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APPROXIMATION OF INVOLUTION IN NEW FUZZY $C^{\star}-$ALGEBRAS: FIXED POINT TECHNIQUE

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Abstract. The ultimate goal of this paper is to investigate the approximation of involution in fuzzy $C^{*}-$ algebra. If A is fuzzy Banach algebra and f is function which defined on it, we prove that under conditions A has unique involution. In this case, fuzzy Banach algebra A can be evolute to Fuzzy $C^{*}-$ algebra. For this purpose, we use the Hyers-Ulam-Rassias stability method and the fixed point technique.

Key Words and Phrases: Stability functional equation, fuzzy normed algebra, fixed point theory, C*-algebra, Hyers-Ulam-Rassias stability.
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1. Introduction

The subject of stability has many applications in various fields such as mathematics, engineering and economics. This issue has been addressed in mathematics in several ways. S. M. Ulam [20] suggested a problem of stability in his own lecture in 1940. He posed, "Suppose that a function satisfies in a functional equation approximately according to some convention. Is it then possible to detect near this function a function satisfying the equation accurately?" Next, Ulam brought up a problem of stability on group homomorphisms in metric groups. D. H. Hyers is the first mathematician who answered the question of Ulam in 1941. He demonstrated the following theorem in [8].

Theorem 1.1. [8] *Let X and Y be two Banach spaces and $f : X \to Y$ be a function so that*

$$\|f(x+y) - f(x) - f(y)\| \le \delta$$

for some $\delta > 0$ *and for every* $x, y \in X$*. Then the limit*

$$A(x) = \lim_{n\to\infty} \frac{f(2^n)x}{2^n}$$

exists for each x *in* X*, moreover* $A(x)$ *is a linear transformation, and the inequality*

$$\|f(x) - A(x)\| \leq \delta$$

is true for every $x \in X$*, furthermore* $A(x)$ *is the only linear transformation satisfying this inequality.*

So far, many mathematicians have developed the Hyers theorem. By changing the control function, space, functional equation, etc., new theorems arise that need to be proven [19]-[17].

Fuzzy norm on vector space was defined in 1984 by Katsaras. Using it, a fuzzy vector topological structure on the space was constructed [10]. Later, some mathematicians from different perspectives defined fuzzy norms in a linear space [6]-[22]. Bag and Samanta [2], following Cheng and Mordeson [4], proposed the idea of a fuzzy norm in such a way that the corresponding fuzzy metric was of the Kramosil and Michalek type [11]. They also prove the theorem of the fuzzy norm decomposition in a family of crisp norms and examined some properties of fuzzy norm spaces.
Recently, considerable attention has been paid to the problem of fuzzy stability of functional equations. Several fuzzy stability results have been investigated in relation to the Cauchy, Jensen, quadratic and cubic functional equations. [14]-[13].
In this research work, we prove that for a function which is approximately Cauchy Jensen in fuzzy Banach $\star$-algebra, there is a **unique involution** near it. Next, we show that under what conditions the involution will be continuous, the fuzzy $\star$-algebra becomes fuzzy C^*-algebra and the fuzzy Banach $\star$-algebra will be self-adjoint.

Definition 1.1. [2] Let X be a real vector space. A function $N : X \times \mathbb{R} \to [0, 1]$ is called a fuzzy norm on X if for all $x, y \in X$ and all $s, t \in \mathbb{R}$

- (N_1) $N(x, t) = 0$, for all $t \leq 0$;
- (N_2) $x = 0$ if and only if $N(x, t) = 1$ for all $t > 0$;
- (N_3) $N(cx, t) = N(x, \frac{t}{|c|})$ if $c \neq 0$;
- (N_4) $N(x + y, s + t) \geq \min\{N(x, s), N(y, t)\}$;
- (N_5) $N(x, .)$ is a non-decreasing function on $\mathbb{R}$ and $\lim_{t\to\infty} N(x, t) = 1$;
- (N_6) For $x \neq 0$, $N(x, .)$ is continuous on $\mathbb{R}$.

The pair (X, N) is called a fuzzy normed vector space.

Definition 1.2. Let (X, N) be a fuzzy normed vector space.

1. A sequence $\{x_n\}$ in X is said to be convergent if there exists an $x \in X$ such that $\lim_{n\to\infty} N(x_n - x, t) = 1, \quad \forall t > 0$. In this case, x is called the limit of the sequence $\{x_n\}$ and we denote it by $N - \lim_{n\to\infty} x_n = x$.
2. A sequence $\{x_n\}$ in X is called Cauchy if for each $\epsilon > 0$ and each $t > 0$ there exists an $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ and all $p > 0$, we have $N(x_{n+p} - x_n, t) > 1 - \epsilon$.

It is known that every convergent sequence in fuzzy normed space is Cauchy. If each Cauchy sequence is convergent, then the fuzzy norm is said to be complete and the fuzzy normed space is called a fuzzy Banach space. We say that a mapping $f : X \to Y$ between fuzzy normed vector spaces X and Y is continuous at a point $x_0 \in X$ if for each sequence $\{x_n\}$ converging to x_0 in X, then the sequence $\{f(x_n)\}$ converges to $f(x_0)$. If $f : X \to Y$ is continuous at each $x_0 \in X$, then $f : X \to Y$ is said to be continuous on X.

Definition 1.3. [18] Let A be an algebra and (A, N) a fuzzy normed space. The fuzzy normed space (A, N) is called a fuzzy normed algebra if

$$N(aa', st) \geq N(a, s).N(a', t) \qquad \forall a, a' \in A, \quad s, t \in \mathbb{R}^+$$

Complete fuzzy normed algebra is called a fuzzy Banach algebra.
If (A, N) be a fuzzy Banch algebra and $\{a_n\}$, $\{b_n\}$ be convergent sequences in (A, N) such that $N - \lim\limits_{n\to\infty} a_n = a$ and $N - \lim\limits_{n\to\infty} b_n = b$. Then

$$\begin{aligned} N(a_nb_n - ab, 2t) &\geq min\{N((a_n - a)b_n, t), N(a(b_n - b), t)\} \\ &\geq min\{N(a_n - a, t), N(b_n - b, t)\} \end{aligned}$$

for all $t > 0$. Therefore $N - \lim_{n\to\infty} a_nb_n = ab$
Let A be a an algebra over $\mathbb{C}$. An involution on A is a mapping

$$\begin{aligned} \star : &A \to A \\ &a \longmapsto a^\star \end{aligned}$$

such that

(i) $(\alpha a + \beta b)^\star = \bar{\alpha}a^\star + \bar{\beta}b^\star \qquad \forall a, b \in A, \quad \forall \alpha, \beta \in \mathbb{C}$
(ii) $(ab)^\star = b^\star a^\star \qquad \forall a, b \in A$
(iii) $a^{\star\star} = a \qquad \forall a \in A$

1. A complex algebra with an involution is a $\star$-algebra.
2. A $C^\star$-algebras is a (non-zero) Banach algebra with an involution, such that

$$||a^\star a|| = ||a||^2.$$

Definition 1.4. Let A be an $\star$-algebra and (A, N) a fuzzy normed algebra. The fuzzy normed algebra (A, N) is called a fuzzy normed $\star$-algebra if

$$N(a^\star, t) = N(a, t) \qquad \forall a, a' \in A, \quad \forall s, t \in \mathbb{R}^+$$

A complete fuzzy normed $*$-algebra is called a fuzzy Banach $*$-algebra.

Definition 1.5. Let (A, N) be an fuzzy Banach $*$-algebra. The (A, N) is a called a fuzzy C^*-algebra if

$$N(a^\star a, st) = N(a^\star, s)N(a, t) \qquad \forall a \in A, \quad \forall s, t \in \mathbb{R}^+$$

Definition 1.6. Let X be a set. A function $d : X^2 \to [0, \infty]$ is a called a generalized metric on X if and only if d satisfies:

(M_1) $d(x, y) = 0$ if and only if $x = y$;
(M_2) $d(x, y) = d(y, x)$, for all $x, y \in X$;
(M_3) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

We now introduce one of the fundamental results of the fixed point theory.

Theorem 1.2. ([3, 5]) *Let (X, d) be a generalized complete metric space. Assume that $G : X \to X$ is a strictly contractive operator with the Lipschitz constant $L < 1$. If there exists a non-negative integer n_0 such that $d(G^{n_0+1}x, G^{n_0}x) < \infty$ for some $x \in X$, then the following statements are correct:*

(*i*) *The sequence $\{G^n x\}$converges to a fixed point x_0 of G;*

(*ii*) *x_0 is the unique fixed point of G in $Y = \{y \in X \mid d(G^{n_0}x, y) < \infty\}$;*

(*iii*) *If $y \in Y$, then*

$$d(y, x_0) \leq \frac{1}{1-L} d(Gy, y).$$

In this article we assume that n_0 is natural number. We also assume that

$$\mathbb{T}^1 = \{z \in \mathbb{C} : |z| = 1\} \text{ and } \mathbb{T}^1_{\frac{1}{m_0}} := \left\{ e^{i\theta}; 0 \leq \theta \leq \frac{2\pi}{m_0} \right\}.$$

Moreover, we suppose that (A, N) is fuzzy Banach algebra. For a given mapping $f : A \to A$, we define

$$D_{\lambda\gamma} f(a, b) = \bar{\lambda} f\left(\frac{a + \gamma b}{2}\right) + \bar{\lambda} f\left(\frac{a - \gamma b}{2}\right) - f(\lambda a) \qquad \forall a, b \in A \quad \text{and} \quad \forall \lambda, \gamma \in \mathbb{C} \tag{1.1}$$

2. Main results

Theorem 2.1. *Let $\psi : A^2 \to [0, \infty)$ be a function such that there exists an $L < \frac{1}{2}$ with $\psi(a, b) \leq \frac{L}{2}\psi(2a, 2b)$ for all $a, b \in A$. In addition, suppose that $f : A \to A$ be a mapping satisfying*

$$N(D_{\lambda\gamma} f(a, b), t) \geq \frac{t}{t + \psi(a, b)}, \tag{2.1}$$

$$N(f(ab) - f(b)f(a)) \geq \frac{t}{t + \psi(a, b)}, \tag{2.2}$$

$$N - \lim_{k\to\infty} 2^k f(2^{-k}(N - \lim_{k\to\infty} 2^k f(2^{-k}a))) = a, \tag{2.3}$$

for all $a, b \in A$, all $t > 0$ and all $\lambda, \gamma \in \mathbb{T}^1_{\frac{1}{m_0}}$. Then there exists a unique involution $H : A \to A$ such that

$$H(a) := N - \lim_{k\to\infty} 2^k f\left(\frac{a}{2^k}\right)$$

and

$$N(f(a) - H(a), t) \geq \frac{(1 - L)t}{(1 - L)t + \psi(a, 0)} \qquad \forall a \in A \quad \forall t > 0. \tag{2.4}$$

Also, if

$$N([N(f(a), t) - N(a, t)]a, t) \geq \frac{t}{t + \psi(a, 0)}. \qquad \forall a \in A, \quad \forall t > 0, \tag{2.5}$$

then (A, N) is a fuzzy Banach $*$-algebra.
Moreover, if for all $s, t > 0$,

$$N([N(f(a)a, st) - N(f(a), s)N(a, t)]a, t) \geq \frac{t}{t + \psi(a, 0)} \qquad \forall a \in A, \tag{2.6}$$

then (A, N) is a fuzzy C^*-algebra with involution $X^* = H(a)$ for all $a \in A$.

Proof. Putting $\lambda = 1$ and $b = 0$ in (2.1), we get

$$N\left(2f\left(\frac{a}{2}\right) - f(a), t\right) \geq \frac{t}{t + \psi(a, 0)} \tag{2.7}$$

Consider the set $S := \{g : A \to A\}$ and introduce the generalized metric on S

$$d(g, h) = \inf\{\delta \in \mathbb{R}^+ : N(g(a) - h(a), \delta t) \geq \frac{t}{t + \psi(a, 0)}, \forall a \in A, \forall t > 0\},$$

where, as usual, $\inf \phi = +\infty$. It is easy to show that (S, d) is complete (see [12]). Now we define mappings $J : S \to S$ by

$$Jg(a) := 2g\left(\frac{a}{2}\right) \qquad \forall a \in A$$

Let $g, h \in S$ be given such that $d(g, h) \neq +\infty$. Then for some $\epsilon > 0$

$$N(g(a) - h(a), \epsilon t) \geq \frac{t}{t + \psi(a, 0)} \qquad \forall a \in A, \forall t > 0$$

Therefore

$$\begin{aligned} N(Jg(a) - Jh(a), L\epsilon t) &= N(2g\left(\frac{a}{2}\right) - 2h\left(\frac{a}{2}\right), L\epsilon t) = N\left(g\left(\frac{a}{2}\right) - h\left(\frac{a}{2}\right), \frac{L}{2}\epsilon t\right) \\ &\geq \frac{\frac{Lt}{2}}{\frac{Lt}{2} + \psi(\frac{a}{2}, 0)} \geq \frac{\frac{Lt}{2}}{\frac{Lt}{2} + \frac{L}{2}\psi(a, 0)} = \frac{t}{t + \psi(a, 0)} \end{aligned}$$

for all $a \in A$ and all $t > 0$. Therefore, using the definition of d metric, we can conclude that $d(Jg, Jh) \leq L\epsilon$. This means that

$$d(Jg, Jh) \leq Ld(g, h) \qquad \forall g, h \in S$$

It follows from (2.7) that $d(f, Jf) \leq 1$. Now, It follows from Theorem 1.2 (i) that there exists a function $H : A \to A$ which is a fixed point of J i.e.,

$$H\left(\frac{a}{2}\right) = \frac{1}{2}H(a),$$

such that $\lim_{n\to\infty} d(J^n f, H) = 0$. Therefore, it can be concluded that

$$N - \lim_{n\to\infty} 2^n f\left(\frac{a}{2^n}\right) = H(a) \qquad \forall a \in A$$

Since the integer n_0 of theorem 1.2(ii) is 0, then $H \in X^*$, where:

$$X^* = \{g \in S : d(f, g) < \infty\}.$$

By Theorem 1.2(iii) and $d(f, Jf) \leq 1$ we obtain

$$d(f, H) \leq \frac{1}{1 - L}d(f, Jf) \leq \frac{1}{1 - L},$$

i.e, the inequality (2.4) is true for all $a \in A$.
Suppose $\lambda = \gamma = 1$ in (2.1), then for all $a, b \in A$ and $t > 0$, we have

$$N\left(2^k f\left(\frac{a + b}{2^{k+1}}\right) + 2^k f\left(\frac{a - b}{2^{k+1}}\right) - 2^k f\left(\frac{a}{2^k}\right), 2^k t\right) \geq \frac{t}{t + \psi(\frac{a}{2^k}, \frac{b}{2^k})}$$

Thus,

$$N\left(2^k f\left(\frac{a+b}{2^{k+1}}\right)+2^k f\left(\frac{a-b}{2^{k+1}}\right)-2^k f\left(\frac{a}{2^k}\right),t\right)\geq\frac{\frac{t}{2^k}}{\frac{t}{2^k}+\frac{L^k}{2^k}\psi(a,b)},$$

Since

$$\lim_{k\to\infty}\frac{\frac{t}{2^k}}{\frac{t}{2^k}+\frac{L^k}{2^k}\psi(a,b)}=1$$

for all $a,b\in A$, all $t>0$ and

$$N\left(H\left(\frac{a+b}{2}\right)+H\left(\frac{a-b}{2}\right)-H(a),t\right)=1\qquad\forall a,b\in A,\quad\forall t>0.$$

We get for all $a,b\in A$ the equality

$$H\left(\frac{a+b}{2}\right)+H\left(\frac{a-b}{2}\right)-H(a)=0,$$

that shows, the map H is a Cauchy additive.
If $b=0$ in (2.1), then for all $a\in A$, $\lambda\in\mathbb{T}^1_{\frac{1}{m_0}}$ and $t>0$, we have

$$N\left(2^k\bar{\lambda}f\left(\frac{a}{2^{k+1}}\right)+2^k\bar{\lambda}f\left(\frac{a}{2^{k+1}}\right)-2^k f\left(\lambda\frac{a}{2^k}\right),2^k t\right)\geq\frac{t}{t+\psi(\frac{a}{2^k},0)},$$

Thus,

$$N\left(2^k\bar{\lambda}f\left(\frac{a}{2^{k+1}}\right)+2^k\bar{\lambda}f\left(\frac{a}{2^{k+1}}\right)-2^k f\left(\lambda\frac{a}{2^k}\right),t\right)\geq\frac{\frac{t}{2^k}}{\frac{t}{2^k}+\frac{L^k}{2^k}\psi(a,0)},$$

Since

$$\lim_{k\to\infty}\frac{\frac{t}{2^k}}{\frac{t}{2^k}+\frac{L^k}{2^k}\psi(a,0)}=1$$

for all $a\in A$, $t>0$, then

$$N\left(2\bar{\lambda}H\left(\frac{a}{2}\right)-H(\lambda a),t\right)=1\qquad\forall a\in A,\quad\forall\lambda\in\mathbb{T}^1_{\frac{1}{m_0}},\quad\forall t>0$$

It follows by last equation and additivity of H that $H(\lambda a)=\bar{\lambda}H(a)$, for all $a\in A$ and all $\lambda\in\mathbb{T}^1_{\frac{1}{m_0}}$.
Now, we show that H is conjugate linear. We have to show that $H(\alpha a)=\bar{\alpha}H(a)$ for all $\alpha\in\mathbb{C}$, $a\in A$. To this end, let $\alpha\in\mathbb{C}$. If α belongs to $\mathbb{T}^1$, then there exists $\theta\in[0,2\pi]$ such that $\alpha=e^{i\theta}$. We set $\alpha_1=e^{\frac{i\theta}{m_0}}$, thus α_1 belongs to $\mathbb{T}^1_{\frac{1}{m_0}}$ and $H(\alpha a)=H(\alpha_1^{m_0}a)=\bar{\alpha}_1^{m_0}H(a)=\bar{\alpha}H(a)$
If α belong to $n\mathbb{T}^1=\{nz;\quad z\in\mathbb{T}^1\}$ for some $n\in\mathbb{N}$, then by additivity of H, $H(\alpha a)=\bar{\alpha}H(a)$ for all $a\in A$.
Let $t\in(0,\infty)$ then by Archimedes property of $\mathbb{C}$, there exists a positive real number n such that the point $(t,0)$ lies in the interior of circle with centre at origin and radius n. Putting $t_1:=t+\sqrt{n^2-t^2}i, t_2:=t-\sqrt{n^2-t^2}i$. Then we have $t=\frac{t_1+t_2}{2}$ and $t_1,t_2\in n\mathbb{T}^1$. It follows that

$$H(ta)=H\left(\frac{t_1+t_2}{2}a\right)=\frac{\bar{t}_1}{2}H(a)+\frac{\bar{t}_2}{2}H(a)=\bar{t}H(a)=tH(a)\qquad\forall a\in A$$

On the other hand, there exists $\theta \in [0, 2\pi]$ such that $\alpha = |\alpha|e^{i\theta}$. It follows that

$$H(\alpha a) = H(|\alpha|e^{i\theta}a) = |\alpha|e^{-i\theta}H(a) = \bar{\alpha}H(a) \qquad \forall a \in A$$

Hence $H : A \to A$ is conjugate $\mathbb{C}$-linear.
By (2.2),

$$N\left(4^k f\left(\frac{ab}{4^k}\right) - 2^k f\left(\frac{b}{2^k}\right) \cdot 2^k f\left(\frac{a}{2^k}\right), 4^k t\right) \geq \frac{t}{t + \psi(\frac{a}{2^k}, \frac{b}{2^k})} \qquad \forall a, b \in A, \quad \forall t > 0.$$

Since

$$\lim_{k\to\infty} \frac{\frac{t}{4^k}}{\frac{t}{4^k} + \frac{L^k}{2^k}\psi(a, b)} = 1$$

for all $a, b \in A$ and all $t > 0$, therefore

$$N(H(ab) - H(b)H(a), t) = 1 \qquad \forall a, b \in A, \quad \forall t > 0$$

Thus, we get for all $a, b \in A$ the equality

$$H(ab) = H(b)H(a)$$

On the other hand by (2.3)

$$H(H(a)) = N - \lim_{k\to\infty} 2^k f(2^{-k}(N - \lim_{k\to\infty} 2^k f(2^{-k}a))) = a$$

for all a in A. Hence $H : A \to A$ is an involution satisfying (2.4).
In addition, we must prove the uniqueness of H. In fact, assume the existence of another such involution H' satisfies (2.4); hence, $H'(\frac{a}{2^k}) = \frac{1}{2^k}H'(a)$, according to (2.4),

$$N\left(2^k f\left(\frac{a}{2^k}\right) - 2^k H'\left(\frac{a}{2^k}\right), t\right) = N\left(f\left(\frac{a}{2^k}\right) - H'\left(\frac{a}{2^k}\right), \frac{t}{2^k}\right)$$
$$\geq \frac{(1-L)\frac{t}{2^k}}{(1-L)\frac{t}{2^k} + \psi(\frac{a}{2^K}, 0)}$$

for all $a \in A$ and $t > 0$. Thus

$$N\left(2^k f\left(\frac{a}{2^k}\right) - 2^k H'\left(\frac{a}{2^k}\right), t\right) \geq \frac{(1-L)\frac{t}{2^k}}{(1-L)\frac{t}{2^k} + \frac{L^K}{2^k}\psi(a, 0)} \qquad \forall a \in A, \quad \forall t > 0$$

Since

$$\lim_{k\to\infty} \frac{(1-L)\frac{t}{2^k}}{(1-L)\frac{t}{2^k} + \frac{L^k}{2^k}\psi(a, 0)} = 1$$

for all $a \in A$, all $t > 0$, then

$$N(H(a) - H'(a), t) = 1 \qquad \forall a \in A, \quad \forall t > 0$$

Therefore, $H(a) = H'(a)$ for all $a \in A$.
Now, suppose H satisfies (2.5), then we have

$$N\left(\left[N\left(2^k f\left(\frac{a}{2^k}\right), 2^k t\right) - N(a, 2^k t)\right] a, 2^k t\right) \geq \frac{t}{t + \psi(\frac{a}{2^k}, 0)} \qquad \forall a \in A, \quad \forall t > 0$$

Thus

$$N\left(\left[N\left(2^k f\left(\frac{a}{2^k}\right)t\right) - N(a,t)\right]a, t\right) \geq \frac{\frac{t}{2^k}}{\frac{t}{2^k} + \frac{L^k}{2^k}\psi(a,0)} \qquad \forall a \in A, \quad \forall t > 0$$

Since

$$\lim_{k\to\infty} \frac{\frac{t}{2^k}}{\frac{t}{2^k} + \frac{L^k}{2^k}\psi(a,0)} = 1$$

for all $a \in A$ and all $t > 0$, therefore

$$\begin{aligned} & N([N(H(a),t) - N(a,t)]a,t) = 1 \qquad \forall a \in A, \quad t > 0 \\ \Longrightarrow \quad & [N(H(a),t) - N(a,t)]a = 0 \qquad \forall a \in A, \quad t > 0 \\ \Longrightarrow \quad & N(H(a),t) - N(a,t) = 0 (or\ N(H(a),t) = N(a,t)), \qquad \forall a \in A, \quad t > 0 \end{aligned}$$

Therefore, (A, N) is a fuzzy Banach $*$-algebra.
Finally, we assume that H satisfies (2.6). Then we have

$$N\left(\left[N\left(2^k f\left(\frac{a}{2^k}\right)a, 2^{2k}st\right) - N\left(2^k f\left(\frac{a}{2^k}\right), 2^k s\right)N(a, 2^k t)\right]a, 2^k t\right) \geq \frac{t}{t + \psi(\frac{a}{2^k}, 0)},$$

for all $a \in A$ and $t, s > 0$. Thus

$$N\left(\left[N(2^k f\left(\frac{a}{2^k}\right)x, st) - N\left(2^k f\left(\frac{a}{2^k}\right), s\right)N(a,t)\right]a, t\right) \geq \frac{\frac{t}{2^k}}{\frac{t}{2^k} + \frac{L^k}{2^k}\psi(\frac{a}{2^k}, 0)},$$

for all $a \in A$ and $s, t > 0$. Again similar to the above can be concluded

$$\begin{aligned} & N([N(H(a)a,st) - N(H(a),s)N(a,t)]a,t) = 1 \qquad \forall a \in A, \quad t > 0 \\ \Longrightarrow \quad & [N(H(a)a,st) - N(H(a),s)N(a,t)]a = 0 \qquad \forall a \in A, \quad t > 0 \\ \Longrightarrow \quad & N(H(a)a,st) - N(H(a),s)N(a,t) = 0 (or N(H(a)a,st) = N(H(a),s)N(a,t)). \end{aligned}$$

Then A is a C^*-algebra with involution $a^* = H(a)$, for all $a \in A$.

Theorem 2.2. *Let $\psi : A^2 \to [0, \infty)$ be a function such that there exists an $L < 1$ with $\psi(a,b) \leq 2L\psi(\frac{a}{2}, \frac{b}{2})$ for all $a, b \in A$. In addition, suppose that $f : A \to A$ be a mapping satisfying*

$$N(D_{\lambda\gamma}f(a,b),t) \geq \frac{t}{t + \psi(a,b)}, \tag{2.8}$$

$$N(f(ab) - f(b)f(a)) \geq \frac{t}{t + \psi(a,b)}, \tag{2.9}$$

$$N - \lim_{k\to\infty} 2^{-k} f(2^k(N - \lim_{k\to\infty} 2^{-k} f(2^k a))) = a, \tag{2.10}$$

for all $a, b \in A$, all $t > 0$ and all $\lambda, \gamma \in \mathbb{T}^1_{\frac{1}{m_0}}$. Then there exists a unique involution $H : A \to A$ such that

$$H(a) := N - \lim_{k\to\infty} \frac{1}{2^k} f(2^k a)$$

and

$$N(f(a) - H(a), t) \geq \frac{(1-L)t}{(1-L)t + L\psi(a,0)} \qquad \forall a \in A \quad \forall t > 0. \tag{2.11}$$

Also, if

$$N([N(f(a),t)-N(a,t)]a,t) \geq \frac{t}{t+\psi(a,0)}. \qquad \forall a \in A, \quad \forall t>0, \tag{2.12}$$

then (A,N) is a fuzzy Banach $*$-algebra.
Moreover, if

$$N([N(f(a)a,st)-N(f(a),s)N(a,t)]a,t) \geq \frac{t}{t+\psi(a,0)} \qquad \forall a \in A, \quad \forall s,t>0, \tag{2.13}$$

then (A,N) is a fuzzy C^*-algebra with involution $X^* = H(a)$ for all $a \in A$.

Let (S,d) be the complete generalized metric space defined in the proof of Theorem 2.1.

Consider the linear mapping $J : S \to S$ by

$$Jg(a) := \frac{1}{2}g(2a) \qquad \forall a \in A$$

It follows from (2.8) that

$$N\left(f(a)-\frac{1}{2}f(2a),\frac{1}{2}t\right) \geq \frac{t}{t+\psi(2a,0)} \geq \frac{t}{t+2L\psi(a,0)} \qquad \forall a \in A \quad t>0,$$

therefore

$$N\left(f(a)-\frac{1}{2}f(2a),Lt\right) \geq \frac{2Lt}{2Lt+2L\psi(a,0)} = \frac{t}{t+\psi(a,0)} \qquad \forall a \in A \quad t>0.$$

So $d(f,Jf) \leq L$ and thus

$$d(f,H) \leq \frac{L}{1-L},$$

which implies that the inequality (2.11) holds.
The rest of the proof is similar to the proof of Theorem 2.1.
We prove the following Hyers-Ulam-Rassias stability problem for involutions on Banach algebras. □

Corollary 2.1. *Let $p \in (0,1)$ and $\theta \in [0,\infty)$ be real numbers. Suppose that $f : A \to A$ with $f(1) = 1$, satisfies satisfying*

$$N(D_{\lambda\gamma}f(a,b),t) \geq \frac{t}{t+\theta(||a||^p+||b||^p)},$$

$$N(f(ab)-f(b)f(a)) \geq \frac{t}{t+\theta(||a||^p+||b||^p)},$$

$$N-\lim_{k\to\infty} 2^{-k}f(2^k(N-\lim_{k\to\infty} 2^{-k}f(2^k a))) = a,$$

for all $a,b \in A$, all $t>0$ and all $\lambda,\gamma \in \mathbb{T}^1_{\frac{1}{m_0}}$. Then there exists a unique involution $H : A \to A$ such that

$$H(a) := N-\lim_{k\to\infty} \frac{1}{2^k}f(2^k a)$$

and

$$N(f(a)-H(a),t) \geq \frac{(1-2^{p-1})t}{(1-2^{p-1})t+2^{p-1}\theta||a||^p} \qquad \forall a \in A \quad \forall t>0.$$

Also, if

$$N([N(f(a),t)-N(a,t)]a,t) \geq \frac{t}{t+\theta||a||^p}. \qquad \forall a \in A, \quad \forall t>0,$$

then (A,N) is a fuzzy Banach $*$-algebra.
Moreover, if

$$N([N(f(a)a,st)-N(f(a),s)N(a,t)]a,t) \geq \frac{t}{t+\theta||a||^p} \qquad \forall a \in A, \quad \forall s,t>0,$$

then (A,N) is a fuzzy C^*-algebra with involution $X^* = H(a)$ for all $a \in A$.

Proof. It follows of Theorem 2.2 by putting $\psi(a,b)=\theta(||a||^p+||b||^p)$ for all $a,b \in A$ and $L=2^{p-1}$. □

Corollary 2.2. *Let $p \in (0,1)$ and $\theta \in [0,\infty)$ be real numbers. Suppose that $f: A \to A$ satisfies satisfying*

$$N(D_{\lambda\gamma}f(a,b),t) \geq \frac{t}{t+\theta(||a||^p||b||^p)},$$
$$N(f(ab)-f(b)f(a)) \geq \frac{t}{t+\theta(||a||^p||b||^p)},$$
$$N-\lim_{k\to\infty} 2^{-k}f(2^k(N-\lim_{k\to\infty} 2^{-k}f(2^k a)))=a,$$

for all $a,b \in A$, all $t>0$ and all $\lambda,\gamma \in \mathbb{T}^1_{\frac{1}{m_0}}$. Then f is an involution on A.
Moreover, if

$$N([N(f(a)a,st)-N(f(a),s)N(a,t)]a,t) \geq \frac{t}{t+\theta||a||^{2p}} \qquad \forall a \in A, \quad \forall s,t>0,$$

Then A is a C^*-algebra with involution f.

Proof. We put $\psi(a,b)=\theta(||a||^p||b||^p)$ for all $a,b \in A$ and $L=2^{2p-1}$ in Theorem 2.2, and then, the result is obtained. □

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