} q(Aw_{2n}) &\leq \langle \zeta_{2n}, Aw_{2n} - P_{Q_{2n}}Aw_{2n} \rangle \\ &\leq \zeta \|(I - P_{Q_{2n}})Aw_{2n}\| \to 0, n \to \infty. \end{aligned} \tag{3.36}$$

Since $\{x_{2n}\}$ is bounded, there exists a subsequence $\{x_{2n_j}\}$ of $\{x_{2n}\}$ such that $x_{2n_j} \rightharpoonup \bar{x} \in H_1$. Thus, the weak continuity of A yields that $Ax_{2n_j} \rightharpoonup A\bar{x}$, which together with the weak lower semicontinuity of q yields

$$q(A\bar{x}) \leq \liminf_{j\to\infty} q(Ax_{2n_j}) = \liminf_{j\to\infty} q(Aw_{2n_j}) \leq 0,$$

that is $A\bar{x} \in Q$. Since $y_{2n_j} \in C_{2n_j}$, then by definition of C_{2n_j}, we have

$$c(w_{2n_j}) + \langle \xi_{2n_j}, y_{2n_j} - w_{2n_j} \rangle \leq 0,$$

where $\xi_{2n_j} \in \partial c(w_{2n_j})$. By the boundedness of $\{\xi_{2n_j}\}$, there exists $\xi > 0$ such that $\|\xi_{2n_j}\| \leq \xi$. Thus form (3.27), we have

$$\begin{aligned} c(x_{2n_j}) = c(w_{2n_j}) &\leq \langle \xi_{2n_j}, w_{2n_j} - y_{2n_j} \rangle \\ &\leq \|\xi_{2n_j}\| \|w_{2n_j} - y_{2n_j}\| \\ &\leq \xi \|w_{2n_j} - y_{2n_j}\| \\ &= \xi \|x_{2n_j} - y_{2n_j}\| \to 0, j \to \infty. \end{aligned} \tag{3.37}$$

Since $x_{2n_j} \rightharpoonup \bar{x}$, then by the weakly lower semicontinuity of c and (3.37), we have

$$c(\bar{x}) \leq \liminf_{j\to\infty} c(x_{2n_j}) \leq 0,$$

which implies $\bar{x} \in C$. Thus, we have $\bar{x} \in \Omega$ and from Lemma 2.5 it follows that $\{x_{2n}\}$ converges weakly to a point in Ω.

On the other hand, from (3.20) and (3.27), we have

$$\|x_{2n+1} - x_{2n}\| \leq \theta\Big(1 + \frac{\tau_{2n}\mu}{\tau_{2n+1}}\Big)\|x_{2n} - y_{2n}\| \to 0, n \to \infty, \tag{3.38}$$

Suppose $\{x_{2n}\}$ converges weakly to $\bar{x} \in \Omega$ and $\{x_{2n}\}$ converges weakly to $x^* \in \Omega$. Then

$$\begin{aligned}
\|\bar{x} - x^*\|^2 &= \langle \bar{x} - x^*, \bar{x} - x^*\rangle \\
&= \langle \bar{x}, \bar{x} - x^*\rangle - \langle x^*, \bar{x} - x^*\rangle \\
&= \lim_{n\to\infty} \langle x_{2n}, \bar{x} - x^*\rangle - \lim_{n\to\infty} \langle x_{2n}, \bar{x} - x^*\rangle \\
&= \lim_{n\to\infty} \langle x_{2n} - x_{2n}, \bar{x} - x^*\rangle = 0.
\end{aligned}$$

Therefore, the weak limit $\bar{x}$ is unique. By definition, we have that for all $y \in H_1$,

$$\lim_{n\to\infty} \langle x_{2n} - \bar{x}, y\rangle = 0.$$

Thus, from $\|x_{2n+1} - x_{2n}\| \to 0, n \to \infty$, we have for all $y \in H_1$

$$\begin{aligned}
|\langle x_{2n+1} - \bar{x}, y\rangle| &= |\langle x_{2n+1} - \bar{x} + x_{2n} - x_{2n}, y\rangle| \\
&\leq |\langle x_{2n} - \bar{x}, y\rangle| + |\langle x_{2n+1} - x_{2n}, y\rangle| \\
&\leq |\langle x_{2n} - \bar{x}, y\rangle| + \|x_{2n+1} - x_{2n}\|\|y\| \to 0, n \to \infty.
\end{aligned}$$

Hence, $\{x_{2n+1}\}$ converges weakly to $\bar{x} \in \Omega$ and therefore, $\{x_n\}$ converges weakly to a point $\bar{x} \in \Omega$. □

We now give an extension of our results to the multiple-sets split feasibility problem (MSFP) [17], which is mathematically formulated as the problem of finding a point x^* such that:

$$x^* \in C := \cap_{i=1}^{t} C_i, \;\; Ax^* \in Q := \cap_{j=1}^{r} Q_j, \tag{3.39}$$

where $t \geq 1$ and $r \geq 1$ are given integers, $\{C_i\}_{i=1}^t$ and $\{Q_j\}_{j=1}^r$ are nonempty closed and convex subsets of Hilbert spaces H_1 and H_2 respectively and $A : H_1 \to H_2$ is a bounded linear operator. Let us denote the solution set of the MSFP (3.39) by Ω_{MSFP}. Our results in this paper can easily be extended to solving the MSFP (3.39) where the nonempty closed and convex subsets $C_i(i = 1, \cdots, t)$ and $Q_j(j = 1 \cdots, r)$ are level sets of convex functions given are follows:

$$C_i := \{x \in H_1 : c_i(x) \leq 0\}$$

and

$$Q_j = \{y \in H_2 : q_j(y) \leq 0\},$$

where $c_i : H_1 \to \mathbb{R}, i = 1, \cdots, t$ and $q_j : H_2 \to \mathbb{R}, j = 1, \cdots, r$ are convex functions such that $c_i(i = 1, \cdots, t)$ and $q_j(j = 1, \cdots, r)$ are subdifferentiable on H_1 and H_2

respectively and $\partial c_i (i = 1, \cdots, t)$ and $\partial q_j (j = 1, \cdots, r)$ are bounded on bounded sets.

Define $C_{i,n}$ and $Q_{j,n}$ as follows:

$$C_{i,n} = \{x \in H_1 : c_i(w_n) \leq \langle \xi_{i,n}, w_n - x \rangle\}, \tag{3.40}$$

where $\xi_{i,n} \in \partial c_i(w_n)$, and

$$Q_{j,n} = \{y \in H_2 : q_j(Aw_n) \leq \langle \zeta_{j,n}, Aw_n - y \rangle\} \tag{3.41}$$

where $\zeta_{j,n} \in \partial q_j(Aw_n)$. Observe that for every $n \geq 0$, $C_i \subseteq C_{i,n}$ and $Q_j \subseteq Q_{j,n}$.
We then modify Algorithm 3.1 accordingly for solving the MSFP (3.39) as follows:

Algorithm 3.5. Initialization : Choose four parameters $\tau_1 > 0, \theta \in (0, 1], \mu \in (0, 1)$ and α_n such that $0 \leq \alpha_n \leq \left(\frac{1-\mu}{1+\mu}\right)^2$. Select initial $x_0, x_1 \in H_1$ and set $n := 1$.
Step 1: Given x_{n-1} and x_n $(n \geq 1)$, Compute

$$w_n = \begin{cases} x_n, \ n = \text{even} \\ x_n + \alpha_n(x_n - x_{n-1}), \ n = \text{odd}, \end{cases} \tag{3.42}$$

Step 2:

$$y_n = w_n - \tau_n P_n(w_n), \tag{3.43}$$

if $y_n = w_n$, then stop and w_n is the solution of the MSFP. Otherwise,
Step 3: Compute

$$x_{n+1} = (1 - \theta)w_n + \theta y_n + \theta \tau_n (P_n(w_n) - P_n(y_n)) \tag{3.44}$$

and update

$$\tau_{n+1} = \begin{cases} \min\left\{\tau_n, \frac{\mu\|w_n - y_n\|}{\|P_n(w_n) - P_n(y_n)\|}\right\}, \text{ if } \|P_n(w_n) - P_n(y_n)\| \neq 0, \\ \tau_n, \ \text{Otherwise}, \end{cases} \tag{3.45}$$

where

$$P_n(x) = \sum_{i=1}^{t} l_i(x - P_{C_{i,n}}(x)) + \sum_{j=1}^{r} \lambda_j A^*(I - P_{Q_{j,n}})Ax.$$

Set $n := n + 1$ and return to **Step 1**.

Remark 3.6. Since P_n is Lipschitz continuous, then just as in Remark 3.2, we have that $\{\tau_n\}$ is monotonically non-increasing and is bounded below by $\min\{\tau_1, \frac{\mu}{L}\}$ $(L = \sum_{i=1}^{t} l_i + \|A\|^2 \sum_{j=1}^{r} \lambda_j)$. Thus there exists $\bar{\tau} > 0$ such that $\lim_{n\to\infty} \tau_n = \bar{\tau}$.

Lemma 3.7. *If $y_n = w_n$ for some $n \geq 1$, then w_n is a solution of the MSFP* (3.39)*, i.e. $w_n \in \Omega_{MSFP}$.*

Proof. Let $y_n = w_n$, and pick $x^* \in \Omega_{MSFP}$. Then from (3.43), we have

$$\langle w_n - \lambda_n P_n(w_n) - w_n, x^* - w_n \rangle = 0,$$

which implies

$$\langle P_n(w_n), w_n - x^* \rangle = 0. \tag{3.46}$$

Since $I - P_{C_{i,n}}, (i = 1, \cdots, t)$ and $I - P_{Q_{j,n}}, (j = 1, \cdots, r)$ are firmly nonexpansive, then

$$\begin{aligned}
& \sum_{i=1}^{t} l_i \|(I - P_{C_{i,n}})w_n\|^2 + \sum_{j=1}^{r} \lambda_j \|(I - P_{Q_{j,n}})Aw_n\|^2 \\
\leq \quad & \sum_{i=1}^{t} l_i \langle (I - P_{C_{i,n}})w_n, w_n - x^* \rangle + \sum_{j=1}^{r} \lambda_j \langle (I - P_{Q_{j,n}})Aw_n, Aw_n - Ax^* \rangle \\
= \quad & \Big\langle \sum_{i=1}^{t} l_i (I - P_{C_{i,n}})w_n + \sum_{j=1}^{r} \lambda_j A^*(I - P_{Q_{j,n}})Aw_n, Aw_n - Ax^* \Big\rangle \\
= \quad & \langle P_n(w_n), w_n - x^* \rangle = 0.
\end{aligned} \tag{3.47}$$

Thus, $w_n \in C_{i,n}, (i = 1, \cdots, t)$ and $Aw_n \in Q_{j,n}, (j = 1, \cdots, r)$. It then follows from the definitions of $C_{i,n}, (i = 1, \cdots, t)$ and $Q_{j,n}, (j = 1, \cdots, r)$ that $c_i(w_n) \leq 0, (i = 1, \cdots, t)$ and $q_j(Aw_n) \leq 0, (j = 1, \cdots, r)$, which implies $w_n \in C_i (i = 1, \cdots, t)$ and $Aw_n \in Q_j, (j = 1, \cdots, r)$. Thus, we have $w_n \in \cap_{i=1}^{t} C_i$ and $Aw_n \in \cap_{j=1}^{r} Q_j$ which means w_n is a solution of MSFP (3.39). □

Theorem 3.8. *Let $\{x_n\}$ be the sequence generated by Algorithm 3.5 and $\Omega_{MSFP} \neq \emptyset$. Then $\{x_n\}$ converges weakly to a point in Ω_{MSFP}.*

Proof. Let $x^* \in \Omega$. From the definition of y_n, we obtain

$$\langle y_{2n+1} - (w_{2n+1} - \tau_{2n+1} P_{2n+1}(w_{2n+1})), y_{2n+1} - x^* \rangle = 0,$$

which gives

$$\begin{aligned}
& 2\langle w_{2n+1} - y_{2n+1}, y_{2n+1} - x^* \rangle - 2\tau_{2n+1} \langle P_{2n+1}(w_{2n+1}) - P_{2n+1}(y_{2n+1}), y_{2n+1} - x^* \rangle \\
& - 2\tau_{2n+1} \langle P_{2n+1}(y_{2n+1}), y_{2n+1} - x^* \rangle = 0.
\end{aligned} \tag{3.48}$$

Using the identity $2\langle a, b \rangle = \|a + b\|^2 - \|a\|^2 - \|b\|^2$ with $a = w_{2n+1} - y_{2n+1}$ and $b = y_{2n+1} - x^*$, we get

$$2\langle w_{2n+1} - y_{2n+1}, y_{2n+1} - x^* \rangle = \|w_{2n+1} - x^*\|^2 - \|w_{2n+1} - y_{2n+1}\|^2 - \|y_{2n+1} - x^*\|^2. \tag{3.49}$$

Moreover, since $x^* \in \Omega$, then $f_{2n+1}(x^*) = 0$. Therefore, it follows from f_{2n+1} being monotone that

$$\langle f_{2n+1}(y_{2n+1}), y_{2n+1} - x^* \rangle \geq \langle f_{2n+1}(x^*), y_{2n+1} - x^* \rangle = 0. \tag{3.50}$$

Combining the relations (3.9), (3.10) and (3.11), we obtain

$$\begin{aligned}
\|y_{2n+1} - x^*\|^2 \leq & \|w_{2n+1} - x^*\|^2 - \|w_{2n+1} - y_{2n+1}\|^2 \\
& - 2\tau_{2n+1} \langle f_{2n+1}(w_{2n+1}) - f_{2n+1}(y_{2n+1}), y_{2n+1} - x^* \rangle.
\end{aligned} \tag{3.51}$$

Thus by similar steps as in (3.13) to (3.26), we have

$$\|x_{2n+2} - x^*\|^2 \quad \leq \quad \|x_{2n} - x^*\|^2, \ \forall n \geq n_0. \tag{3.52}$$

Therefore, we have that $\lim_{n\to\infty}\|x_{2n}-x^*\|$ exists and that $\{x_{2n}\}$ is bounded. Furthermore, we get from (3.25) that

$$\lim_{n\to\infty}\|x_{2n}-y_{2n}\|=0 \tag{3.53}$$

and

$$\lim_{n\to\infty}\|w_{2n+1}-y_{2n+1}\|=0. \tag{3.54}$$

Again, from the definition of y_n, we have

$$\langle w_{2n}-\tau_{2n}P_{2n}(w_{2n})-y_{2n},x^*-y_{2n}\rangle=0,$$

which implies

$$\begin{aligned}\langle P_{2n}(w_{2n}),y_{2n}-x^*\rangle &= \frac{1}{\tau_{2n}}\langle w_{2n}-y_{2n},y_{2n}-x^*\rangle\\ &\le \frac{1}{\tau_{2n}}\|w_{2n}-y_{2n}\|\|y_{2n}-x^*\|. \end{aligned}\tag{3.55}$$

Furthermore, since P_{2n} is $L=\sum_{i=1}^t l_i+\|A\|^2\sum_{j=1}^r\lambda_j$-Lipschitz, then

$$\begin{aligned}\langle P_{2n}(w_{2n}),w_{2n}-y_{2n}\rangle &\le \|P_{2n}(w_{2n})-P_{2n}(x^*)\|\|y_{2n}-w_{2n}\|\\ &\le L\|w_{2n}-x^*\|\|y_{2n}-w_{2n}\|. \end{aligned}\tag{3.56}$$

Combining (3.55) and (3.56), we have

$$\langle P_{2n}(w_{2n}),w_{2n}-x^*\rangle\le K\|y_{2n}-w_{2n}\|, \tag{3.57}$$

where $K=\sup\{L\|w_{2n}-x^*\|+\frac{1}{\tau_{2n}}\|y_{2n}-x^*\|\}$. However, since P_{2n} is $\frac{1}{L}$-ism, we have

$$\begin{aligned}\langle P_{2n}(w_{2n}),w_{2n}-x^*\rangle &= \langle P_{2n}(w_{2n})-P_{2n}(x^*),w_{2n}-x^*\rangle\\ &\ge \frac{1}{L}\|P_{2n}(w_{2n})-P_{2n}(x^*)\|^2\\ &= \frac{1}{L}\|P_{2n}(w_{2n})\|^2. \end{aligned}\tag{3.58}$$

From (3.57) and (3.58), we get

$$\begin{aligned}\|P_{2n}(w_{2n})\|^2 &\le LK\|y_{2n}-w_{2n}\|\\ &= LK\|y_{2n}-x_{2n}\|\to 0, n\to\infty. \end{aligned}\tag{3.59}$$

Since $I - P_{C_{i,2n}}, (i = 1, \cdots, t)$ and $I - P_{Q_{j,2n}}, (j = 1, \cdots, r)$ are firmly nonexpansive, then

$$\begin{aligned}
& \sum_{i=1}^{t} l_i \|(I - P_{C_{i,2n}})w_{2n}\|^2 + \sum_{j=1}^{r} \lambda_j \|(I - P_{Q_{j,2n}})Aw_{2n}\|^2 \\
\leq \; & \sum_{i=1}^{t} l_i \langle (I - P_{C_{i,2n}})w_{2n}, w_{2n} - x^* \rangle + \sum_{j=1}^{r} \lambda_j \langle (I - P_{Q_{j,2n}})Aw_{2n}, Aw_{2n} - Ax^* \rangle \\
= \; & \Big\langle \sum_{i=1}^{t} l_i (I - P_{C_{i,2n}})w_{2n} + \sum_{j=1}^{r} \lambda_j A^*(I - P_{Q_{j,2n}})Aw_{2n}, Aw_{2n} - Ax^* \Big\rangle \\
= \; & \langle P_{2n}(w_{2n}), w_{2n} - x^* \rangle \\
\leq \; & \|P_{2n}(w_{2n})\| \|w_{2n} - x^*\| \to 0, n \to \infty, \qquad (3.60)
\end{aligned}$$

which gives

$$\|(I - P_{C_{i,2n}})w_{2n}\| \to 0, n \to \infty, (i = 1, \cdots, t) \qquad (3.61)$$

and

$$\|(I - P_{Q_{j,2n}})Aw_{2n}\| \to 0, n \to \infty, (j = 1, \cdots, r). \qquad (3.62)$$

Since, $\partial c_i, (i = 1, \cdots, t)$ are bounded on bounded sets, we have that there exists $\xi_i > 0, (i = 1, \cdots, t)$ such that $\|\xi_{i,2n}\| \leq \xi_i$. Since $P_{C_{i,2n}} w_{2n} \in C_{i,2n}$, we obtain from (3.40) and (3.61) that,

$$\begin{aligned}
c_i(w_{2n}) \;&\leq\; \langle \xi_{i,2n}, w_{2n} - P_{C_{i,2n}} w_{2n} \rangle \\
&\leq\; \xi_i \|(I - P_{C_{i,2n}})w_{2n}\| \to 0, n \to \infty, (i = 1, \cdots, t). \qquad (3.63)
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
q_j(Aw_{2n}) \;&\leq\; \langle \zeta_{j,2n}, Aw_{2n} - P_{Q_{j,2n}} Aw_{2n} \rangle \\
&\leq\; \zeta_j \|(I - P_{Q_{j,2n}})Aw_{2n}\| \to 0, n \to \infty, (j = 1, \cdots, r). \qquad (3.64)
\end{aligned}$$

Since $\{x_{2n}\}$ is bounded, there exists a subsequence $\{x_{2n_k}\}$ of $\{x_{2n}\}$ such that $x_{2n_k} \rightharpoonup \bar{x} \in H_1$. Therefore by the weakly lower semicontinuity of c_i and (3.63), we have

$$c_i(\bar{x}) \leq \liminf_{k \to \infty} c(x_{2n_k}) \leq 0,$$

which implies $\bar{x} \in C_i, (i = 1, \cdots, t)$, that is $\bar{x} \in \cap_{i=1}^{t} C_i$

Also, the weak continuity of A yields that $Ax_{2n_j} \rightharpoonup A\bar{x}$, which together with the weak lower semicontinuity of $q_j, (j = 1, \cdots, r)$, yields

$$q_j(A\bar{x}) \leq \liminf_{k \to \infty} q(Ax_{2n_k}) = \liminf_{j \to \infty} q(Aw_{2n_k}) \leq 0,$$

that is $A\bar{x} \in Q_j.(j = 1, \cdots, r)$, that is $A\bar{x} \in \cap_{j=1}^{r} Q_j$. Since $x_{2n_k} \rightharpoonup \bar{x}$, and $\bar{x} \in \Omega_{MSFP}$, then from Lemma 2.5 it follows that $\{x_{2n}\}$ converges weakly to a point in Ω_{MSFP}.

Suppose $\{x_{2n}\}$ converges weakly to $\bar{x} \in \Omega$ and $\{x_{2n}\}$ converges weakly to $x^* \in \Omega$ and since just as in (3.38), we have

$$\|x_{2n+1} - x_{2n}\| \leq \theta\Big(1 + \frac{\tau_{2n}\mu}{\tau_{2n+1}}\Big)\|x_{2n} - y_{2n}\| \to 0, n \to \infty, \tag{3.65}$$

Then

$$\begin{aligned} \|\bar{x} - x^*\|^2 &= \langle \bar{x} - x^*, \bar{x} - x^* \rangle \\ &= \langle \bar{x}, \bar{x} - x^* \rangle - \langle x^*, \bar{x} - x^* \rangle \\ &= \lim_{n\to\infty} \langle x_{2n}, \bar{x} - x^* \rangle - \lim_{n\to\infty} \langle x_{2n}, \bar{x} - x^* \rangle \\ &= \lim_{n\to\infty} \langle x_{2n} - x_{2n}, \bar{x} - x^* \rangle = 0. \end{aligned}$$

Therefore, the weak limit $\bar{x}$ is unique. By definition, we have that for all $y \in H_1$,

$$\lim_{n\to\infty} \langle x_{2n} - \bar{x}, y \rangle = 0.$$

Thus, from $\|x_{2n+1} - x_{2n}\| \to 0, n \to \infty$, we have for all $y \in H_1$

$$\begin{aligned} |\langle x_{2n+1} - \bar{x}, y \rangle| &= |\langle x_{2n+1} - \bar{x} + x_{2n} - x_{2n}, y \rangle| \\ &\leq |\langle x_{2n} - \bar{x}, y \rangle| + |\langle x_{2n+1} - x_{2n}, y \rangle| \\ &\leq |\langle x_{2n} - \bar{x}, y \rangle| + \|x_{2n+1} - x_{2n}\|\|y\| \to 0, n \to \infty. \end{aligned}$$

Hence, $\{x_{2n+1}\}$ converges weakly to $\bar{x} \in \Omega$ and therefore, $\{x_n\}$ converges weakly to a point $\bar{x} \in \Omega$. □

4. Numerical Examples

In this section, in order to show the validity of Algorithms 3.1 and 3.5, we present some preliminary numerical results for solving the SFP and MSFP respectively in the setting of finite dimensional Hilbert spaces. All the codes are written in MATLAB R2015a and run on HP Intel(R) Core(TM) i3-5005U CPU @ 2.00GHz 2.00GHz; 4.00GB Ram laptop.

Example 4.1. Let $H_1 = H_2 = \mathbb{R}^3$. We take

$$C = \{x = (x_1, x_2, x_3)^T \in \mathbb{R}^3 | x_1 + \frac{1}{2}x_2^2 + x_3^2 \leq 0\},$$

$$Q = \{x = (x_1, x_2, x_3)^T \in \mathbb{R}^3 | x_1^2 + x_2 + \frac{1}{2}x_3^2 \leq 0\}$$

and

$$A = \begin{pmatrix} 3 & 1 & -2 \\ 3 & 2 & 2 \\ 2 & 0 & 1 \end{pmatrix}.$$

For this example, we take $\alpha_n = \left(\frac{1-\mu}{1+\mu}\right)^2 \frac{n+1}{n+5}$, $\mu = \frac{1}{5}$, $\theta = 0.6$, $\tau_1 = 1$ and stopping rule $\|w_n - y_n\| < \epsilon$. For the algorithm (1.3) of Yang [44], we take $\tau_n = \frac{0.04}{\|A\|^2}$ and for the algorithms López et al. [27] and Kesornprom et al. [26], we take $\rho_n = 0.01$ and $\theta_n = \frac{1}{n+1}$. We for the algorithms of Yang [44], López et al. [27] and Kesornprom et al. [26], we take the stopping rule $\|x_{n+1} - x_n\| < \epsilon$.

TABLE 1. $\epsilon = 10^{-5}$

	No. of iterations	CPU Time.
Alg. 3.1	144	0.045504
Yang [44]	287	0.033093
López et al. [27]	820	0.092272
Kesornprom et al. [26]	498	0.088831

TABLE 2. $\epsilon = 10^{-7}$

	No. of iterations	CPU Time.
Alg. 3.1	237	0.045504
Yang [44]	557	0.064735
López et al. [27]	1740	0.19157
Kesornprom et al. [26]	3085	0.54775

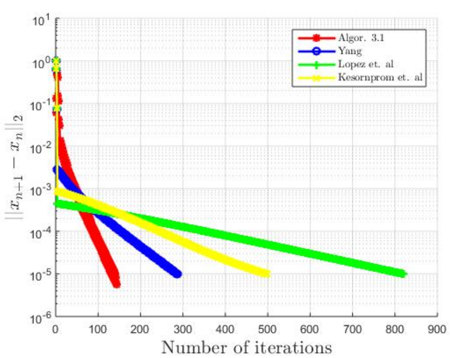

FIGURE 1. $\epsilon = 10^{-5}$

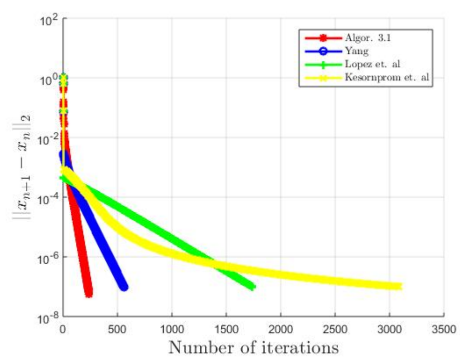

FIGURE 2. $\epsilon = 10^{-7}$

Example 4.2. In this example, we compare the performance of Algorithm 3.5 and Algorithm of Tang et al. [40] Let $H_1 = H_2 = \mathbb{R}^3$ and $r = t = 2$. We take

$$C_1 = \{x = (x_1, x_2, x_3)^T \in \mathbb{R}^3 | x_1 + x_2^2 + 2x_3 \leq 0\},$$

$$C_2 = \{x = (x_1, x_2, x_3)^T \in \mathbb{R}^3 | \frac{x_1^2}{16} + \frac{x_2^2}{9} + \frac{x_3^2}{4} - 1 \leq 0\},$$

$$Q_1 = \{x = (x_1, x_2, x_3)^T \in \mathbb{R}^3 | x_1^2 + x_2 - x_3 \leq 0\},$$

$$Q_2 = \{x = (x_1, x_2, x_3)^T \in \mathbb{R}^3 | \frac{x_1^2}{4} + \frac{x_2^2}{4} + \frac{x_3^2}{9} - 1 \leq 0\}$$

and

$$A = \begin{pmatrix} 3 & 1 & -2 \\ 3 & 2 & 2 \\ 2 & 0 & 1 \end{pmatrix}.$$

For this example, we take

$$\alpha_n = \Big(\frac{1-\mu}{1+\mu}\Big)^2 \frac{n+1}{n+5},\ \mu = \frac{1}{5},\ \theta = 0.6,\ \tau_1 = 1,$$

$$l_1 = l_2 = \lambda_1 = \lambda_2 = \frac{1}{2}$$

and stopping rule $\|w_n - y_n\| < 10^{-2}$. Algorithm of Tang et al. [40], we take

$$\rho_1^k = \rho_1^n = \rho_2^n = \frac{1}{5},\ \alpha_1 = \alpha_2 = \beta_1 = \beta_2 = \frac{1}{2}$$

and stopping rule $\|x_{n+1} - x_n\| < 10^{-2}$. We choose $x_0 = [\frac{1}{2}, \frac{1}{2}, \frac{1}{2}]^T$, $x_1 = [1, 1, 1]^T$ for each case but take $x_1 = [1, 1, 1]^T$ and $x_0 = [\frac{1}{2}, \frac{1}{2}, \frac{1}{2}]^T$ respectively.

Table 3. $x_1 = [1, 1, 1]^T$ for Tang et al. [40]

	No. of iterations	CPU Time.
Alg. 3.5	9	0.0039859
Tang et al [40]	16	0.0054634

Table 4. $x_1 = [\frac{1}{2}, \frac{1}{2}, \frac{1}{2}]^T$ for Tang et al. [40]

	No. of iterations	CPU Time.
Alg. 3.5	9	0.0061113
Tang et al [40]	12	0.0023727

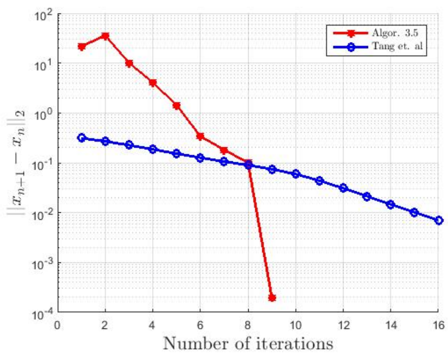

FIGURE 3. $x_1 = [1, 1, 1]^T$ for Tang et al. [40]

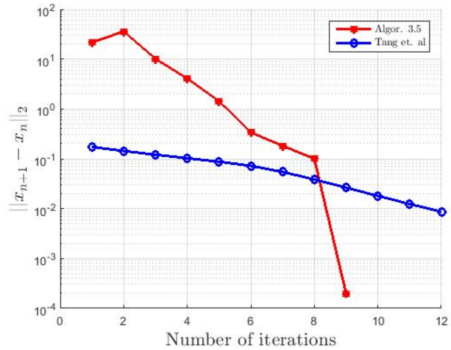

FIGURE 4. $x_1 = [\frac{1}{2}, \frac{1}{2}, \frac{1}{2}]^T$ for Tang et al. [40]

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Received: May 23, 2021; Accepted: January 27, 2022.