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A SELF-ADAPTIVE INERTIAL ALGORITHM FOR BILEVEL PSEUDO-MONOTONE VARIATIONAL INEQUALITY PROBLEMS WITH NON-LIPSCHITZ MAPPINGS

YINGLIN LUO*, JINGJING FAN** AND SONGXIAO LI***

*School of Mathematics and Statistics,
Huizhou University, Huishou 516007, China
E-mail: luoyinglink@163.com

**School of Applied Sciences,
Taiyuan University of Sciences and Technology, Taiyuan 030024, China
E-mail: fanjingjing0324@163.com

***Department of Mathematics, Shantou University, Shantou 515063, China
E-mail: jyulsx@163.com (Corresponding author)

Abstract. In this paper, a self-adaptive algorithm involving inertial technique will be introduced to solve bilevel pseudo-monotone variational inequality problems in Hilbert spaces. The main advantages of our algorithm is that the strong convergence theorem of the stated iterative method is proved without Lipschitz continuity condition of the associated mapping. Finally, the effectiveness and superiority of the proposed algorithm are proposed by numerical experiments.
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1. Introduction

Let H be a Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and induced norm $\| \cdot \|$, and let C be a closed convex and nonempty subset of H. Suppose that $A : H \to H$ and $F : H \to H$ are two single-valued mappings. In this paper, we investigate the following bilevel inequality variational problem (BVIP) in space H:

$$\text{find } \ p \in VI(C, A) \text{ such that } \ \langle Fp, y - p \rangle \geq 0, \ \ \forall y \in VI(C, A), \tag{BVIP}$$

where $VI(C, A)$ denotes the solution set of the following variational inequality problem (VIP):

$$\text{find } \ x^* \in C \text{ such that } \ \langle Ax^*, z - x^* \rangle \geq 0, \ \ \forall z \in C. \tag{VIP}$$

It is known that the (VIP) is a hot spot in various research fields by its particular advantage in analysis and applications, see [3, 6, 9, 10, 16, 26, 27, 28, 29]. There are many methods have been proposed to solve the (VIP), and one of the most common ways is the gradient projection method:

$$x_{n+1} = P_C(x_n - \lambda A x_n),$$

where $\lambda \in (0, \frac{2\alpha}{L^2})$ and L is the Lipschitz constant of the operator A. One knows that the sequence generated by the gradient projection method converges to some solution of the (VIP) when A is L-Lipschitz continuous and α-strongly monotone. In order to weaken the constraint of the operator A, Korpelevich [15] introduced the extragradient method which bases on calculating two projections onto the feasible set:

$$\begin{cases} y_n = P_C(x_n - \lambda A x_n), \\ x_{n+1} = P_C(x_n - \lambda A y_n), \end{cases}$$

where $\lambda \in (0, \frac{1}{L})$ and A is L-Lipschitz continuous and monotone. Thereafter, this method has been studied and extended in many ways, see [4, 7, 8] and the references therein. Since each iteration needs to compute two projections onto the feasible set, it may seriously affect the efficiency of the extragradient method if the projection onto C is hard to evaluate. To overcome this barrier, Tseng [25] proposed the Tseng's extragradient method, and Censor, Gibali and Reich [5] proposed the subgradient extragradient method. In their algorithms, each iteration only needs to compute one projection on the feasible set. But there is a faultiness in these methods: when the associated mapping in the (VIP) is not Lipschitz continuous or the Lipschitz constant is very difficult to compute, the methods mentioned above are not applicable to implement because the step size cannot be determined. For avoiding the use of Lipschitz continuous condition, Thong, Shehu and Iyiola [22] proposed a strong convergence algorithm for solving the (VIP) with non-Lipschitz continuous of associated mappings. The iterative scheme of their algorithm is devised as follows:

Initialization: Given $\gamma > 0$, $l \in (0,1)$, $\mu \in (0,1)$. Let $x_1 \in H$ be arbitrary.

Step 1. Compute $y_n = P_C(x_n - \lambda_n A x_n)$, where $\lambda_n = \gamma l^{m_n}$ with m_n is the smallest nonnegative integer m satisfying

$$\gamma l^{m_n} \langle A x_n - A y_n, x_n - y_n \rangle \leq \mu \|x_n - y_n\|^2.$$

Step 2. Compute $x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n) P_{C_n}(x_n)$, where

$$C_n := \{x \in H : s_n(x) \leq 0\} \text{ and } s_n(x) = \langle x_n - y_n - \lambda_n (A x_n - A y_n), x - y_n \rangle.$$

Set $n \leftarrow n+1$ and go to **Step 1**.

As a modeling tool, bilevel variational inequality problem plays an important role in many fields, such as economy, signal processing, engineering mechanics, convex programming models and so on, see [21, 24]. Therefore, it is necessary to study some fast and effective iterative methods to solve the (BVIP). In recent years, many iterative methods for solving the (BVIP) and related applications have been constructed, see [1, 12, 17, 20, 23]. In 2020, Thong et al. [23] proposed an extragradient method for solving the (BVIP) in real Hilbert spaces. They declared that their iterative method converges strongly under certain assumptions on parameters. Base on this method, Tan, Liu and Qin [20] introduced an inertial extragradient algorithm for solving the

(BVIP) in real Hilbert spaces. For any two initial values, the iterative scheme of their algorithm is devised as follows:

$$\begin{cases} w_n = x_n + \theta_n(x_n - x_{n+1}), \\ y_n = P_C(w_n - \lambda_n A w_n), \\ z_n = y_n - \lambda_n(A y_n - A w_n), \\ x_{n+1} = z_n - \alpha_n \gamma F z_n, \end{cases}$$

where θ_n and λ_n are updated as

$$\theta_n = \begin{cases} \min\{\frac{\varepsilon_n}{\|x_n - x_{n-1}\|}, \theta\}, & \text{if} \quad x_n \neq x_{n-1}, \\ \theta, & \text{otherwise,} \end{cases}$$

and

$$\lambda_{n+1} = \begin{cases} \min\left\{\frac{\mu(\|w_n - y_n\|)}{\|A w_n - A y_n\|}, \lambda_n\right\}, & \text{if} \quad A w_n - A y_n \neq 0, \\ \lambda_n, & \text{otherwise,} \end{cases}$$

respectively. The sequence $\{x_n\}$ generated by their algorithm also converges strongly. However, we find that all the algorithms mentioned above have a constraint: the operator A is Lipschitz-continuous whether the Lipschitz constant needs to be known or not. Therefore, a natural question is arisen:

How to design a algorithm to solve the (BVIP) **without the Lipschitz continuity of A?**

The purposes of this paper is to answer this question in the affirmative. In order to answer the question, we give a new self adaptive inertial algorithm for solving the (BVIP) without Lipschitz continuity of pseudo-monotone mapping A. More precisely, the contributions of this paper are stated as follows.

(1) We construct a new algorithm for solving the (BVIP) that converges strongly under a weaker condition in infinite-dimensional Hilbert spaces. Moreover, our proposed algorithm uses the inertial technique and a self adaptive Armijo-type linesearch to accelerate the convergence speed.
(2) The associated mapping A in our algorithm is not a monotone and Lipschitz continuous mapping, but a pseudo-monotone and uniformly continuous mapping.
(3) We give several numerical experiments to illustrate the convergence of our proposed algorithm. By comparing our algorithm with the related algorithm in the literature, we find that our algorithm outperforms the comparison algorithm according to the numerical results.

The structure of the paper is stated as follows. In the next section, we give some conclusions and definitions, which will be used in our analysis. Section 3 deals with the convergence analysis of our proposed algorithm. In Section 4, we illustrate the effectiveness of the proposed algorithm and compare our algorithm with previously known algorithm. In the last section, Section 5, a concluding remark is given.

2. Preliminaries

Throughout this paper, we suppose that H is a Hilbert space and C is a nonempty closed convex subset of H. The inner product and norm of H are denoted by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$, respectively. The sequence $\{x_n\}$ converges weakly to x^* is denoted by $x_n \rightharpoonup x^*$, and $\{x_n\}$ converges strongly to x^* is denoted by $x_n \to x^*$. The following are some definitions and lemmas that will be used in our paper.

Definition 2.1. Let $A : H \to H$ be a mapping. For all $x, y \in H$, A is said to be

(1) strongly monotone if there exists $\eta > 0$ such that
$$\langle Ax - Ay, x - y \rangle \geq \eta \|x - y\|^2.$$
(2) monotone if
$$\langle Ax - Ay, x - y \rangle \geq 0.$$
(3) maximal monotone if A is monotone and there exists no monotone mapping B such that $gra(B)$ properly contains $gra(A)$, where $gra(A)$ and $gra(B)$ denote the graph of A and B, respectively.
(4) pseudomonotone if
$$\langle Ax, y - x \rangle \geq 0 \Rightarrow \langle Ay, x - y \rangle \leq 0.$$
(5) L-Lipshhitz continuous if there exists $L > 0$ such that
$$\|Ax - Ay\| \leq L\|x - y\|, \quad \forall x, y \in C.$$

Lemma 2.2. [22] *Let τ be a number in $(0, 1]$ and $\lambda > 0$. Let $F : H \to H$ be a L-Lipschitz and α-strongly monotone mapping with $0 < \alpha \leq L$ and let $S : H \to H$ be a nonexpansive mapping. Define a mapping $S^\lambda : H \to H$ by*
$$S^\lambda x := (I - \tau\lambda F)(Sx), \forall x \in H.$$
Then S^λ is a contraction provided $\lambda < \dfrac{2\alpha}{L^2}$, that is,
$$\|S^\lambda x - S^\lambda y\| \leq (1 - \tau\theta)\|x - y\|, \forall x, y \in H,$$
where $\theta = 1 - \sqrt{1 - \lambda(2\alpha - \lambda L^2)} \in (0, 1)$.

Lemma 2.3. [2] *Given $x, u \in H$ and $u \neq 0$. Let $S = \{y \in H, \langle u, y - x \rangle \leq 0\}$. Then, for all $z \in H$, the projection $P_S(z)$ is defined by*
$$P_S(z) = z - \max\left\{0, \frac{\langle u, z - x \rangle}{\|u\|^2}\right\} u.$$
If $z \notin S$, then we have
$$P_S(z) = z - \frac{\langle u, z - x \rangle}{\|u\|^2} u.$$

Lemma 2.4. [11] *Let f be a real-valued function on H. Let $C := \{x \in H : f(x) \leq 0\}$. If C is nonempty and f is Lipschitz continuous on H with modulus L. Then $Dist(x, C) \geq L^{-1} \max\{f(x), 0\}$, $\forall x \in H$, where $Dist(x, C)$ denotes the distance function from x to C.*

Lemma 2.5. [18] *Let $\{s_n\}$ be a sequence of nonnegative real numbers and let $\{c_n\}$ be a sequence of real numbers. Let $\{\delta_n\}$ be a sequence in $(0,1)$ with $\sum_{n=1}^{\infty} \delta_n = \infty$. Assume that*

$$s_{n+1} \leq (1-\delta_n)s_n + \delta_n c_n, \forall n \geq 1.$$

If $\limsup_{k\to\infty} c_{n_k} \leq 0$ for every subsequence $\{s_{n_k}\}$ of $\{s_n\}$ satisfying $\liminf_{k\to\infty}(s_{n_k+1} - s_{n_k}) \geq 0$, then $\lim_{n\to\infty} s_n = 0$.

3. Main results

In this section, we introduce a new self adaptive inertial algorithm for solving the (BVIP) without Lipschitz continuity of associated mappings. In order to obtain our main results, we assume that the following conditions hold.

Condition 3.1.

(1) The feasible set C is a nonempty closed convex subset of H and the solution set of (VIP) is nonempty, that is, $VI(C,A) \neq \emptyset$.
(2) $A : H \to H$ is pseudo-monotone, uniformly continuous on H and sequentially weakly continuous on C. In finite dimensional spaces, it is sufficient to assume that $A : H \to H$ is continuous pseudo-monotone on H.
(3) $F : H \to H$ is η-strongly monotone and L-Lipschitz continuous on H such that $L \geq \eta$. In addition, we denote x^* the unique solution of the (BVIP).
(4) Let $\{\theta_n\}$ be a positive sequence such that $\lim_{n\to\infty} \frac{\theta_n}{\alpha_n} = 0$, where $\{\alpha_n\} \subset (0,1)$ satisfying $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\lim_{n\to\infty} \alpha_n = 0$.

Based on this formulation, our algorithm is constructed as follows.

Algorithm 1

Initialization: Choose $\delta \geq 0$, $\gamma > 0$, $l \in (0,1)$, $\mu \in (0,1)$ and $0 < \tau < \frac{2\eta}{L^2}$. Let $x_0, x_1 \in H$ be arbitrary.
Iterative Steps: Calculate x_{n+1} as follows:
Step 1. Compute $w_n = x_n + \delta_n(x_n - x_{n-1})$, where

$$\delta_n = \begin{cases} \min\left\{\delta, \dfrac{\theta_n}{\|x_n - x_{n-1}\|}\right\}, & x_n - x_{n-1} \neq 0, \\ \delta, & \textit{otherwise}. \end{cases} \tag{3.1}$$

Step 2. Compute $y_n = P_C(w_n - \lambda_n A w_n)$, where $\lambda_n = \gamma l^{r_n}$ and r_n is the smallest nonnegative integer such that

$$\lambda_n \langle Aw_n - Ay_n, w_n - y_n \rangle \leq \mu \|w_n - y_n\|^2. \tag{3.2}$$

Step 3. Compute $z_n = P_{Q_n}(w_n)$, where $Q_n := \{x \in H : s_n(x) \leq 0\}$ and

$$s_n(x) = \langle w_n - y_n - \lambda_n(Aw_n - Ay_n), x - y_n \rangle. \tag{3.3}$$

Step 4. Compute

$$x_{n+1} = z_n - \alpha_n \tau F z_n.$$

Set $n \leftarrow n+1$ and go to **Step 1**.

Remark 3.2. Since $\lim_{n\to\infty}\frac{\theta_n}{\alpha_n}=0$, from (3.1), it is easy to verify that $\lim_{n\to\infty}\delta_n\|x_n-x_{n-1}\|=0$ and $\lim_{n\to\infty}\frac{\delta_n}{\alpha_n}\|x_n-x_{n-1}\|\leq\lim_{n\to\infty}\frac{\theta_n}{\alpha_n}=0$.

Now we give some lemmas, which are essential to show the convergence analysis of our algorithm.

Lemma 3.3. [22, Lemma 3.1] *Suppose that Condition 3.1 holds. Then Armijo-line search rule* (3.2) *is well defined.*

Lemma 3.4. [22, Lemma 3.2] *Suppose that Condition 3.1 holds and $p\in VI(C,A)$. Let $s_n(x)=\langle w_n-y_n-\lambda_n(Aw_n-Ay_n),w_n-y_n\rangle$. Then*

$$s_n(p)\leq 0 \text{ and } s_n(w_n)\geq(1-\mu)\|w_n-y_n\|^2.$$

In particular, if $w_n\neq y_n$, then $s_n(w_n)>0$.

Remark 3.5. We get from Lemma 3.4 that $w_n\notin Q_n$. Hence, by Lemma 2.3, we have

$$z_n=w_n-\frac{\langle w_n-y_n-\lambda_n(Aw_n-Ay_n),w_n-y_n\rangle}{\|w_n-y_n-\lambda_n(Aw_n-Ay_n)\|^2}(w_n-y_n-\lambda_n(Aw_n-Ay_n)).$$

Lemma 3.6. [22, Lemma 3.3] *Let $\{x_n\}$ be a sequence generated by Algorithm 1. Suppose that Condition 3.1 holds. If there exists a subsequence $\{w_{n_j}\}$ of $\{w_n\}$ such that $\{w_{n_j}\}$ converges weakly to $\tilde{p}\in H$ and $\lim_{j\to\infty}\|w_{n_j}-y_{n_j}\|=0$, then $\tilde{p}\in VI(C,A)$.*

The following theorem states the convergence of the sequence generated by Algorithm 1.

Theorem 3.7. *Suppose that Condition 3.1 holds. Then the sequence $\{x_n\}$ generated by Algorithm 1 converges strongly to the unique solution of $(BVIP)$.*

Proof. The proof is split into three steps.
Step 1. We show that $\{x_n\}$ is bounded. For any $x^*\in VI(C,A)$, we have

$$\begin{aligned}\|z_n-x^*\|^2&=\|P_{Q_n}w_n-x^*\|^2\\&\leq\|w_n-x^*\|^2-\|P_{Q_n}w_n-w_n\|^2\\&=\|w_n-x^*\|^2-d^2(w_n,Q_n),\end{aligned}\tag{3.4}$$

which implies that

$$\|z_n-x^*\|\leq\|w_n-x^*\|.\tag{3.5}$$

By the construction of w_n, we obtain

$$\|w_n-x^*\|\leq\|x_n-x^*\|+\delta_n\|x_n-x_{n-1}\|.\tag{3.6}$$

Substituting (3.6) back into (3.5) gives

$$\|z_n-x^*\|\leq\|x_n-x^*\|+\delta_n\|x_n-x_{n-1}\|.\tag{3.7}$$

According to Remark 3.2, we see that $\lim_{n\to\infty}\frac{\delta_n}{\alpha_n}\|x_n-x_{n-1}\|=0$. Hence, there exists an $M_0>0$ such that $\frac{\delta_n}{\alpha_n}\|x_n-x_{n-1}\|\leq M_0$, $\forall n\geq 1$. So, by (3.7) we get

$$\|z_n-x^*\|\leq\|w_n-x^*\|\leq\|x_n-x^*\|+\alpha_nM_0.\tag{3.8}$$

By Lemma 2.2 and (3.7), we have

$$\begin{aligned}
\|x_{n+1} - x^*\| &= \|(I - \alpha_n \tau F)z_n - (I - \alpha_n \tau F)x^* - \alpha_n \tau F x^*\| \\
&\leq \|(I - \alpha_n \tau F)z_n - (I - \alpha_n \tau F)x^*\| + \|\alpha_n \tau F x^*\| \\
&\leq (1 - \alpha_n \theta)\|z_n - x^*\| + \alpha_n \theta \cdot \frac{\tau}{\theta}\|Fx^*\| \\
&\leq (1 - \alpha_n \theta)\|x_n - x^*\| + \alpha_n \theta \cdot \frac{M_0}{\theta} + \alpha_n \theta \cdot \frac{\tau}{\theta}\|Fx^*\| \\
&\leq \max\{\|x_n - x^*\|, \frac{M_0 + \tau\|Fx^*\|}{\theta}\} \\
&\leq \cdots \leq \max\{\|x_0 - x^*\|, \frac{M_0 + \tau\|Fx^*\|}{\theta}\},
\end{aligned}$$

where $\theta = 1 - \sqrt{1 - \tau(2\eta - \tau L^2)} \in (0, 1)$. This implies that $\{x_n\}$ is bounded. Hence, $\{w_n\}$, $\{Aw_n\}$ and $\{z_n\}$ are also bounded. Since $y_n = P_C(w_n - \lambda_n A w_n)$, we deduce that $\{y_n\}$ and $\{Ay_n\}$ are also bounded.

Step 2. We show that

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 \leq (1 - \alpha_n \theta)\|x_n - x^*\|^2 + \alpha_n \theta [\frac{2\|x_n - x^*\|}{\theta} \cdot \frac{\delta_n \|x_n - x_{n-1}\|}{\alpha_n} \\
+ \frac{\delta}{\theta} \cdot \frac{\delta_n \|x_n - x_{n-1}\|^2}{\alpha_n} + \frac{2\tau}{\theta}\langle Fx^*, x^* - x_{n+1}\rangle].
\end{aligned}$$

Since $\{w_n\}$, $\{Aw_n\}$, $\{y_n\}$ and $\{Ay_n\}$ are bounded and A is uniformly continuous, by the choice of $\{\lambda_n\}$, we see that there exists $M_1 > 0$ such that

$$\|w_n - y_n - \lambda_n(Aw_n - Ay_n)\| \leq M_1, \forall n \geq 1,$$

from which we deduce that

$$\begin{aligned}
\|s_n(x) - s_n(y)\| &= \|\langle w_n - y_n - \lambda_n(Aw_n - Ay_n), x - y\rangle\| \\
&\leq \|w_n - y_n - \lambda_n(Aw_n - Ay_n)\|\|x - y\| \\
&\leq M_1\|x - y\|, \forall x, y \in H.
\end{aligned}$$

Thus $s_n(\cdot)$ is M_1-Lipschitz continuous on H, which together with Lemma 2.4 gives

$$d(w_n, Q_n) \geq \frac{1}{M_1} s_n(w_n).$$

Therefore, one yields from (3.4) that

$$d(w_n, Q_n) \geq \frac{1}{M_1}(1 - \mu)\|w_n - y_n\|^2. \tag{3.9}$$

Combining (3.5) and (3.9), we get

$$\|z_n - x^*\|^2 \leq \|w_n - x^*\|^2 - \left[\frac{1}{M_1}(1 - \mu)\|w_n - y_n\|^2\right]^2. \tag{3.10}$$

By the boundedness of $\{x_n\}$, we see that

$$\begin{aligned}\|x_{n+1}-x^*\|^2 &= \|z_n-\alpha_n F z_n - x^*\|^2\\ &= \|(1-\alpha_n\tau)Fz_n-(1-\alpha_n\tau)Fx^*-\alpha_n\tau Fx^*\|^2\\ &\le \|(1-\alpha_n\tau)Fz_n-(1-\alpha_n\tau)Fx^*\|^2+2\alpha_n\tau\langle Fx^*,x^*-x_{n+1}\rangle \quad (3.11)\\ &\le \|z_n-x^*\|^2+2\alpha_n\tau\langle Fx^*,x^*-x_{n+1}\rangle\\ &\le \|z_n-x^*\|^2+\alpha_n M_3,\end{aligned}$$

for some $M_3>0$. Substituting (3.10) into (3.11) gives

$$\left[\frac{1}{M_1}(1-\mu)\|w_n-y_n\|^2\right]^2 \le \|x_n-x^*\|^2-\|x_{n+1}-x^*\|^2+\alpha_n M_2. \tag{3.12}$$

On the other hand, by (3.8), one yields the inequality

$$\begin{aligned}\|x_{n+1}-x^*\|^2 &\le \|(1-\alpha_n\tau)Fz_n-(1-\alpha_n\tau)Fx^*\|^2+2\alpha_n\tau\langle Fx^*,x^*-x_{n+1}\rangle\\ &\le (1-\alpha_n\theta)^2\|z_n-x^*\|^2+2\alpha_n\tau\langle Fx^*,x^*-x_{n+1}\rangle\\ &\le (1-\alpha_n\theta)\|x_n-x^*\|^2+\alpha_n\theta[\frac{2\|x_n-x^*\|}{\theta}\cdot\frac{\delta_n\|x_n-x_{n-1}\|}{\alpha_n} \quad (3.13)\\ &\quad+\frac{\delta}{\theta}\cdot\frac{\delta_n\|x_n-x_{n-1}\|^2}{\alpha_n}+\frac{2\tau}{\theta}\langle Fx^*,x^*-x_{n+1}\rangle].\end{aligned}$$

Step 3. Now, we show that $\{\|x_n-x^*\|\}$ converges to zero. By Lemma 2.5 and Condition 3.1, we only need to show that $\limsup_{k\to\infty}\langle Fx^*,x^*-x_{n_k+1}\rangle\le 0$ for every subsequence $\{\|x_{n_k}-x^*\|\}$ of $\{\|x_n-x^*\|\}$ satisfying

$$\liminf_{k\to\infty}(\|x_{n_k+1}-x^*\|-\|x_{n_k}-x^*\|)\ge 0.$$

To this end, we assume that $\{\|x_{n_k}-x^*\|\}$ is a subsequence of $\|x_n-x^*\|$ such that $\liminf_{k\to\infty}(\|x_{n_k+1}-x^*\|-\|x_{n_k}-x^*\|)\ge 0$. Then, we can estimate that

$$\begin{aligned}&\liminf_{k\to\infty}(\|x_{n_k+1}-x^*\|^2-\|x_{n_k}-x^*\|^2)\\ &=\liminf_{k\to\infty}\,[(\|x_{n_k+1}-x^*\|+\|x_{n_k}-x^*\|)\times(\|x_{n_k+1}-x^*\|-\|x_{n_k}-x^*\|)]\ge 0.\end{aligned} \tag{3.14}$$

By the inequality (3.12), we obtain

$$\begin{aligned}&\limsup_{k\to\infty}\left[\frac{1}{M_1}(1-\mu)\|w_{n_k}-y_{n_k}\|^2\right]\\ &\quad\le \limsup_{k\to\infty}\left[\|x_{n_k}-x^*\|^2-\|x_{n_k+1}-x^*\|^2+\alpha_{n_k}M_3\right]\le 0.\end{aligned}$$

Thus,

$$\lim_{k\to\infty}\|w_{n_k}-y_{n_k}\|=0. \tag{3.15}$$

On the other hand, we see that

$$\|z_n-x^*\|^2=\|P_{Q_n}(w_n)-x^*\|^2\le\|w_n-x^*\|^2-\|z_n-w_n\|^2. \tag{3.16}$$

Substituting (3.11) into (3.16), it is easy to get

$$\|x_{n+1}-x^*\|^2\le\|w_n-x^*\|^2-\|z_n-w_n\|^2+\alpha_n M_2.$$

That is,

$$\|z_n - w_n\|^2 \le \|w_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 + \alpha_n M_2. \tag{3.17}$$

Hence it follows from (3.6) and Remark 3.2 that

$$\lim_{k\to\infty} \|z_{n_k} - w_{n_k}\| = 0. \tag{3.18}$$

Since

$$\lim_{n\to\infty} \|w_n - x_n\| = \lim_{n\to\infty} \delta_n \|x_n - x_{n-1}\| = 0,$$

from (3.18) we have

$$\lim_{k\to\infty} \|z_{n_k} - x_{n_k}\| \le \lim_{k\to\infty} (\|z_{n_k} - w_{n_k}\| + \|w_{n_k} - x_{n_k}\|) = 0. \tag{3.19}$$

In addition, we see that

$$\lim_{k\to\infty} \|x_{n_k+1} - z_{n_k}\| \le \lim_{k\to\infty} \alpha_{n_k} \tau \|F z_{n_k}\| = 0. \tag{3.20}$$

From (3.19) and (3.20), we get

$$\lim_{k\to\infty} \|x_{n_k+1} - x_{n_k}\| \le \lim_{k\to\infty} (\|x_{x_k+1} - z_{n_k}\| + \|x_{n_k} - z_{n_k}\|) = 0.$$

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_{k_i}}\}$ of $\{x_{n_k}\}$, which converges weakly to some $\tilde{p} \in H$, such that

$$\limsup_{k\to\infty} \langle Fx^*, x^* - x_{n_k}\rangle = \limsup_{i\to\infty} \langle Fx^*, x^* - x_{n_{k_i}}\rangle = \langle Fx^*, x^* - \tilde{p}\rangle. \tag{3.21}$$

According to (3.6), we have that $\{w_{n_{k_i}}\}$ also converges weakly to $\tilde{p}$. According to (3.15) and the fact of Lemma 3.6, we get that $\tilde{p} \in VI(C, A)$. Since x^* is the unique solution of the (BVIP), we obtain

$$\limsup_{k\to\infty} \langle Fx^*, x^* - x_{n_k}\rangle = \langle Fx^*, x^* - \tilde{p}\rangle \le 0. \tag{3.22}$$

Due to (3.21) and (3.22), one gets

$$\begin{aligned} \limsup_{k\to\infty} \langle Fx^*, x^* - x_{n_k+1}\rangle &= \limsup_{k\to\infty} \langle Fx^*, x^* - x_{n_k}\rangle \\ &= \langle Fx^*, x^* - \tilde{p}\rangle \le 0. \end{aligned} \tag{3.23}$$

Then Lemma 2.5 together with (3.13) and (3.23) gives $\lim_{n\to\infty} \|x_n - x^*\| = 0$. □

Let $F(x) = x - \tilde{x}$, where $\tilde{x} \in H$. Then we see that F is 1-Lipschitz continuous and 1-strongly monotone on H. Take $\tau = 1$. We get the following algorithm.

Algorithm 2

Initialization: Choose $\delta \geq 0$, $\gamma > 0$, $l \in (0,1)$ and $\mu \in (0,1)$. Let $x_0, x_1 \in H$ be arbitrary.

Iterative Steps: Calculate x_{n+1} as follows:

$$\begin{cases} w_n = x_n + \delta_n(x_n - x_{n-1}), \\ y_n = P_C(w_n - \lambda_n A w_n), \\ z_n = P_{Q_n}(w_n), \\ x_{n+1} = \alpha_n \tilde{x} + (1-\alpha_n) z_n, \end{cases}$$

where the calculation of $\{\delta_n\}$, $\{\lambda_n\}$ and $\{\alpha_n\}$ is the same as Algorithm 1.

In this situation, we get the following result.

Corollary 3.8. *Suppose that Condition 3.1 holds. Then the sequence $\{x_n\}$ generated by Algorithm 2 converges strongly to $x^* \in VI(C,A)$, where $x^* = P_{VI(C,A)} \circ \tilde{x}$.*

Remark 3.9. Let $\delta = 0$ in Corollary 3.8. Then we get the Corollary 3.7 in [22].

Let $F(x) = x - f(x)$, where $f : H \to H$ is a contraction with constant $k \in [0,1)$. Then we see that F is $(1+k)$-Lipschitz continuous and $(1-k)$-strongly monotone on H. Take $\tau = 1$, we get the following algorithm.

Algorithm 3

Initialization: Choose $\delta \geq 0$, $\gamma > 0$, $l \in (0,1)$ and $\mu \in (0,1)$. Let $x_0, x_1 \in H$ be arbitrary.

Iterative Steps: Calculate x_{n+1} as follows:

$$\begin{cases} w_n = x_n + \delta_n(x_n - x_{n-1}), \\ y_n = P_C(w_n - \lambda_n A w_n), \\ z_n = P_{Q_n}(w_n), \\ x_{n+1} = \alpha_n f(z_n) + (1-\alpha_n) z_n, \end{cases}$$

where the calculation of $\{\delta_n\}$, $\{\lambda_n\}$ and $\{\alpha_n\}$ is the same as Algorithm 1.

Corollary 3.10. *Suppose that Condition 3.1 holds. Then the sequence $\{x_n\}$ generated by Algorithm 3 converges strongly to $x^* \in VI(C,A)$, where $x^* = P_{VI(C,A)} \circ f(x^*)$.*

Remark 3.11. Let $\delta = 0$ in Algorithm 3. Then we get the Algorithm 3 in [22].

4. Numerical examples

In this section, we provide some numerical examples to show the numerical behavior of our proposed algorithms and compare with Algorithm 3.2 in [20]. All the programs were implemented in MATLAB 2018a on a Intel(R) Core(TM) i5-8250U CPU @ 1.60GHz computer with RAM 8.00GB.

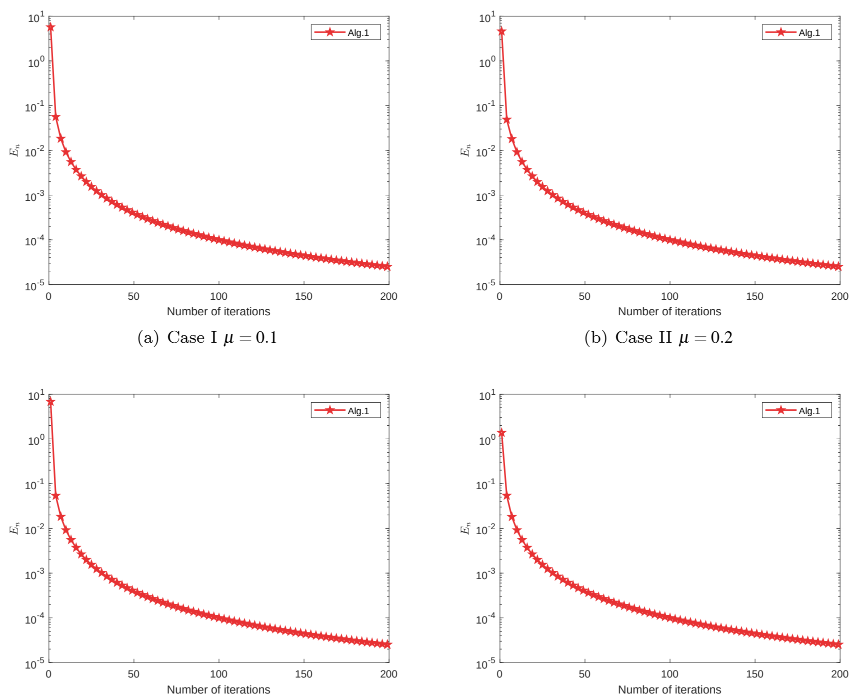
Example 4.1. In this example, the convergence of our algorithms is illustrated by numerical experiments. Set $F(x) = 0.5x$. Then F is 0.5-strongly monotone and 0.5-Lipschitz continuous. The choice of the operator A is as follows:

$$A(x) = \begin{bmatrix} \frac{1}{2}x_1x_2 - 2x_2 - 10^7 \\ -4x_1 + \frac{1}{10}x_2^2 - 10^7 \end{bmatrix},$$

and the feasible set C is defined by

$$C = \{x \in R^2 : (x_1 - 2)^2 + (x_2 - 2)^2 \leq 1\}.$$

It is known that A is pseudo-monotone on C but not monotone, for details, see [13]. In the numerical experiment, the choice of initial values are random and the parameters of the algorithm are as follows. We set $\tau = \frac{1.5\eta}{L^2}$, $\delta = 0.1$, $\alpha_n = \frac{1}{(n+1)}$, and $theta_n = \frac{10}{(n+1)^2}$, $l = 0.2$ and $\gamma = 0.1$. The numerical results are showed in Fig. 1.



(a) Case I $\mu = 0.1$

(b) Case II $\mu = 0.2$

(c) Case III $\mu = 0.6$

(d) Case IV $\mu = 0.8$

FIGURE 1. The behavior of Algorithm 1 under different μ

According to Fig. 1, we find that Algorithm 1 is efficient and easy to implement. In this example, the choice of initial values are random. Hence, we see that the choice of initial value has no significant effect on the convergence of the proposed algorithm. It is worth noting that Algorithm 1 do not need the Lipshchitz continuous condition of the associated mapping. So it is not necessary to know the Lipshchitz constant of the associated mappings in the implementation of numerical experiments.

Example 4.2. In this example, we compare the convergence behavior of our algorithm with the related algorithm in [20]. We consider the following classical fractional programming problem, which has been considered by many authors, see [20, 23].

$$\min f(x) = \frac{x^T N x + a^T x + a_0}{b^T x + b_0},$$
$$\text{subject to } x \in C := \{x \in R^4 : b^T x + b_0 > 0\},$$

where

$$N = \begin{pmatrix} 5 & -1 & 2 & 0 \\ -1 & 5 & -1 & 3 \\ 2 & -1 & 3 & 0 \\ 0 & 3 & 0 & 5 \end{pmatrix}, \quad a = \begin{bmatrix} 1 \\ -2 \\ -2 \\ 1 \end{bmatrix}, \quad b = \begin{bmatrix} 2 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \quad a_0 = -2, \quad b_0 = 4.$$

It can be easily verify that N is symmetric and positive define and f is pseudo-convex on $C = \{x \in R^4 : b^T x + b_0 > 0\}$. Set

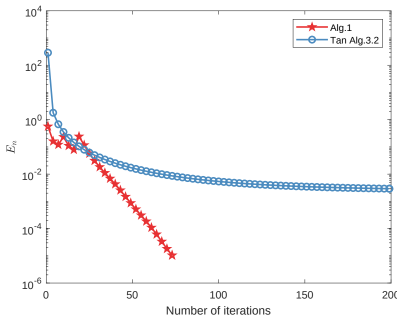
$$Ax := \nabla f(x) = \frac{(b^T x + b_0)(2Nx + a) - b(x^T N x + a^T x + a_0)}{(b^T x + b_0)^2}.$$

It is known that A is pseudo-monotone, for details, see [14, 19]. The operator $F : R^m \to R^m$ $(m = 4)$ is defined as $F(x) = Mx + q_0$, where $M = BB^T + D + H$, B is an $m \times m$ matrix with their entries in $(0, 1)$, D is a $m \times m$ skew-symmetric matrix with their entries in $(-1, 1)$, H is a $m \times m$ diagonal matrix, whose diagonal entries are positive in $(0, 1)$ and $q_0 \in R^m$ is a vector with entries in $(0, 1)$. It is easy to verify that M is positive semidefinite and F is α-monotone and L-Lipschitz continuous with $\alpha = \min\{\text{eig}(M)\}$, $L = \max\{\text{eig}(M)\}$, where $\text{eig}(M)$ represents all eigenvalues of M.

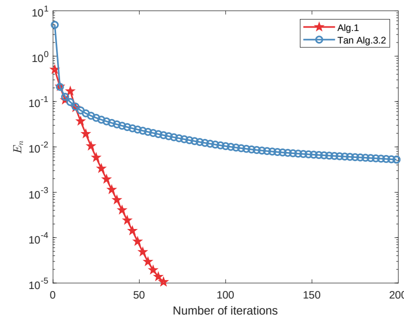
We compare our Algorithm 1 with Algorithm 3.2 in [20], which proposed by Tan, Liu and Qin. Here, we denote their algorithm by Tan Alg.3.2. In the numerical experiment, the choice of initial values are random and the parameters of the two algorithms are as follows. We take $\alpha_n = \frac{1}{(n+1)^2}$ and $\theta_n = \frac{10}{(n+1)^{2.5}}$ in the numerical example, and other parameters are the same as example 4.1. In Tan Alg.3.2, we set $\lambda_1 = 0.3$. We use $E_n = \|x_n - x_{n-1}\|^2$ to denote the error of the n-th iteration of the two algorithms, and the maximal iteration is 200, as the stopping criterion. The numerical results are showed in Fig. 2 and Table 1.

Table 1. Compare the behavior of Algorithm 1 and Tan Alg.3.2 under different μ

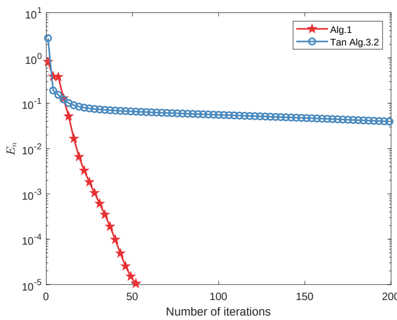
Algorithm	Algorithm 1			Tan Alg.3.2		
	CPU(time)	Iter.	E_n	CPU(time)	Iter.	E_n
$\mu = 0.1$	2.5666	74	10^{-5}	5.0532	200	2.912×10^{-3}
$\mu = 0.2$	2.0581	65	10^{-5}	5.0660	200	5.227×10^{-3}
$\mu = 0.6$	1.5218	53	10^{-5}	5.1411	200	3.9497×10^{-2}
$\mu = 0.8$	3.0745	68	10^{-5}	5.7479	200	1.6384×10^{-2}



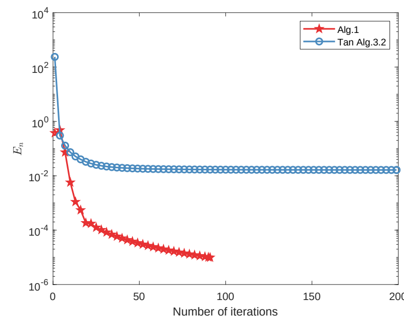
(a) Case I $\mu = 0.1$



(b) Case II $\mu = 0.2$



(c) Case III $\mu = 0.6$



(d) Case IV $\mu = 0.8$

FIGURE 2. Compare the behavior of Algorithm 1 and Tan Alg. 3.2 under different μ

According to Fig. 2 and Table 1, we find that Algorithm 1 performs better than Tan Alg.3.2 both in the number of iterations and CPU time. It is worth noting that Algorithm 1 do not need the Lipshchitz continuous condition of the associated mapping, while Tan Alg.3.2 needs this condition.

5. Conclusions

In this paper, we proposed a new self-adaptive inertial algorithm for finding the solution of the bilevel pseudo-monotone variational inequality problem in real Hilbert spaces. Our approaches can solve the (BVIP) without Lipschitz continuity condition on the associated mappings. The strong convergence theorem of the new algorithm is proved under some suitable conditions. Numerical results were present to demonstrate the performance of our proposed algorithm.

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