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φ-CONTRACTIVE PARENT-CHILD POSSIBLY INFINITE IFSS AND ORBITAL $\varphi-$ CONTRACTIVE POSSIBLY INFINITE IFSS

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Abstract. In this paper we introduce the notions of φ-contractive parent-child possibly infinite iterated function system (pcIIFS) and orbital φ-contractive possibly infinite iterated function system (oIIFS) and we prove that the corresponding fractal operator is weakly Picard. The corresponding notions of shift space, canonical projection and their properties are also treated.
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1. Introduction

The concept of iterated function system (IFS for short) was introduced forty years ago by J. Hutchinson in [9], popularized by Michael Barnsley in [1] and it represents one of the most important methods of constructing fractals. In the last years there have been considered many generalizations of this concept. A first direction of generalization consists of using weaker contractivity conditions. Along these lines of research let us mention some papers. For example, in [10] L. Ioana and A. Mihail introduced and studied IFSs consisting of φ-contractions and in [19] I. Savu studied IFSs consisting of continuous functions satisfying Banach's orbital condition. In [6] F. Georgescu generalized the concept of IFS consisting of convex contractions and in [21] N. A. Secelean introduced a new type of IFS, namely IFS consisting of F-contractions. The fractal operator associated to IFSs with weaker contractivity conditions could have the same properties as the component functions (see [12]) or not (see [17]). In this case, it may appear some difficulties. For example, in [23] N. Van Dung and A. Petruşel pointed out the problems in providing some results of IFSs consisting of Kannan maps, Reich maps and Chatterjea type maps.

A second way to generalize the notion of IFS was to consider systems with an arbitrary number (finite or infinite) of functions. For example, in [5] H. Fernau studied infinite IFSs, in [7] G. Gwóźdź-Lukowska and J. Jachymski presented the Hutchinson-Barnsley theory for infinite IFSs and in [4] D. Dumitru studied arcwise connected attractors of infinite IFSs. Also, I. Jaksztas in [11] studied the infinite IFSs depending on a parameter and F. Mendivil in [15] constructed a generalization of IFS with probabilities to infinitely many maps. The infinite IFSs were also studied in [2], [8], [13] and [16].

Another way to generalize the IFSs was to change the structure of component functions or the structure of space. For example, R. D. Mauldin and M. Urbański in [14] studied graph directed Markov systems and in [18] the authors studied the canonical projection of generalized IFSs.

Related to the concept of IFS is the notion of shift (or code) space. The shift space of an iterated function system and the address of the points lying on the attractor of the IFS are very good tools to get a more precise description of the invariant dynamics of the IFS and of the topological properties of the attractor. For example, in [3] D. Dumitru studied the topological properties of the attractors of IFS and in [22] F. Strobin studied this problem in the framework of generalized iterated function systems.

In this paper we use the notion of parent-child contractivity condition to define the notion of φ-contractive parent-child possibly infinite iterated function system (pcIIFS). The parent-child contractivity condition was also used in [24] by R. Zaharopol to study IFS with probabilities. Another notion introduced in this paper is that of orbital φ-contractive possibly infinite iterated function system (oIIFS).

In the first part of the main results we study the fractal operator associated to a pcIIFS and we prove that it is weakly Picard. Also, we construct the canonical projection and we study its properties. We define a continuous function Θ (which is uniformly continuous on bounded sets) which describes the dynamics of a possibly infinite iterated function system (IIFS for short) better than the canonical projection. Using the function Θ, we obtain the canonical projection and we define an extended canonical projection, π^t.

If we consider a pcIIFS $\mathcal{S} = \left((X,d), (f_i)_{i\in I}\right)$ such that the fractal operator associated to $\mathcal{S}$ is a Picard operator, then as $S_{\Lambda(I)} = \left(\Lambda(I), (F_i)_{i\in I}\right)$ is a universal model for the pcIIFS S restricted to its fixed point, we have that the system $S_{\Lambda^t(I)} = (\Lambda^t(I) \times X, (F_i^t)_{i\in I})$ is a universal model for the pcIIFS S (see Remark 3.27).

The second part of the main results is dedicated to the study of oIIFSs. We study similar properties with those presented in the first part.

We now present a simple example. Let us consider the normed space $\left(\mathbb{R}^2, \|.\|_2\right)$, $I = \{0,1\}$ and the functions $f_i : \mathbb{R}^2 \to \mathbb{R}^2$ given by $f_i(x,y) = \left(\frac{x}{4} + \frac{3i}{4}, y\right)$ for all $(x,y) \in \mathbb{R}^2$ and $i \in I$. Then, $\mathcal{S} = \left((X,d), (f_i)_{i\in I}\right)$ is a pcIIFS. We note that

$$f_{i_1} \circ f_{i_2} \circ ... \circ f_{i_n}(x,y) = \left(\frac{1}{4^n}x + 3\left(\frac{i_1}{4} + \frac{i_2}{4^2} + ... + \frac{i_n}{4^n}\right), y\right)$$

for all $n \in \mathbb{N}^*$, $i_1, i_2, ..., i_n \in I$ and $(x,y) \in \mathbb{R}^2$.

If $\alpha = i_1 i_2 ... i_n ...$, we have

$$a_\alpha (x, y) = \lim_{n\to\infty} f_{i_1} \circ f_{i_2} \circ ... \circ f_{i_n} (x, y) = \left(3 \sum_{n\geq 1} \frac{i_n}{4^n}, y \right)$$

for all $(x, y) \in \mathbb{R}^2$. Therefore,

$$\Theta (\alpha, B) = \begin{cases} \{a_\alpha (x, y) \mid (x, y) \in B\}, \text{ if } \alpha = i_1 i_2 ... i_n ... \\ f_\alpha (B), \text{ if } \alpha = i_1 i_2 ... i_n \end{cases}$$

for all $B \in P_{cl,b} (X)$.

2. Preliminaries

Notations and terminology

Given a set X, a function $f : X \to X$ and $n \in \mathbb{N}^*$, by f^n we mean $f \circ f \circ ... \circ f$ for n times. By $Id_X : X \to X$ we mean the function defined by $Id_X (x) = x$ for every $x \in X$.

Given a metric space (X, d), by:

- $diam (A)$ we mean the diameter of the subset A of X;
- $P_b (X)$ we mean the set of non-empty bounded subsets of X;
- $P_{cl,b} (X) = \{A \in P_b (X) \mid A \text{ is closed}\}$;
- $B [A, r] = \{x \in X \mid d (A, x) \leq r\}$, where $A \in P_b (X)$ and $r > 0$;
- a weakly Picard operator we mean a function $f : X \to X$ having the property that, for every $x \in X$, the sequence $(f^n (x))_{n\in\mathbb{N}}$ is convergent to a fixed point of f;
- a Picard operator we mean a function $f : X \to X$ having the property that the sequence $(f^n (x))_{n\in\mathbb{N}}$ is convergent to the unique fixed point of f, for all $x \in X$;
- a continuous weakly Picard operator we mean a function $f : X \to X$ which is a weakly Picard operator and the function $x \to \lim_{n\to\infty} f^n (x)$ for all $x \in X$ is continuous.

Let (X, d_X) and (Y, d_Y) be two metric spaces. By

- $\mathcal{C} (X, Y)$ we mean the set of continuous functions from X to Y;
- $\mathcal{C}_b (X, Y) = \{f \in \mathcal{C} (X, Y) \mid f \text{ is bounded}\}$;
- $d_{\max}$ we mean the metric on $X \times Y$ defined by $d_{\max} ((x_1, y_1), (x_2, y_2)) = \max \{d_X (x_1, x_2), d_Y (y_1, y_2)\}$ for all $(x_1, y_1), (x_2, y_2) \in X \times Y$.

Given a metric space (X, d) and $f, g : X \to X$, by $d_u (f, g)$ we mean the uniform distance between f and g, namely $d_u (f, g) = \sup_{x\in X} d (f (x), g (x))$.

Definition 2.1. Let (X, d_X) and (Y, d_Y) be two metric spaces. A family of functions $(f_i)_{i\in I}$ is said to be

1) bounded if the set $\cup_{i\in I} f_i (B) \in P_b (X)$ for every $B \in P_b (X)$,

2) equi-uniformly continuous if for every $\varepsilon > 0$ there exists $\delta_\varepsilon > 0$ such that for all $x, y \in X$ with $d_X (x, y) < \delta_\varepsilon$ we have $d_Y (f_i (x), f_i (y)) < \varepsilon$, for all $i \in I$.

Definition 2.2. For a metric space (X, d), we consider the generalized Hausdorff-Pompeiu pseudometric $h : P_b (X) \times P_b (X) \to [0, \infty)$ defined by

$$h (A, B) = \max \{d (A, B), d (B, A)\}$$

for all $A, B \in P_b(X)$, where $d(A,B) = \sup_{x\in A} \inf_{y\in B} d(x,y)$.

Definition 2.3. The restriction of h to $P_{cl,b}(X)$ is called the Hausdorff-Pompeiu metric and it is also denoted by h.

Results regarding the Hausdorff-Pompeiu semidistance

Proposition 2.4. [see [20]] *For a metric space (X,d), we have:*

1) If H, $K \in P_b(X)$, then

$$h(H,K) = h\left(\overline{H}, \overline{K}\right); \tag{2.1}$$

2) If $(H_i)_{i\in I}$ and $(K_i)_{i\in I}$ are families of elements from $P_b(X)$ such that $\cup_{i\in I} H_i \in P_b(X)$ and $\cup_{i\in I} K_i \in P_b(X)$, then

$$h(\cup_{i\in I} H_i, \cup_{i\in I} K_i) \leq \sup_{i\in I} h(H_i, K_i); \tag{2.2}$$

3) If $f: X \to X$ is a uniformly continuous function, $A \in P_b(X)$ and $(A_n)_{n\in\mathbb{N}} \subset P_b(X)$ with $\lim_{n\to\infty} h(A_n, A) = 0$, then $\lim_{n\to\infty} h(f(A_n), f(A)) = 0$.

Proposition 2.5. [see [20]] *If the metric space (X,d) is complete, then the metric space $(P_{cl,b}(X), h)$ is complete.*

Notations and terminology for the shift space

$\mathbb{N}$ denotes the natural numbers, $\mathbb{N}^* = \mathbb{N} \setminus \{0\}$ and $\mathbb{N}_n^* = \{1, 2, ..., n\}$, where $n \in \mathbb{N}^*$. Given two sets A and B, by B^A we mean the set of all functions from A to B.

For a set I, by $\Lambda(I)$ we mean the set $I^{\mathbb{N}^*}$. The elements of $\Lambda(I)$ can be written as infinite words, namely $\omega = \omega_1\omega_2...\omega_n...$. For $\omega \in \Lambda(I)$ and $n \in \mathbb{N}^*$, by $[\omega]_n$ we mean the word formed with the first n letters from ω. By $\Lambda_n(I)$ we mean the set $I^{\mathbb{N}_n^*}$. The elements of $\Lambda_n(I)$ are finite words with n letters: $\omega = \omega_1\omega_2...\omega_n$. In this case, n is called the length of ω and it is denoted by $|\omega|$. For $\omega \in \Lambda_m(I)$ and $n \in \mathbb{N}^*$, by $[\omega]_n$ we mean the word formed with the first n letters from ω if $m \geq n$, or the word ω if $m \leq n$. For two words $\alpha \in \Lambda_n(I)$ and $\beta \in \Lambda_m(I)$ or $\beta \in \Lambda(I)$, by $\alpha\beta$ we mean the concatenation of α and β, i.e. $\alpha\beta = \alpha_1\alpha_2...\alpha_n\beta_1\beta_2...\beta_m$ and $\alpha\beta = \alpha_1\alpha_2...\alpha_n\beta_1\beta_2...\beta_m\beta_{m+1}...$ respectively.

For a family of functions $(f_i)_{i\in I}$, where $f_i : X \to X$ and $\omega = \omega_1\omega_2...\omega_n \in \Lambda_n(I)$, we use the following notation: $f_\omega = f_{\omega_1} \circ ... \circ f_{\omega_n}$. For a set $B \subset X$ and $\omega \in \Lambda_n(I)$ we use the notation $B_\omega = f_\omega(B)$. For $y, z \in B$, we say that z is a child of y (or y is a parent of z) if there exist $n \in \mathbb{N}^*$, $\omega_1, \omega_2, ..., \omega_{n+1} \in I$ and $x \in B$ such that $y = f_{\omega_1} \circ ... \circ f_{\omega_n}(x)$ and $z = f_{\omega_1} \circ ... \circ f_{\omega_n} \circ f_{\omega_{n+1}}(x)$.

By $\Lambda^*(I)$ we mean the set of all finite words, $\Lambda^*(I) = (\cup_{n\in\mathbb{N}^*} \Lambda_n(I)) \cup \{\lambda\}$, where λ is the empty word. By $\Lambda^t(I)$ we mean the set of all words with letters from I, namely the set $\Lambda^*(I) \cup \Lambda(I)$.

For a fixed element $\tau \notin I$, we denote by $\widetilde{I} = I \cup \{\tau\}$. Let us consider $c \in [0,1)$. We define $d_c : \Lambda(\widetilde{I}) \times \Lambda(\widetilde{I}) \to [0,\infty)$ by $d_c(\alpha,\beta) = \sum_{n\geq 1} c^n d(\alpha_n, \beta_n)$, for all $\alpha, \beta \in \Lambda(\widetilde{I})$, where by α_n we mean the letter on position n in α and

$$d(\alpha_n,\beta_n)=\begin{cases}0, \text{ if } \alpha_n=\beta_n\\ 1, \text{ if } \alpha_n\neq\beta_n\end{cases}$$

for all $n\in\mathbb{N}^*$. $\Lambda^t(I)$ can be seen as a subset of $\Lambda(\widetilde{I})$, by defining the injective function $\iota:\Lambda^t(I)\to\Lambda(\widetilde{I})$,

$$\iota(\alpha)=\begin{cases}\alpha, \text{ if } \alpha\in\Lambda(I)\\ \alpha\tau\tau...\tau... \text{ if } \alpha\in\Lambda^*(I)\\ \tau\tau...\tau... \text{ if } \alpha=\lambda\end{cases}.$$

$(\Lambda^t(I), d_c)$ is a complete metric space, $\Lambda(I)$ is a closed subset of $\Lambda^t(I)$ and $\Lambda^*(I)$ contains only isolated points.

Possibly infinite iterated function systems

Definition 2.6. Let (X,d) be a metric space. A function $\varphi:[0,\infty)\to[0,\infty)$ is called

1) comparison function if $\varphi(r)<r$ for all $r>0$ and φ is an increasing function on $[0,\infty)$;

2) summable comparison function if φ is a comparison function and $\sum_{n=0}^{\infty}\varphi^n(r)$ is convergent for every $r>0$;

3) right continuous function if $\lim_{\substack{r\to r_0\\ r>r_0}}\varphi(r)=\varphi(r_0)$ for all $r_0\in[0,\infty)$.

Remark 2.7. If $\varphi:[0,\infty)\to[0,\infty)$ is a summable or right continuous comparison function, then $\varphi(0)=0$ and $\lim_{n\to\infty}\varphi^n(r)=0$ for every $r>0$.

Definition 2.8. Let (X,d) be a metric space and $\varphi:[0,\infty)\to[0,\infty)$ a comparison function. $f:X\to X$ is called φ-contraction if for every $x,y\in X$

$$d(f(x),f(y))\leq\varphi(d(x,y)).$$

Theorem 2.9. *Let (X,d) be a complete metric space, $\varphi:[0,\infty)\to[0,\infty)$ a right continuous comparison function and $f:X\to X$ a φ-contraction. Then, f has a unique fixed point denoted by η. For every $x_0\in X$, the sequence $(f^n(x_0))_{n\in\mathbb{N}}$ is convergent to η and for all $m\in\mathbb{N}$, we have*

$$d(f^m(x_0),\eta)\leq\varphi^m(d(x_0,\eta)).$$

Definition 2.10. Let (X,d) be a complete metric space and $(f_i)_{i\in I}$ a family of functions, where $f_i:X\to X$ for all $i\in I$. The pair denoted by $\mathcal{S}=\left((X,d),(f_i)_{i\in I}\right)$ is called possibly infinite iterated function system (IIFS) if

i) $f_i:X\to X$ is a continuous function for every $i\in I$,

ii) the family $(f_i)_{i\in I}$ is equi-uniformly continuous on bounded sets, i.e. for every $B\in P_b(X)$ and every $\varepsilon>0$ there exists $\delta_{\varepsilon,B}>0$ such that for all $x,y\in B$ with $d(x,y)<\delta_{\varepsilon,B}$ we have $d(f_i(x),f_i(y))<\varepsilon$, for all $i\in I$,

iii) $(f_i)_{i\in I}$ is a bounded family of functions.

Given an IIFS $\mathcal{S}=\left((X,d),(f_i)_{i\in I}\right)$, we consider the fractal operator $F_{\mathcal{S}}:P_b(X)\to P_b(X)$ defined by

$$F_{\mathcal{S}}(B)=\overline{\cup_{i\in I}f_i(B)}$$

for every $B\in P_b(X)$. $F_{\mathcal{S}}$ restricted to $P_{cl,b}(X)$ will still be denoted by $F_{\mathcal{S}}$.

Notation 2.11. For a set $B\in P_b(X)$, by the orbit of B we mean the set $\mathcal{O}(B)=\cup_{n\in\mathbb{N}}F^n_{\mathcal{S}}(B)$. If $B=\{x\}$, we denote its orbit by $\mathcal{O}(x)$.

Definition 2.12. Let $\mathcal{S}=\left((X,d),(f_i)_{i\in I}\right)$ be an IIFS. $\mathcal{S}$ is called

1) φ-contractive parent-child possibly infinite iterated function system (pcIIFS) if $\varphi:[0,\infty)\to[0,\infty)$ is a summable comparison function and

$$d(f_\omega(x),f_{\omega i}(x))\le\varphi^{|\omega|}(d(x,f_i(x))),\tag{2.3}$$

for every $i\in I$, $\omega\in\Lambda^*(I)$ and $x\in X$;

2) orbital φ-contractive possibly infinite iterated function system (oIIFS) if $\varphi:[0,\infty)\to[0,\infty)$ is a right-continuous comparison function and

$$d(f_i(y),f_i(z))\le\varphi(d(y,z))$$

for every $i\in I$, $x\in X$ and $y,z\in\mathcal{O}(x)$.

Remark 2.13. If $\mathcal{S}=\left((X,d),(f_i)_{i\in I}\right)$ is an oIIFS, then

$$d(f_\omega(y),f_\omega(z))\le\varphi^{|\omega|}(d(y,z)),\tag{2.4}$$

for every $\omega\in\Lambda^*(I)$, $x\in X$ and $y,z\in\mathcal{O}(x)$.

Remark 2.14. If $B\in P_b(X)$ and $(B_n)_{n\in\mathbb{N}}\subset P_b(X)$ such that $\lim\limits_{n\to\infty}h(B_n,B)=0$, we deduce that the sequence $(B_n)_n$ is bounded. Therefore, there exists a set $M\in P_b(X)$ such that $(\cup_{n\in\mathbb{N}}B_n)\cup B\subset M$.

Let $\mathcal{S}=\left((X,d),(f_i)_{i\in I}\right)$ be an IIFS. Using ii) and iii) from Definition 2.10, we deduce that the family $(f_\alpha)_{\alpha\in\Lambda_n(I)}$ is equi-uniformly continuous on M and as a consequence, $F^n_{\mathcal{S}}$ is uniformly continuous on $P_b(M)$ for every $n\in\mathbb{N}$.

3. Main results

$\varphi-$Contractive parent-child possibly infinite iterated function systems (pcIIFSs)

Theorem 3.1. *For a pcIIFS $\mathcal{S}=\left((X,d),(f_i)_{i\in I}\right)$, $F_{\mathcal{S}}$ is a weakly Picard operator. More precisely, for every $B\in P_{cl,b}(X)$, there exists $A_B\in P_{cl,b}(X)$ such that $\lim\limits_{n\to\infty}F^n_{\mathcal{S}}(B)=A_B$ and $F_{\mathcal{S}}(A_B)=A_B$. Moreover, for all $m\in\mathbb{N}$,*

$$h(F^m_{\mathcal{S}}(B),A_B)\le\sum_{k\ge m}\varphi^k(diam(B\cup F_{\mathcal{S}}(B))).\tag{3.1}$$

Proof. We note that $\sup\limits_{x\in B}\left(\sup\limits_{i\in I}d(x,f_i(x))\right)$ is finite for all $B\in P_{cl,b}(X)$.

Claim 3.2. For all $n\in\mathbb{N}$ and $x\in B$, we have

$$h\left(F^n_{\mathcal{S}}(\{x\}),F^{n+1}_{\mathcal{S}}(\{x\})\right)\le\varphi^n\left(\sup_{i\in I}d(x,f_i(x))\right).\tag{3.2}$$

Justification: Indeed, for all $n \in \mathbb{N}$,

$$h\left(F_{\mathcal{S}}^{n}\left(\{x\}\right), F_{\mathcal{S}}^{n+1}\left(\{x\}\right)\right) = h\left(\cup_{\alpha\in\Lambda_n(I)} \cup_{i\in I} \{f_\alpha(x)\}, \cup_{\alpha\in\Lambda_n(I)} \cup_{i\in I} \{f_\alpha(f_i(x))\}\right)$$

$$\overset{(2.2)}{\leq} \sup_{\alpha\in\Lambda_n(I)} \sup_{i\in I} h\left(\{f_\alpha(x)\}, \{f_\alpha(f_i(x))\}\right) \leq \varphi^n\left(\sup_{i\in I} d\left(x, f_i(x)\right)\right).$$

Claim 3.3. $\left(F_{\mathcal{S}}^{n}(B)\right)_{n\in\mathbb{N}}$ is a Cauchy sequence. Moreover,

$$h\left(F_{\mathcal{S}}^{m}(B), F_{\mathcal{S}}^{n}(B)\right) \leq \sum_{k=m}^{n-1} \varphi^k\left(diam\left(B \cup F_{\mathcal{S}}(B)\right)\right) \tag{3.3}$$

for all $B \in P_{cl,b}(X)$ and $m, n \in \mathbb{N}$, $m < n$.

Justification: Let $B \in P_{cl,b}(X)$. For each $n \in \mathbb{N}$, we have

$$h\left(F_{\mathcal{S}}^{n}(B), F_{\mathcal{S}}^{n+1}(B)\right) = h\left(\cup_{x\in B} F_{\mathcal{S}}^{n}(\{x\}), \cup_{x\in B} F_{\mathcal{S}}^{n+1}(\{x\})\right)$$

$$\overset{(2.2)}{\leq} \sup_{x\in B} h\left(F_{\mathcal{S}}^{n}(\{x\}), F_{\mathcal{S}}^{n+1}(\{x\})\right) \overset{(3.2)}{\leq} \sup_{x\in B} \varphi^n\left(\sup_{i\in I} d\left(x, f_i(x)\right)\right)$$

$$\overset{\text{Def 2.6 1)}}{\leq} \varphi^n\left(\sup_{x\in B} \sup_{i\in I} d\left(x, f_i(x)\right)\right) \leq \varphi^n\left(diam\left(B \cup F_{\mathcal{S}}(B)\right)\right).$$

From the triangle inequality we obtain (3.3) and from Definition 2.6 2) we deduce that $\left(F_{\mathcal{S}}^{n}(B)\right)_{n\in\mathbb{N}}$ is Cauchy.

Claim 3.4. $\overline{\mathcal{O}(C)}$ is bounded for every $C \in P_b(X)$.

Justification: It follows from

$$h\left(\overline{\mathcal{O}(C)}, C\right) \overset{(2.1)}{=} h\left(\cup_{n\in\mathbb{N}} F_{\mathcal{S}}^{n}(C), \cup_{n\in\mathbb{N}} C\right) \overset{(2.2)}{\leq} \sup_{n\in\mathbb{N}} h\left(F_{\mathcal{S}}^{n}(C), C\right)$$

$$\overset{(3.3)}{\leq} \sup_{n\in\mathbb{N}} \sum_{k=0}^{n-1} \varphi^k\left(diam\left(C \cup F_{\mathcal{S}}(C)\right)\right) = \sum_{k=0}^{\infty} \varphi^k\left(diam\left(C \cup F_{\mathcal{S}}(C)\right)\right) < \infty.$$

Claim 3.5. For every $B \in P_b(X)$, there exists $A_B \in P_{cl,b}(X)$ such that

$$\lim_{n\to\infty} F_{\mathcal{S}}^{n}(B) = A_B \text{ and } F_{\mathcal{S}}(A_B) = A_B.$$

Justification: We have that $\left(F_{\mathcal{S}}^{n}((B))\right)_{n\in\mathbb{N}}$ is a Cauchy sequence. From Proposition 2.5 and Definition 2.10 iii), there exists $A_B \in P_{cl,b}(X)$ such that $\lim\limits_{n\to\infty} F_{\mathcal{S}}^{n}(B) = A_B$. Definition 2.10 ii) implies that $F_{\mathcal{S}}$ is continuous on $\overline{\mathcal{O}(B)}$. Thus, $F_{\mathcal{S}}(A_B) = A_B$. By passing to limit as $n \to \infty$ in (3.3), we obtain (3.1). □

Remark 3.6. If we consider $B = \{x\}$, then there exists $A_{\{x\}} \in P_{cl,b}(X)$ such that $\lim\limits_{n\to\infty} F_{\mathcal{S}}^{n}(\{x\}) = A_{\{x\}}$. For the sake of simplicity, we will denote $A_{\{x\}}$ by A_x. In this case, for every $m \in \mathbb{N}$, we have

$$h\left(F_{\mathcal{S}}^{m}(\{x\}), A_x\right) \leq \sum_{k\geq m} \varphi^k\left(diam\left(\{x\} \cup F_{\mathcal{S}}(\{x\})\right)\right). \tag{3.4}$$

Proposition 3.7. *Let* $\mathcal{S} = \left((X,d),(f_i)_{i\in I}\right)$ *be a pcIIFS. Then,* $A_B = \overline{\cup_{x\in B} A_x}$ *for every* $B \in P_{cl,b}(X)$.

Proof. Let us consider $B \in P_{cl,b}(X)$. We have

$$h\left(F_{\mathcal{S}}^n(B), \overline{\cup_{x\in B} A_x}\right) \overset{(2.1),(2.2)}{\leq} \sup_{x\in B} h\left(F_{\mathcal{S}}^n(\{x\}), A_x\right)$$

$$\overset{(3.4)}{\leq} \sup_{x\in B} \sum_{k\geq n} \varphi^k\left(diam\left(\{x\}\cup F_{\mathcal{S}}(\{x\})\right)\right) \overset{\text{Def 2.6 1)}}{\leq} \sum_{k\geq n} \varphi^k\left(diam\left(B\cup F_{\mathcal{S}}(B)\right)\right)$$

for all $n \in \mathbb{N}$. Hence,

$$h\left(F_{\mathcal{S}}^n(B), \overline{\cup_{x\in B} A_x}\right) \leq \sum_{k\geq n} \varphi^k\left(diam\left(B\cup F_{\mathcal{S}}(B)\right)\right) \tag{3.5}$$

for all $n \in \mathbb{N}$. We deduce

$$h\left(A_B, \overline{\cup_{x\in B} A_x}\right) \leq h\left(A_B, F_{\mathcal{S}}^n(B)\right) + h\left(F_{\mathcal{S}}^n(B), \overline{\cup_{x\in B} A_x}\right)$$

$$\overset{(3.1),(3.5)}{\leq} 2\cdot \sum_{k\geq n} \varphi^k\left(diam\left(B\cup F_{\mathcal{S}}(B)\right)\right),$$

for all $n \in \mathbb{N}$. By passing to limit as $n \to \infty$ and applying 2) from Definition 2.6, we conclude that $A_B = \overline{\cup_{x\in B} A_x}$. □

The following result shows that $F_{\mathcal{S}}$ is a continuous weakly Picard operator.

Proposition 3.8. *Let* $\mathcal{S} = \left((X,d),(f_i)_{i\in I}\right)$ *be a pcIIFS and* $F : P_{cl,b}(X) \to P_{cl,b}(X)$ *given by* $F(B) = A_B$*, for all* $B \in P_{cl,b}(X)$*. Then,* F *is continuous on* $P_{cl,b}(X)$ *and uniformly continuous on* $P_{cl,b}(B)$*, for all* $B \in P_{cl,b}(X)$.

Proof. Using relation (3.5) and Proposition 3.7, we deduce that $\left(F_{\mathcal{S}}^n\right)_{n\in\mathbb{N}}$ converges uniformly to F on $P_{cl,b}(B)$, for all $B \in P_{cl,b}(X)$. As $F_{\mathcal{S}}$ is continuous on $P_{cl,b}(X)$ and uniformly continuous on $P_{cl,b}(B)$, for all $B \in P_{cl,b}(X)$, it results that F is uniformly continuous on $P_{cl,b}(B)$, for all $B \in P_{cl,b}(X)$. Applying Remark 2.14, we obtain that F is continuous on $P_{cl,b}(X)$. □

Proposition 3.9. *Let* $\mathcal{S} = \left((X,d),(f_i)_{i\in I}\right)$ *be a pcIIFS. Then, for every* $B \in P_b(X)$ *and* $\alpha \in \Lambda(I)$*, the sequence* $\left(\overline{f_{[\alpha]_n}(B)}\right)_n$ *is convergent. If we denote by*

$$a_\alpha(B) = \lim_{n\to\infty} \overline{f_{[\alpha]_n}(B)},$$

then, for all $m \in \mathbb{N}$*, we have*

$$h\left(\overline{f_{[\alpha]_m}(B)}, a_\alpha(B)\right) = h\left(f_{[\alpha]_m}(B), a_\alpha(B)\right) \leq \sum_{k=m}^{\infty} \varphi^k\left(diam\left(\mathcal{O}(B)\right)\right). \tag{3.6}$$

Proof. Let $B \in P_b(X)$. We have

$$h\left(\overline{f_{[\alpha]_n}(B)}, \overline{f_{[\alpha]_{n+1}}(B)}\right) = h\left(\overline{\cup_{x\in B}\left\{f_{[\alpha]_n}(x)\right\}}, \overline{\cup_{x\in B}\left\{f_{[\alpha]_{n+1}}(x)\right\}}\right)$$

$$\overset{(2.1),(2.2)}{\leq} \sup_{x\in B} h\left(\left\{f_{[\alpha]_n}(x)\right\}, \left\{f_{[\alpha]_{n+1}}(x)\right\}\right) = \sup_{x\in B} d\left(f_{[\alpha]_n}(x), f_{[\alpha]_{n+1}}(x)\right)$$

$$\overset{\text{Def 2.12 1)}}{\leq} \sup_{x\in B}\varphi^{n}\left(\sup_{i\in I} d\left(x, f_{i}\left(x\right)\right)\right) \overset{\text{Def 2.6 1)}}{\leq} \varphi^{n}\left(diam\left(\mathcal{O}\left(B\right)\right)\right), \tag{3.7}$$

for every $n \in \mathbb{N}$. Applying (3.7) and Definition 2.6, we have that $\left(\overline{f_{[\alpha]_n}\left(B\right)}\right)_n$ is Cauchy and from Proposition 2.5 we obtain that $\left(\overline{f_{[\alpha]_n}\left(B\right)}\right)_n$ is convergent. Using Definition 2.10, we deduce that there exists $a_\alpha\left(B\right) \in P_b\left(X\right)$ such that

$$\lim_{n\to\infty}\overline{f_{[\alpha]_n}\left(B\right)} = a_\alpha\left(B\right).$$

Let $m, n \in \mathbb{N}$, with $m < n$. We have

$$h\left(\overline{f_{[\alpha]_m}\left(B\right)}, \overline{f_{[\alpha]_n}\left(B\right)}\right) \overset{(3.7)}{\leq} \sum_{k=m}^{n-1}\varphi^{k}\left(diam\left(\mathcal{O}\left(B\right)\right)\right)$$

for every $m, n \in \mathbb{N}$, $m < n$. By passing to limit as $n \to \infty$ and using the fact that $\sum_{k=m}^{\infty}\varphi^{k}\left(diam\left(\mathcal{O}\left(B\right)\right)\right)$ is convergent, we obtain (3.6). □

Remark 3.10. If $B = \{x\}$, then there exists a set $a_\alpha(\{x\})$ such that

$$\lim_{n\to\infty}\overline{f_{[\alpha]_n}\left(\{x\}\right)} = a_\alpha\left(\{x\}\right).$$

Since $f_{[\alpha]_n}\left(\{x\}\right) = \{f_{[\alpha]_n}\left(x\right)\}$ and this set has one element, it follows that $a_\alpha\left(\{x\}\right)$ has one element denoted by $a_\alpha(x)$.

Lemma 3.11. *Let $\mathcal{S} = \left(\left(X, d\right), \left(f_i\right)_{i\in I}\right)$ be a pcIIFS. Then, $a_\alpha\left(B\right) = \overline{\cup_{x\in B}\left\{a_\alpha\left(x\right)\right\}}$ for every $B \in P_b\left(X\right)$ and $\alpha \in \Lambda\left(I\right)$.*

Proof. Let $B \in P_b\left(X\right)$ and $\alpha \in \Lambda\left(I\right)$. We have

$$\begin{aligned} h\left(a_\alpha\left(B\right), \overline{\cup_{x\in B}\left\{a_\alpha\left(x\right)\right\}}\right) &\leq h\left(a_\alpha\left(B\right), \overline{f_{[\alpha]_n}\left(B\right)}\right) + h\left(\overline{f_{[\alpha]_n}\left(B\right)}, \overline{\cup_{x\in B}\left\{a_\alpha\left(x\right)\right\}}\right) \\ &\overset{(3.6),(2.1),(2.2)}{\leq} \sum_{k=n}^{\infty}\varphi^{k}\left(diam\left(\mathcal{O}\left(B\right)\right)\right) + \sup_{x\in B} h\left(\left\{f_{[\alpha]_n}\left(x\right)\right\}, \left\{a_\alpha\left(x\right)\right\}\right) \\ &\overset{(3.6)}{\leq} 2\cdot\sum_{k=n}^{\infty}\varphi^{k}\left(diam\left(\mathcal{O}\left(B\right)\right)\right) \end{aligned}$$

for every $n \in \mathbb{N}$. By passing to limit as $n \to \infty$, we obtain the conclusion. □

Lemma 3.12. *Let $\mathcal{S} = \left(\left(X, d\right), \left(f_i\right)_{i\in I}\right)$ be a pcIIFS. Then, for every $\alpha \in \Lambda\left(I\right)$ and $B \in P_{cl,b}\left(X\right)$, the function a_α is uniformly continuous on B.*

Proof. Let $B \in P_{cl,b}\left(X\right)$ and $\alpha \in \Lambda\left(I\right)$. Since

$$d\left(a_\alpha\left(x\right), f_{[\alpha]_m}\left(x\right)\right) \leq \sum_{k=m}^{\infty}\varphi^{k}\left(diam\left(\mathcal{O}\left(B\right)\right)\right)$$

for every $m \in \mathbb{N}$ and $x \in B$, we have $f_{[\alpha]_m} \overset{u.c.}{\to} a_\alpha$. Applying ii) from Definition 2.10, it results that $f_{[\alpha]_m}$ is uniformly continuous on B for all $m \in \mathbb{N}$. We obtain that the function a_α is uniformly continuous on B. □

Lemma 3.13. *Let $\mathcal{S}=\left((X,d),(f_i)_{i\in I}\right)$ be a pcIIFS. Then, $f_\omega(a_\alpha(B))=a_{\omega\alpha}(B)$ for every $\alpha\in\Lambda(I)$, $\omega\in\Lambda_n(I)$, $n\in\mathbb{N}^*$ and $B\in P_b(X)$.*

Proof. Let $B\in P_b(X)$, $i\in I$ and $\alpha\in\Lambda(I)$. As $\lim_{n\to\infty} f_{[\alpha]_n}(B)=a_\alpha(B)$, applying Definition 2.10 and Proposition 2.4 3), we deduce that $\lim_{n\to\infty} f_i\left(f_{[\alpha]_n}(B)\right)=f_i(a_\alpha(B))$. Uniqueness of the limit assures us that $a_{i\alpha}(B)=f_i(a_\alpha(B))$. By mathematical induction, we have

$$f_\omega(a_\alpha(B))=a_{\omega\alpha}(B) \tag{3.8}$$

for every $\alpha\in\Lambda(I)$, $\omega\in\Lambda_n(I)$, $n\in\mathbb{N}^*$ and $B\in P_b(X)$. □

Lemma 3.14. *Let $\mathcal{S}=\left((X,d),(f_i)_{i\in I}\right)$ be a pcIIFS. Then, $a_\alpha(x)=a_\alpha(y)$ for every $x\in X$, $y\in\overline{\mathcal{O}(x)}$ and $\alpha\in\Lambda(I)$.*

Proof. Let $x\in X$, $y\in\overline{\mathcal{O}(x)}$ and $\alpha\in\Lambda(I)$. As

$$\overline{\mathcal{O}(x)}=\overline{\cup_{n\in\mathbb{N}}F_{\mathcal{S}}^n(\{x\})}=\overline{\cup_{n\in\mathbb{N}}\overline{\cup_{\omega\in\Lambda_n(I)}\{f_\omega(x)\}}}=\overline{\cup_{n\in\mathbb{N}}\cup_{\omega\in\Lambda_n(I)}\{f_\omega(x)\}},$$

we distinguish two cases.

Case 1. There exist $m\in\mathbb{N}^*$ and $\omega\in\Lambda_m(I)$ such that $y=f_\omega(x)$. For all $n\in\mathbb{N}$, we have

$$\begin{aligned} d(a_\alpha(x),a_\alpha(f_\omega(x)))&\le d\left(a_\alpha(x),f_{[\alpha]_n}(x)\right)\\ &+d\left(f_{[\alpha]_n}(x),f_{[\alpha]_n}((f_\omega(x)))\right)+d\left(f_{[\alpha]_n}(f_\omega(x)),a_\alpha(f_\omega(x))\right). \end{aligned} \tag{3.9}$$

Using the triangle inequality and (2.3), we deduce

$$\begin{aligned} d\left(f_{[\alpha]_n}(x),f_{[\alpha]_n}((f_\omega(x)))\right)&\le\sum_{k=0}^{m-1}d\left(f_{[\alpha]_n[\omega]_k}(x),f_{[\alpha]_n[\omega]_{k+1}}(x)\right)\\ &\le\sum_{k=n}^{n+m-1}\varphi^k(diam(\mathcal{O}(x))) \end{aligned} \tag{3.10}$$

for every $n\in\mathbb{N}$. Applying relation (3.6) for $B=\{f_\omega(x)\}$ and $B=\{x\}$, relations (3.9) and (3.10), we have

$$d(a_\alpha(x),a_\alpha(f_\omega(x)))\le\sum_{k=n}^{\infty}\varphi^k(diam(\mathcal{O}(x)))$$

for every $n\in\mathbb{N}$. We conclude that $a_\alpha(x)=a_\alpha(f_\omega(x))$.

Case 2. There exists a sequence $(y_m)_{m\in\mathbb{N}}\subset\cup_{n\in\mathbb{N}}\cup_{\omega\in\Lambda_n(I)}\{f_\omega(x)\}$ such that $\lim_{m\to\infty} d(y_m,y)=0$. For every $m\in\mathbb{N}$, we have

$$d(a_\alpha(y),a_\alpha(x))\le d(a_\alpha(y),a_\alpha(y_m))+d(a_\alpha(y_m),a_\alpha(x)).$$

Using the first case, we have $a_\alpha(y_m)=a_\alpha(x)$ for every $m\in\mathbb{N}$. Hence,

$$d(a_\alpha(y),a_\alpha(x))\le d(a_\alpha(y),a_\alpha(y_m))$$

for every $m\in\mathbb{N}$ and applying Lemma 3.12 we infer that $a_\alpha(y)=a_\alpha(x)$. □

Theorem 3.15. *Let* $\mathcal{S} = \left((X,d), (f_i)_{i\in I}\right)$ *be a pcIIFS. Then,*

$$A_B = \overline{\cup_{\alpha\in\Lambda(I)} a_\alpha (B)} = \overline{\cup_{x\in B} \cup_{\alpha\in\Lambda(I)} \{a_\alpha (x)\}}$$

for every $B \in P_{cl,b}(X)$.

Proof. Let us consider $B \in P_{cl,b}(X)$ and $x \in B$. We have

$$h\left(F_{\mathcal{S}}^n(\{x\}), \overline{\{a_\alpha (x) \mid \alpha \in \Lambda(I)\}}\right) \overset{(2.1),(2.2)}{\leq} \sup_{\alpha\in\Lambda(I)} h\left(\left\{f_{[\alpha]_n}(x)\right\}, \{a_\alpha(x)\}\right)$$

$$= \sup_{\alpha\in\Lambda(I)} d\left(f_{[\alpha]_n}(x), a_\alpha(x)\right) \overset{(3.6)}{\leq} \sum_{k=n}^{\infty} \varphi^k (diam(\mathcal{O}(B))) \tag{3.11}$$

for every $n \in \mathbb{N}$. We deduce

$$h\left(A_x, \overline{\{a_\alpha(x) \mid \alpha\in\Lambda(I)\}}\right) \leq h\left(A_x, F_{\mathcal{S}}^n(\{x\})\right)$$
$$+h\left(F_{\mathcal{S}}^n(\{x\}), \overline{\{a_\alpha(x) \mid \alpha\in\Lambda(I)\}}\right)$$
$$\overset{(3.4),(3.11)}{\leq} \sum_{k=n}^{\infty} \varphi^k (diam(B \cup F_{\mathcal{S}}(B))) + \sum_{k=n}^{\infty} \varphi^k (diam(\mathcal{O}(B)))$$

for every $n \in \mathbb{N}$. By passing to limit as $n \to \infty$, we obtain that

$$A_x = \overline{\{a_\alpha(x) \mid \alpha\in\Lambda(I)\}} \tag{3.12}$$

and

$$A_B \overset{\text{Prop 3.7}}{=} \overline{\cup_{x\in B} A_x} \overset{(3.12)}{=} \overline{\cup_{x\in B}\overline{\{a_\alpha(x) \mid \alpha\in\Lambda(I)\}}} \overset{\text{Lemma 3.11}}{=} \overline{\cup_{\alpha\in\Lambda(I)} a_\alpha(B)}.$$

□

Proposition 3.16. *Let* $\mathcal{S} = \left((X,d), (f_i)_{i\in I}\right)$ *be a pcIIFS. Then,* $A_B = A_x$ *for every* $x \in X$ *and* $B \in P_{cl,b}\left(\overline{\mathcal{O}(x)}\right)$.

Proof. Let $x \in X$ and $B \in P_{cl,b}\left(\overline{\mathcal{O}(x)}\right)$. As $\lim_{n\to\infty} F_{\mathcal{S}}^n(\{x\}) = A_x$, we deduce that $\lim_{n\to\infty} \cup_{k\geq n} F_{\mathcal{S}}^k(\{x\}) = A_x$. But $\cup_{k\geq n} F_{\mathcal{S}}^k(\{x\}) = F_{\mathcal{S}}^n(\mathcal{O}(x))$ and we obtain that $\lim_{n\to\infty} F_{\mathcal{S}}^n\left(\overline{\mathcal{O}(x)}\right) = A_x$. As $B \subset \overline{\mathcal{O}(x)}$ it results that $F_{\mathcal{S}}^n(B) \subset F_{\mathcal{S}}^n\left(\overline{\mathcal{O}(x)}\right)$ for every $n \in \mathbb{N}$. As $\lim_{n\to\infty} F_{\mathcal{S}}^n(B) = A_B$, we infer that $A_B \subset A_x$. Now, let us consider $y \in B$. Using Lemma 3.14 and relation (3.12), it results that $A_x = A_y$. As $y \in B$, we have that $A_y \subset A_B$ and we deduce that $A_x \subset A_B$. The conclusion holds from the two inclusions. □

Proposition 3.17. *Let* $\mathcal{S} = \left((X,d), (f_i)_{i\in I}\right)$ *be a pcIIFS. Then, for every* $x, y \in X$ *such that* $\overline{\mathcal{O}(x)} \cap \overline{\mathcal{O}(y)} \neq \varnothing$, *we have* $A_x = A_y$. *In particular, if* $\overline{\mathcal{O}(x)} \cap \overline{\mathcal{O}(y)} \neq \varnothing$ *for all* $x, y \in X$, *we have that* $F_{\mathcal{S}}$ *is a Picard operator.*

Proof. Let $z \in \overline{\mathcal{O}(x)} \cap \overline{\mathcal{O}(y)}$. It results that $z \in \overline{\mathcal{O}(x)}$. From Proposition 3.16 we have $A_x = A_z$. Similarly, $A_y = A_z$. We conclude that $A_x = A_y$. □

Proposition 3.18. *Let $\mathcal{S} = \left((X,d),(f_i)_{i\in I}\right)$ be a pcIIFS. Then, the sequence $\left(diam\left(A_{[\alpha]_n,x}\right)\right)_{n\in\mathbb{N}}$ is convergent to* 0.

Proof. We are using the notation $C = \{a_\alpha(x) \mid \alpha \in \Lambda(I)\}$. We have

$$diam\left(A_{[\alpha]_n,x}\right) \overset{(3.12)}{=} diam\left(f_{[\alpha]_n}\left(\overline{C}\right)\right) = diam\left(\overline{f_{[\alpha]_n}\left(\overline{C}\right)}\right)$$

$$= diam\left(\overline{f_{[\alpha]_n}(C)}\right) = diam\left(f_{[\alpha]_n}(C)\right) = \sup_{u,v\in C} d\left(f_{[\alpha]_n}(u), f_{[\alpha]_n}(v)\right)$$

for every $n \in \mathbb{N}$. As $u, v \in C$, we deduce that there exist $\beta,\gamma \in \Lambda(I)$ such that $u = a_\beta(x)$ and $v = a_\gamma(x)$. For every $n \in \mathbb{N}$, we obtain

$$d\left(f_{[\alpha]_n}(u), f_{[\alpha]_n}(v)\right) = d\left(f_{[\alpha]_n}(a_\beta(x)), f_{[\alpha]_n}(a_\gamma(x))\right) \overset{(3.8)}{=} d\left(a_{[\alpha]_n\beta}(x), a_{[\alpha]_n\gamma}(x)\right)$$

$$\leq d\left(a_{[\alpha]_n\beta}(x), f_{[\alpha]_n}(x)\right) + d\left(f_{[\alpha]_n}(x), a_{[\alpha]_n\gamma}(x)\right) \overset{(3.6)}{\leq} 2\cdot\sum_{k=n}^{\infty}\varphi^k\left(diam\left(\mathcal{O}(x)\right)\right).$$

By passing to limit as $n\to\infty$, we have $\lim_{n\to\infty} diam\left(A_{[\alpha]_n,x}\right) = 0$. □

Proposition 3.19. *Let $\mathcal{S} = \left((X,d),(f_i)_{i\in I}\right)$ be a pcIIFS. Then*

$$\{a_\alpha(x)\} = \lim_{n\to\infty} A_{[\alpha]_n,x}$$

for every $x \in X$ and $\alpha \in \Lambda(I)$.

Proof. Let $x \in X$ and $\alpha \in \Lambda(I)$. Easily, one can prove that $\overline{A_{[\alpha]_{n+1},x}} \subset \overline{A_{[\alpha]_n,x}}$ for every $n \in \mathbb{N}$. From $\lim_{n\to\infty} diam\left(A_{[\alpha]_n,x}\right) = 0$, it results that there exists an element $c_\alpha(x) \in X$ such that $\cap_{n\geq 1}\overline{A_{[\alpha]_n,x}} = \{c_\alpha(x)\}$. Thus, $\lim_{n\to\infty}\overline{f_{[\alpha]_n}(A_x)} = \{c_\alpha(x)\}$. We prove that $c_\alpha(x) = a_\alpha(x)$. We consider $\alpha = \alpha_1\alpha_2...\alpha_n\alpha_{n+1}...$ and $\omega_n = \alpha_{n+1}\alpha_{n+2}.... \in \Lambda(I)$ for every $n \in \mathbb{N}^*$. As $a_{\omega_n}(x) \in \{a_\alpha(x) \mid \alpha \in \Lambda(I)\}$ for every $n \in \mathbb{N}^*$, applying relation (3.12) we deduce $f_{[\alpha]_n}(a_{\omega_n}(x)) \in A_{[\alpha]_n,x}$ for every $n \in \mathbb{N}^*$. But $f_{[\alpha]_n}(a_{\omega_n}(x)) = a_\alpha(x)$ for every $n \in \mathbb{N}^*$ and we obtain $a_\alpha(x) \in A_{[\alpha]_n,x}$ for all $n \in \mathbb{N}^*$. Therefore, $\{a_\alpha(x)\} \subset \cap_{n\geq 1}\overline{A_{[\alpha]_n,x}} = \{c_\alpha(x)\}$ and in conclusion $c_\alpha(x) = a_\alpha(x)$. □

Theorem 3.20. *Let $\mathcal{S} = \left((X,d),(f_i)_{i\in I}\right)$ be a pcIIFS. Then the function $\Theta : \Lambda^t(I)\times P_{cl,b}(X) \to P_{cl,b}(X)$ defined by*

$$\Theta(\alpha,B) = \begin{cases} a_\alpha(B)\,,\ \text{if } \alpha\in\Lambda(I) \\ f_\alpha(B)\,,\ \text{if } \alpha\in\Lambda^*(I)\setminus\{\lambda\} \\ B,\ \text{if } \alpha=\lambda \end{cases}$$

for all $(\alpha,B) \in \Lambda^t(I)\times P_{cl,b}(X)$ is uniformly continuous on bounded sets. In particular, Θ is continuous.

Proof. For $\alpha \in \Lambda^*(I)$ we make the notation $a_\alpha(B) := f_\alpha(B)$. Let us consider $M \in P_{cl,b}(X)$, B,$C \in P_{cl,b}(M) \subset P_{cl,b}(X)$, α,$\beta \in \Lambda^t(I)$ and $\varepsilon > 0$.

Remarks.

1) If $\alpha \in \Lambda(I)$, then using (3.6), we have that for every $m \in \mathbb{N}$,

$$h\left(\Theta\left(\alpha, B\right), f_{[\alpha]_m}\left(B\right)\right) = h\left(a_\alpha\left(B\right), f_{[\alpha]_m}\left(B\right)\right) \leq \sum_{k=m}^{\infty} \varphi^k\left(diam\left(\mathcal{O}\left(M\right)\right)\right).$$

2) If $\alpha \in \Lambda^*(I)$, then using again (3.6), we have that for every $m \in \mathbb{N}$,

$$h\left(\Theta\left(\alpha, B\right), f_{[\alpha]_m}\left(B\right)\right) = h\left(f_\alpha\left(B\right), f_{[\alpha]_m}\left(B\right)\right)$$

$$\leq \begin{cases} 0 \text{ if } |\alpha| \leq m \\ \displaystyle\sum_{k=m}^{|\alpha|} \varphi^k\left(diam\left(\mathcal{O}\left(M\right)\right)\right) \end{cases} \leq \sum_{k=m}^{\infty} \varphi^k\left(diam\left(\mathcal{O}\left(M\right)\right)\right).$$

For $\alpha, \beta \in \Lambda^t(I)$ and $B, C \in P_{cl,b}(M)$, using the triangle inequality, we obtain

$$h\left(\Theta\left(\alpha, B\right), \Theta\left(\beta, C\right)\right) = h\left(a_\alpha\left(B\right), a_\beta\left(C\right)\right) \leq h\left(a_\alpha\left(B\right), f_{[\alpha]_m}\left(B\right)\right)$$

$$+h\left(f_{[\alpha]_m}\left(B\right), f_{[\beta]_m}\left(B\right)\right) + h\left(f_{[\beta]_m}\left(B\right), f_{[\beta]_m}\left(C\right)\right) + h\left(f_{[\beta]_m}\left(C\right), a_\beta\left(C\right)\right)$$

for every $m \in \mathbb{N}$. Applying the above remarks, we deduce

$$h\left(\Theta\left(\alpha, B\right), \Theta\left(\beta, C\right)\right) \leq \sum_{k=m}^{\infty} \varphi^k\left(diam\left(\mathcal{O}\left(M\right)\right)\right) + h\left(f_{[\alpha]_m}\left(B\right), f_{[\beta]_m}\left(B\right)\right)$$

$$+h\left(f_{[\beta]_m}\left(B\right), f_{[\beta]_m}\left(C\right)\right) + \sum_{k=m}^{\infty} \varphi^k\left(diam\left(\mathcal{O}\left(M\right)\right)\right) \tag{3.13}$$

for every $m \in \mathbb{N}$. We take m_ε such that

$$\sum_{k=m}^{\infty} \varphi^k\left(diam\left(\mathcal{O}\left(M\right)\right)\right) < \frac{\varepsilon}{3} \tag{3.14}$$

for every $m \geq m_\varepsilon$. Let us fix $m \geq m_\varepsilon$. If $d_c(\alpha, \beta) < c^m$, we have $[\alpha]_m = [\beta]_m$ which implies

$$h\left(f_{[\alpha]_m}\left(B\right), f_{[\beta]_m}\left(B\right)\right) = 0. \tag{3.15}$$

As M is bounded, the function $f_{[\alpha]_m}$ is uniformly continuous on M. So, there exists $\delta_\varepsilon > 0$ such that for all $x, y \in M$ with $d(x, y) < \delta_\varepsilon$ we have

$$d(f_{[\beta]_m}(x), f_{[\beta]_m}(y)) < \frac{\varepsilon}{3}.$$

Hence, for $h(B, C) < \delta_\varepsilon$ we deduce that

$$h(f_{[\beta]_m}(B), f_{[\beta]_m}(C)) \leq \frac{\varepsilon}{3}. \tag{3.16}$$

Therefore, if $d_c(\alpha, \beta) < c^m$ and $h(B, C) < \delta_\varepsilon$, using relations (3.13), (3.14), (3.15) and (3.16), it results

$$h\left(\Theta\left(\alpha, B\right), \Theta\left(\beta, C\right)\right) < \frac{\varepsilon}{3} + 0 + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

In conclusion, Θ is uniformly continuous on M. □

Corollary 3.21. *Let $\mathcal{S} = \left((X,d), (f_i)_{i\in I}\right)$ be a pcIIFS. We consider the functions $g : X \to P_{cl,b}(X)$ defined by $g(x) = \{x\}$ for every $x \in X$ and $h : g(X) \to X$ defined by $h(\{x\}) = x$. We now consider a function $\pi : \Lambda(I) \times X \to P_{cl,b}(X)$ given by $\pi = h \circ \Theta \circ \left(Id_{\Lambda(I)} \times g\right)$. This function is uniformly continuous on bounded sets.*

Remark 3.22. 1) We note that

$$\pi(\alpha, x) = h \circ \Theta \circ \left(Id_{\Lambda(I)} \times g\right)(\alpha, x) = h \circ \Theta(\alpha, \{x\}) = h(\{a_\alpha(x)\}) = a_\alpha(x).$$

2) If S has only one attractor, then π is independent of x and it represents the canonical projection for a classical IIFS.

Corollary 3.23. *Let $\mathcal{S} = \left((X,d), (f_i)_{i\in I}\right)$ be a pcIIFS. Let us consider the function $Id_{\Lambda^t(I)} \times F : \Lambda^t(I) \times P_{cl,b}(X) \to \Lambda^t(I) \times P_{cl,b}(X)$ defined by*

$$Id_{\Lambda(I)} \times F(\alpha, B) = (\alpha, A_B)$$

for all $(\alpha, B) \in \Lambda^t(I) \times P_{cl,b}(X)$. We define $\Psi : \Lambda^t(I) \times P_{cl,b}(X) \to P_{cl,b}(X)$ given by $\Psi = \Theta \circ Id_{\Lambda(I)} \times F$. Then Ψ is continuous.

Remark 3.24. We note that for all $(\alpha, B) \in \Lambda^t(I) \times P_{cl,b}(X)$,

$$\Psi(\alpha, B) = \left(\Theta \circ Id_{\Lambda(I)} \times F\right)(\alpha, B) = \Theta(\alpha, A_B)$$
$$= \begin{cases} a_\alpha(A_B) \text{ if } \alpha \in \Lambda(I) \\ f_\alpha(A_B) \text{ if } \alpha \in \Lambda^*(I) \setminus \{\lambda\} \\ A_B \text{ if } \alpha = \lambda \end{cases}.$$

Corollary 3.25. *Let $\mathcal{S} = \left((X,d), (f_i)_{i\in I}\right)$ be a pcIIFS. We consider the functions $g : X \to P_{cl,b}(X)$ defined by $g(x) = \{x\}$ for every $x \in X$ and $h : g(X) \to X$ defined by $h(\{x\}) = x$. Then, the function $\pi^t : \Lambda^t(I) \times X \to P_{cl,b}(X)$ given by $\pi^t = h \circ \Theta \circ \left(Id_{\Lambda^t(I)} \times g\right)$ is continuous.*

Remark 3.26. We note that for all $(\alpha, x) \in \Lambda^t(I) \times X$,

$$\pi^t(\alpha, x) = h \circ \Theta \circ \left(Id_{\Lambda^t(I)} \times g\right)(\alpha, x) = h \circ \Theta(\alpha, \{x\})$$
$$= \begin{cases} a_\alpha(x), \text{ if } \alpha \in \Lambda(I) \\ f_\alpha(x), \text{ if } \alpha \in \Lambda^*(I) \setminus \{\lambda\} \\ x, \text{ if } \alpha = \lambda \end{cases}.$$

Remark 3.27. Let us consider $\mathcal{S} = \left((X,d), (f_i)_{i\in I}\right)$ a pcIIFS such that the fractal operator associated to $\mathcal{S}$ is a Picard operator. We denote its fixed point by A. For every $i \in I$, we consider the functions $F_i : \Lambda(I) \to \Lambda(I)$ defined by $F_i(\alpha) = i\alpha$, for all $\alpha \in \Lambda^t(I)$ and $F_i^t : \Lambda^t(I) \times X \to \Lambda^t(I) \times X$ given by $F_i^t(\alpha, x) = (i\alpha, x)$ for all $(\alpha, x) \in \Lambda^t(I) \times X$. Then the following diagrams are commutative:

$$\begin{array}{ccc} \Lambda(I) & \xrightarrow{F_i} & \Lambda(I) \\ \downarrow \pi & & \downarrow \pi \\ A & \xrightarrow{f_i} & A \end{array} \qquad \begin{array}{ccc} \Lambda^t(I) \times X & \xrightarrow{F_i^t} & \Lambda^t(I) \times X \\ \downarrow \pi^t & & \downarrow \pi^t \\ X & \xrightarrow{f_i} & X \end{array}$$

This fact reveals that as $S_{\Lambda(I)} = \left(\Lambda(I), (F_i)_{i\in I}\right)$ is a universal model for the pcIIFS S restricted to its fixed point, the system $S_{\Lambda^t(I)} = (\Lambda^t(I) \times X, (F_i^t)_{i\in I})$ is a universal model for the pcIIFS S.

Orbital $\varphi-$ contractive possibly infinite iterated function systems (oIIFSs)

Proposition 3.28. [see [12]] *Let $\mathcal{S} = \left((X,d), (f_i)_{i\in I}\right)$ be an oIIFS. For every $x \in X$, we consider the IIFS denoted by $\mathcal{S}_x = \left(\left(\overline{\mathcal{O}(x)}, d\right), (f_i)_{i\in I}\right)$ and let $F_{\mathcal{S}_x}$ be its fractal operator. Then, $\mathcal{S}_x$ is a system consisting of $\varphi-$contractions and for all $B,C \in P_{cl,b}\left(\overline{\mathcal{O}(x)}\right)$, we have $h(F_{\mathcal{S}_x}(B), F_{\mathcal{S}_x}(C)) \le \varphi(h(B,C))$.*

Remark 3.29. Applying Proposition 3.28, we obtain that $F_{\mathcal{S}_x}$ is a φ-contraction on $P_{cl,b}\left(\overline{\mathcal{O}(x)}\right)$. As (X,d) is a complete metric space, it results that $\left(\overline{\mathcal{O}(x)}, d\right)$ is a complete metric space. From Proposition 2.5 and Theorem 2.9 we deduce that $F_{\mathcal{S}_x}$ has a unique fixed point (which will be denoted by A_x). Moreover, the sequence $\left(F^n_{\mathcal{S}_x}(B)\right)_{n\in\mathbb{N}}$ is convergent to A_x and

$$h\left(F^n_{\mathcal{S}_x}(B), A_x\right) \le \varphi^n(h(B, A_x)) \tag{3.17}$$

for every $n \in \mathbb{N}$ and $B \in P_{cl,b}\left(\overline{\mathcal{O}(x)}\right)$. In particular, for every $x \in X$, $\left(F^n_{\mathcal{S}_x}(\{x\})\right)_{n\in\mathbb{N}}$ is convergent to A_x and it results that $\mathcal{O}(x)$ is bounded.

Remark 3.30. If $\mathcal{S} = \left((X,d), (f_i)_{i\in I}\right)$ is an oIIFS, then $\left(diam\left(\overline{f_{[\alpha]_n}(\mathcal{O}(x))}\right)\right)_{n\in\mathbb{N}}$ is convergent to 0 for every $x \in X$. Hence, there exists an element $a_\alpha(x) \in X$ such that $\lim\limits_{n\to\infty} \overline{f_{[\alpha]_n}(\mathcal{O}(x))} = \{a_\alpha(x)\}$. In this case, for all $m \in \mathbb{N}$, we have

$$h\left(f_{[\alpha]_m}(\mathcal{O}(x)), \{a_\alpha(x)\}\right) \le \varphi^m(diam(\mathcal{O}(x))). \tag{3.18}$$

Lemma 3.31. [see [12]] *Let $\mathcal{S} = \left((X,d), (f_i)_{i\in I}\right)$ be an oIIFS, $x \in X$ and $a_\alpha(x) \in X$ such that $\lim\limits_{n\to\infty} \overline{f_{[\alpha]_n}(\mathcal{O}(x))} = \{a_\alpha(x)\}$. Then, $\lim\limits_{n\to\infty} f_{[\alpha]_n}(B) = \{a_\alpha(x)\}$ for all $\alpha \in \Lambda(I)$ and $B \in P_b\left(\overline{\mathcal{O}(x)}\right)$.*
In particular, if $B = \{y\} \subset \mathcal{O}(x)$, we have $\lim\limits_{n\to\infty} f_{[\alpha]_n}(y) = a_\alpha(x)$ and for every $m \in \mathbb{N}$,

$$d\left(f_{[\alpha]_m}(y), a_\alpha(x)\right) \le \varphi^m(diam(\mathcal{O}(x))). \tag{3.19}$$

Let $B \in P_b(X)$. $\hat{C}_B := \{f : B \to X \mid f \text{ is continuous and bounded}\}$. On $\hat{C}_B$ we consider the metric d_u defined by $d_u(f,g) = \sup\limits_{x\in B} d(f(x), g(x))$. For every $i \in I$, we define the function $\hat{F}_i : \hat{C}_B \to \hat{C}_B$ given by $\hat{F}_i(f) = f_i \circ f$ for every $f \in \hat{C}_B$. The orbit of a function $h \in \hat{C}_B$ is defined by $\mathcal{O}(\{h\}) = \cup_{n\ge 0} \hat{F}^n_{\mathcal{S}}(\{h\})$, where $\hat{F}_{\mathcal{S}}$ is the fractal operator associated to the system $\hat{S} = \left((\hat{C}_B, d_u), (\hat{F}_i)_{i\in I}\right)$.

Proposition 3.32. *Let $\mathcal{S} = \left((X,d), (f_i)_{i\in I}\right)$ be an oIIFS. Then the system*

$$\hat{S} = \left((\hat{C}_B, d_u), (\hat{F}_i)_{i\in I}\right)$$

is an oIIFS.

Proof. Let $h \in \hat{C}_B$ and $g \in \mathcal{O}(\{h\})$.

Claim 3.33. $g(x) \in \mathcal{O}(\{h(x)\})$ for every $g \in \mathcal{O}(\{h\})$ and $x \in B$.

Justification: As $g \in \mathcal{O}(\{h\})$, we have

$$g \in \cup_{n\geq 0} \hat{F}^n_{\mathcal{S}}(\{h\}) = \cup_{n\geq 0} \overline{\cup_{\alpha\in\Lambda_n(I)} \hat{F}_\alpha(\{h\})} = \cup_{n\geq 0} \overline{\cup_{\alpha\in\Lambda_n(I)} \{f_\alpha \circ h\}}.$$

Thus, there exists $n_0 \in \mathbb{N}$ such that $g \in \overline{\cup_{\alpha\in\Lambda_{n_0}(I)} \{f_\alpha \circ h\}}$. We distinguish two cases:

1) $g \in \cup_{\alpha\in\Lambda_{n_0}(I)} \{f_\alpha \circ h\}$. It results that there exists $\alpha \in \Lambda_{n_0}(I)$ such that

$$g = f_\alpha \circ h.$$

Hence, $g(x) = f_\alpha(h(x)) \subset F^{n_0}_S(\{h(x)\}) \in \mathcal{O}(\{h(x)\})$. So, $g(x) \in \mathcal{O}(\{h(x)\})$.

2) $g \notin \cup_{\alpha\in\Lambda_{n_0}(I)} \{f_\alpha \circ h\}$. In this case, there exists a sequence $(g_m)_{m\in\mathbb{N}} \subset \cup_{\alpha\in\Lambda_{n_0}(I)} \{f_\alpha \circ h\}$ such that $\lim\limits_{m\to\infty} d_u(g_m, g) = 0$. We have that for every $m \in \mathbb{N}$, there is $\alpha_m \in \Lambda_{n_0}(I)$ such that $g_m = f_{\alpha_m} \circ h$. We have $g_m(x) = f_{\alpha_m}(h(x)) \in F^{n_0}_S(\{h(x)\})$ and using the fact that $F^{n_0}_S(\{h(x)\})$ is a closed set, we deduce that $g(x) \in F^{n_0}_S(\{h(x)\}) \in \mathcal{O}(\{h(x)\})$. So, $g(x) \in \mathcal{O}(\{h(x)\})$.

Claim 3.34. $\hat{S} = \left((\hat{C}_B, d_u), (\hat{F}_i)_{i\in I}\right)$ is an oIIFS.

Justification: Let us consider $h \in \hat{C}_B$ and $f, g \in \mathcal{O}(\{h\})$. Then,

$$f(x), g(x) \in \mathcal{O}(\{h(x)\})$$

for all $x \in B$. Using this, we have

$$\begin{aligned} d_u\left(\hat{F}_{i_1\ldots i_n}(f), \hat{F}_{i_1\ldots i_n}(g)\right) &= \sup_{x\in B} d\left(f_{i_1\ldots i_n}(f(x)), f_{i_1\ldots i_n}(g(x))\right) \\ &\overset{(2.4)}{\leq} \sup_{x\in B} \varphi^n\left(d(f(x), g(x))\right) = \varphi^n\left(d_u(f,g)\right) \end{aligned}$$

for every $n \in \mathbb{N}$. We deduce that $\hat{S} = \left((\hat{C}_B, d_u), (\hat{F}_i)_{i\in I}\right)$ is an oIIFS. □

Remark 3.35. The set $\mathcal{O}(\{h\})$ is bounded for every $h \in \hat{C}_B$.

Lemma 3.36. *Let $\mathcal{S} = \left((X,d), (f_i)_{i\in I}\right)$ be an oIIFS. Then, $\mathcal{O}(B)$ is bounded for every $B \in P_b(X)$.*

Proof. Let $B \in P_b(X)$ and $\alpha \in \Lambda^*(I)$. We have $\hat{F}_\alpha(Id_B) = f_\alpha \circ Id_B \in \mathcal{O}(\{Id_B\})$. As $\mathcal{O}(\{Id_B\})$ is bounded, we deduce that $\cup_{\alpha\in\Lambda^*(I)} \{f_\alpha \circ Id_B\}$ is bounded, so $\cup_{\alpha\in\Lambda^*(I)} \{f_\alpha \circ Id_B(B)\}$ is bounded. But $\overline{\cup_{\alpha\in\Lambda^*(I)} \{f_\alpha \circ Id_B(B)\}} = \overline{\mathcal{O}(B)}$ and we infer that $\mathcal{O}(B)$ is bounded. □

Theorem 3.37. *Let $\mathcal{S} = \left((X,d), (f_i)_{i\in I}\right)$ be an oIIFS and $F_{\mathcal{S}}$ the associated fractal operator. Then, $F_{\mathcal{S}}$ is a weakly Picard operator. Moreover,*

$$h\left(F^n_{\mathcal{S}}(B), A_B\right) \leq \varphi^n\left(diam\left(\overline{\mathcal{O}(B)}\right)\right) \tag{3.20}$$

for all $n \in \mathbb{N}$ and $B \in P_{cl,b}(X)$, where $A_B = \overline{\cup_{x\in B} A_x}$.

Proof. Let us consider $B \in P_{cl,b}(X)$ and $A_B = \overline{\cup_{x\in B} A_x}$. We have

$$h\left(F_{\mathcal{S}}^{n}\left(\{x\}\right), A_x\right) \overset{(3.17)}{\leq} \varphi^{n}\left(h\left(\{x\}, A_x\right)\right) \leq \varphi^{n}\left(diam\left(\mathcal{O}\left(x\right)\right)\right) \tag{3.21}$$

for every $n \in \mathbb{N}$. We deduce

$$h\left(F_{\mathcal{S}}^{n}\left(B\right), A_B\right) = h\left(\cup_{x\in B} F_{\mathcal{S}}^{n}\left(\{x\}\right), \cup_{x\in B} A_x\right) \overset{(2.2)}{\leq} \sup_{x\in B} h\left(F_{\mathcal{S}}^{n}\left(\{x\}\right), A_x\right)$$

$$\overset{(3.21)}{\leq} \sup_{x\in B} \varphi^{n}\left(diam\left(\overline{\mathcal{O}\left(x\right)}\right)\right) \overset{\text{Def 2.6 1)}}{\leq} \varphi^{n}\left(diam\left(\overline{\mathcal{O}\left(B\right)}\right)\right)$$

for every $n \in \mathbb{N}$. By passing to limit as $n \to \infty$, we have $\lim_{n\to\infty} h\left(F_{\mathcal{S}}^{n}\left(B\right), A_B\right) = 0$. Using ii) from Definition 2.10, we deduce that $F_{\mathcal{S}}$ is uniformly continuous on bounded subsets of X. Hence, $F_{\mathcal{S}}\left(A_B\right) = A_B$. □

Remark 3.38. If $\mathcal{S} = \left((X,d), (f_i)_{i\in I}\right)$ is an oIIFS, then $A_B = A_x$ for every $x \in X$ and $B \in P_{cl,b}\left(\overline{\mathcal{O}\left(x\right)}\right)$.

Proposition 3.39. *Let $\mathcal{S} = \left((X,d), (f_i)_{i\in I}\right)$ be an oIIFS. Then, for all $B \in P_b(X)$ and $\alpha \in \Lambda(I)$, the sequence $\left(f_{[\alpha]_n}(B)\right)_{n\in\mathbb{N}}$ is convergent. If we denote its limit by $a_\alpha(B)$, then for all $m \in \mathbb{N}$, we have*

$$h\left(f_{[\alpha]_m}(B), a_\alpha(B)\right) \leq \varphi^{m}\left(diam\left(\mathcal{O}\left(B\right)\right)\right). \tag{3.22}$$

Proof. Let us consider $B \in P_b(X)$ and $\alpha \in \Lambda(I)$. We use the notation

$$a_\alpha(B) = \overline{\cup_{x\in B}\{a_\alpha(x)\}}.$$

Then, for all $n \in \mathbb{N}$,

$$h\left(f_{[\alpha]_n}(B), a_\alpha(B)\right) = h\left(\cup_{x\in B}\left\{f_{[\alpha]_n}(x)\right\}, \cup_{x\in B}\{a_\alpha(x)\}\right)$$

$$\overset{(2.2)}{\leq} \sup_{x\in B} d\left(f_{[\alpha]_n}(x), a_\alpha(x)\right) \overset{(3.19)}{\leq} \sup_{x\in B} \varphi^{m}\left(diam\left(\mathcal{O}\left(x\right)\right)\right).$$

We deduce that for all $n \in \mathbb{N}$,

$$h\left(f_{[\alpha]_n}(B), a_\alpha(B)\right) \leq \sup_{x\in B} \varphi^{n}\left(diam\left(\mathcal{O}\left(x\right)\right)\right) \overset{\text{Def 2.6 1)}}{\leq} \varphi^{n}\left(diam\left(\mathcal{O}\left(B\right)\right)\right).$$

Thus, we obtain that the sequence $\left(f_{[\alpha]_n}(B)\right)_{n\in\mathbb{N}}$ is convergent to $a_\alpha(B)$. □

Using (3.19), (3.20) and a technique similar with the one used for a pcIIFS, one can prove the following:

Proposition 3.40. *Let $\mathcal{S} = \left((X,d), (f_i)_{i\in I}\right)$ be an oIIFS and $F : P_{cl,b}(X) \to P_{cl,b}(X)$ defined by $F(B) = A_B$ for all $B \in P_{cl,b}(X)$. Then, F is continuous.*

Lemma 3.41. *Let $\mathcal{S} = \left((X,d), (f_i)_{i\in I}\right)$ be an oIIFS. For every $\alpha \in \Lambda(I)$ and $B \in P_{cl,b}(X)$, the function a_α is uniformly continuous on B.*

Lemma 3.42. *Let $\mathcal{S} = \left((X,d), (f_i)_{i\in I}\right)$ be an oIIFS. Then, $f_i\left(a_\alpha(B)\right) = a_{i\alpha}(B)$ for every $B \in P_b(X)$, $\alpha \in \Lambda(I)$ and $i \in I$.*

Theorem 3.43. *Let* $\mathcal{S} = \left((X,d), (f_i)_{i\in I}\right)$ *be an oIIFS. Then,*

$$A_B = \overline{\cup_{\alpha\in\Lambda(I)} a_\alpha(B)} = \overline{\cup_{x\in B}\cup_{\alpha\in\Lambda(I)}\{a_\alpha(x)\}}$$

for every $B \in P_{cl,b}(X)$.

Proof. Using a technique similar with the one used for a pcIIFS, one can prove that

$$h\left(F_{\mathcal{S}}^n(\{x\}), \overline{\{a_\alpha(x) \mid \alpha\in\Lambda(I)\}}\right) \overset{(3.22)}{\leq} \varphi^n(diam(\mathcal{O}(x))) \tag{3.23}$$

and

$$h\left(A_x, \overline{\{a_\alpha(x) \mid \alpha\in\Lambda(I)\}}\right) \leq h(A_x, F_{\mathcal{S}}^n(\{x\}))$$

$$+h\left(F_{\mathcal{S}}^n(\{x\}), \overline{\{a_\alpha(x) \mid \alpha\in\Lambda(I)\}}\right) \overset{(3.21),(3.23)}{\leq} 2\cdot\varphi^n(diam(\mathcal{O}(x)))$$

for every $n \in \mathbb{N}$. Hence,

$$A_x = \overline{\{a_\alpha(x) \mid \alpha\in\Lambda(I)\}} \tag{3.24}$$

and we obtain that

$$A_B = \overline{\cup_{x\in B}\overline{\{a_\alpha(x) \mid \alpha\in\Lambda(I)\}}} = \overline{\cup_{x\in B}\cup_{\alpha\in\Lambda(I)}\{a_\alpha(x)\}}$$

$$= \overline{\cup_{\alpha\in\Lambda(I)}\overline{\{a_\alpha(x) \mid x\in B\}}} = \overline{\cup_{\alpha\in\Lambda(I)} a_\alpha(B)}.$$

□

Using Remark 3.38, relations (3.18), (3.24) and a technique similar with the one used for a pcIIFS, one can prove the following:

Lemma 3.44. *Let* $\mathcal{S} = \left((X,d), (f_i)_{i\in I}\right)$ *be an oIIFS. Then* $a_\alpha(x) = a_\alpha(y)$ *for all* $y \in \overline{\mathcal{O}(x)}$. *Moreover, for every* $x, y \in X$ *such that* $\overline{\mathcal{O}(x)} \cap \overline{\mathcal{O}(y)} \neq \varnothing$, *we have* $A_x = A_y$.

Proposition 3.45. *Let* $\mathcal{S} = \left((X,d), (f_i)_{i\in I}\right)$ *be an oIIFS. Then the sequence* $\left(diam\left(A_{[\alpha]_n,x}\right)\right)_{n\in\mathbb{N}}$ *is convergent to* 0.

Proposition 3.46. *Let* $\mathcal{S} = \left((X,d), (f_i)_{i\in I}\right)$ *be an oIIFS. Then*

$$\{a_\alpha(x)\} = \lim_{n\to\infty} A_{[\alpha]_n,x}$$

for every $x \in X$ *and* $\alpha \in \Lambda(I)$.

Theorem 3.47. *Let* $\mathcal{S} = \left((X,d), (f_i)_{i\in I}\right)$ *be an oIIFS. Then the function* $\Theta : \Lambda^t(I) \times P_{cl,b}(X) \to P_{cl,b}(X)$ *defined by*

$$\Theta(\alpha, B) = \begin{cases} a_\alpha(B),\ if\ \alpha\in\Lambda(I) \\ f_\alpha(B),\ if\ \alpha\in\Lambda^*(I)\setminus\{\lambda\} \\ B,\ if\ \alpha = \lambda \end{cases}$$

for all $(\alpha, B) \in \Lambda^t(I) \times P_{cl,b}(X)$ *is uniformly continuous on bounded sets.*

Proof. Let $B \in P_{cl,b}(X)$. We have

$$h\left(a_{\alpha}(B), f_{[\alpha]_m}(B)\right) \leq \varphi^m\left(diam\left(\mathcal{O}(B)\right)\right)$$

for every $m \in \mathbb{N}$, $(\alpha, B) \in \Lambda(I) \times P_{cl,b}(X)$ and

$$h\left(f_{\alpha}(B), f_{[\alpha]_m}(B)\right) \leq \begin{cases} 0 \text{ if } |\alpha| \leq m \\ \varphi^m\left(diam\left(\mathcal{O}(B)\right)\right) \end{cases} \leq \varphi^m\left(diam\left(\mathcal{O}(B)\right)\right)$$

for every $m \in \mathbb{N}$ and $(\alpha, B) \in \Lambda^*(I) \times P_{cl,b}(X)$. Using these relations and a technique similar with the one used in Theorem 3.20, one can prove that Θ is uniformly continuous. □

Remark 3.48. Corollaries 3.21, 3.23 and 3.25 remain true for an oIIFS.

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