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(α, β)-NONEXPANSIVE MAPPINGS AND PICARD OPERATORS

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Abstract. Let C be a nonempty closed bounded (not necessary convex) subset of a Banach space X and let $T : C \to C$ be an (α, β)-nonexpansive mapping with $\alpha > 0$, $\beta > 0$ and $\alpha + \beta < 1$. In this paper, we show that T has a unique fixed point. Moreover, T is a Picard operator if and only if T is asymptotically regular.
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1. Introduction and Preliminaries

Let C be a nonempty subset of a Banach space $(X, \|.\|)$ and let $T : C \to C$. A point $p \in C$ is called a fixed point for T when $Tp = p$ and the fixed point set of T is denoted by $\mathrm{Fix}(T)$. The mapping T is said to be a *Picard operator*, if $\mathrm{Fix}(T) = \{p\}$ and for each $x \in C$, $T^n x \to p$ as $n \to \infty$, [13, 14]. The sequence (x_n) in C is called an *approximate fixed point sequence* for T provided that $\|x_n - Tx_n\| \to 0$ as $n \to \infty$. T is said to be *asymptotically regular* provided that for every $x \in C$, $\|T^{n+1}x - T^n x\| \to 0$ as $n \to \infty$, [11].
In recent years, several generalizations of nonexpansive mappings have been introduced and their fixed point theory have been studied by many authors; see [1, 3, 5, 6, 7, 8, 9, 10, 12, 11, 15] and the references therein.
Now we recall some definitions and results which will be used throughout the paper. Let l_∞ denote the Banach space of bounded real sequences with the supremum norm. There exists a bounded linear functional μ on l_∞, called *Banach limit*, that satisfies the following conditions:

(i) If $(t_n) \in l_\infty$ with $t_n \geq 0$ for every $n \in \mathbb{N}$, then $\mu(t_n) \geq 0$.
(ii) If $t_n = 1$ for every $n \in \mathbb{N}$, then $\mu(t_n) = 1$.
(iii) For every $(t_n) \in l_\infty$, $\mu(t_n) = \mu(t_{n+1})$.

It is well-known that for every Banach limit μ and every $(t_n) \in l_\infty$,

$$\liminf_{n\to\infty} t_n \leq \mu(t_n) \leq \limsup_{n\to\infty} t_n.$$

In 2010, Aoyama et al. [4] introduced the class of λ-hybrid mappings in Hilbert spaces. This class contains the class of nonexpansive mappings, nonspreading mappings, and hybrid mappings in Hilbert spaces.

Defininition 1.1. [4] Let C be a nonempty subset of a Hilbert space H and let $\lambda \in \mathbb{R}$. A mapping $T : C \to H$ is said to be λ-hybrid if, for each $x, y \in C$,

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + 2(1 - \lambda)\langle x - Tx, y - Ty\rangle.$$

In 2011, the class of α-nonexpansive mappings was introduced by Aoyama and Kohsaka [3] in the setting of Banach spaces.

Defininition 1.2. [3] Let C be a nonempty subset of a Banach space X and let α be a real number such that $\alpha < 1$. A mapping $T : C \to X$ is said to be *α-nonexpansive mapping* if, for each $x, y \in C$,

$$\|Tx - Ty\|^2 \leq \alpha\|y - Tx\|^2 + \alpha\|x - Ty\|^2 + (1 - 2\alpha)\|x - y\|^2.$$

Let H be a Hilbert space, let $C \subseteq H$, let $T : C \to H$ be a mapping and let $\lambda < 2$. Aoyama and Kohsaka [3] showed that T is λ-hybrid if and only if T is an α-nonexpansive mapping, where $\alpha = \frac{1-\lambda}{2-\lambda}$.

2. Main results

In [2], the authors introduced a two parametric class of nonlinear mappings which is properly larger than the class of α-nonexpansive mappings.

Defininition 2.1. [2] Let C be a nonempty subset of a Banach space X and let $\alpha, \beta \in \mathbb{R}$. A mapping $T : C \to X$ is said to be *(α, β)-nonexpansive mapping* if for each $x, y \in C$,

$$\begin{aligned}\|Tx - Ty\|^2 \leq{}& \alpha\|y - Tx\|^2 + \alpha\|x - Ty\|^2 \\ &+ \beta\|x - Tx\|^2 + \beta\|y - Ty\|^2 + (1 - 2\alpha - 2\beta)\|x - y\|^2.\end{aligned}$$

To obtain a characterization of (α, β)-nonexpansive mappings in Hilbert spaces, we now introduce the class of (λ, μ)-hybrid mappings in Hilbert spaces.

Definition 2.2. Let H be a Hilbert space and let C be a nonempty subset of H. Let λ and μ be two real numbers. A mapping $T : C \to H$ is said to be *(λ, μ)-hybrid* if

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + 2(1 - \lambda)\langle x - Tx, y - Ty\rangle + 2(1 - \mu)\langle x - Ty, y - Tx\rangle,$$

for each $x, y \in C$.

Proposition 2.3. *Let C be a nonempty subset of a Hilbert space H and let $T : C \to H$ be a mapping. Let λ and μ be two real numbers such that $\lambda+\mu < 3$ and put $\alpha = \frac{1-\lambda}{3-\lambda-\mu}$ and $\beta = \frac{1-\mu}{3-\lambda-\mu}$. Then T is (λ, μ)-hybrid if and only if T is (α, β)-nonexpansive.*

Proof. Let $x, y \in C$. Then we have

$$\begin{aligned}
&\|x-y\|^2 + 2(1-\lambda)\langle x-Tx, y-Ty\rangle + 2(1-\mu)\langle x-Ty, y-Tx\rangle - \|Tx-Ty\|^2\\
&= \|x-y\|^2 + (1-\lambda)(\|x-Ty\|^2 + \|Tx-y\|^2 - \|x-y\|^2 - \|Tx-Ty\|^2)\\
&+ (1-\mu)(\|x-Tx\|^2 + \|y-Ty\|^2 - \|x-y\|^2 - \|Tx-Ty\|^2) - \|Tx-Ty\|^2\\
&= (1-\lambda)(\|x-Ty\|^2 + \|Tx-y\|^2) + (1-\mu)(\|x-Tx\|^2 + \|y-Ty\|^2)\\
&+ (-1+\lambda+\mu)\|x-y\|^2 - (3-\lambda-\mu)\|Tx-Ty\|^2\\
&= (3-\lambda-\mu)(\alpha(\|x-Ty\|^2 + \|Tx-y\|^2) + \beta(\|x-Tx\|^2 + \|y-Ty\|^2)\\
&+ (1-2\alpha-2\beta)\|x-y\|^2 - \|Tx-Ty\|^2).
\end{aligned}$$

Since $3 > \lambda + \mu$, we get the conclusion. □

The following theorem is the main result of this paper.

Theorem 2.4. *Let C be a nonempty closed bounded (not necessary convex) subset of a Banach space X and let $T : C \to C$ be an (α,β)-nonexpansive mapping with $\alpha > 0$, $\beta > 0$ and $\alpha + \beta < 1$. Then T has a unique fixed point $x^* \in C$. Furthermore, the following statements hold:*

(i) *for each $x \in C$, $(T^n x)$ has a subsequence which is convergent to x^*.*
(ii) *T is a Picard operator if and only if T is an asymptotically regular mapping.*
(iii) *if $\alpha + \beta \le \frac{1}{2}$, then T is a Picard operator.*

Proof. Let $M = diam(C)$ and set $K_n := \{x \in C : \|x - Tx\| \le \frac{1}{n}\}$, for each $n \in \mathbb{N}$. By Theorem 2.1 of [2], $\inf_{x\in C} \|x - Tx\| = 0$, and therefore $K_n \ne \emptyset$, for each $n \in \mathbb{N}$. We first show that $diam(K_n) \to 0$ as $n \to \infty$. Since T is (α,β)-nonexpansive, then for each $x, y \in K_n$ we have

$$\begin{aligned}
\|x-y\|^2 &\le (\|x-Tx\| + \|Tx-Ty\| + \|y-Ty\|)^2\\
&\le (\frac{2}{n} + \|Tx-Ty\|)^2 \le \frac{4}{n^2} + \frac{4M}{n} + \|Tx-Ty\|^2\\
&\le \frac{4}{n^2} + \frac{4M}{n} + \frac{2\beta}{n^2} + \alpha\|y-Tx\|^2\\
&+ \alpha\|x-Ty\|^2 + (1-2\alpha-2\beta)\|x-y\|^2\\
&\le \frac{4+2\beta}{n^2} + \frac{4M}{n} + \alpha(\|x-y\| + \|x-Tx\|)^2\\
&+ \alpha(\|x-y\| + \|y-Ty\|)^2 + (1-2\alpha-2\beta)\|x-y\|^2\\
&\le \frac{4+2(\alpha+\beta)}{n^2} + \frac{4M(1+\alpha)}{n} + (1-2\beta)\|x-y\|^2,
\end{aligned}$$

and so

$$\|x-y\|^2 \le \frac{1}{\beta}\left(\frac{2+\alpha+\beta}{n^2} + \frac{2M(1+\alpha)}{n}\right), \text{ for each } x, y \in K_n. \tag{2.1}$$

From (2.1), we get that $diam(K_n) \to 0$ as $n \to \infty$. Since $K_{n+1} \subseteq K_n$ and $diam(\overline{K_n}) = diam(K_n)$, for each $n \in \mathbb{N}$, then $(\overline{K_n})$ is a decreasing sequence of closed sets and

$diam(\overline{K_n}) \to 0$ as $n \to \infty$. Thus, by the Cantor's intersection theorem, $\bigcap_{n\in\mathbb{N}} \overline{K_n} = \{x^*\}$, for some $x^* \in C$. Hence, for each $n \in \mathbb{N}$, there exists $x_n \in K_n$ such that $\| x_n - x^* \| < \frac{1}{n}$. Then (x_n) in C, is an approximate fixed point sequence for T, and $x_n \to x^*$ as $n \to \infty$. Since T is (α, β)-nonexpansive then for each $n \in \mathbb{N}$, we have

$$\|Tx^* - Tx_n\|^2 \le \alpha\|x_n - Tx^*\|^2 + \alpha\|x^* - Tx_n\|^2 + \beta\|x^* - Tx^*\|^2 + \beta\|x_n - Tx_n\|^2 + (1 - 2\alpha - 2\beta)\|x^* - x_n\|^2. \tag{2.2}$$

Since $x_n \to x^*$ as $n \to \infty$ and $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$, then $Tx_n \to x^*$ as $n \to \infty$, and so we get $\lim_{n\to\infty} \|Tx^* - Tx_n\| = \|Tx^* - x^*\|$ and $\lim_{n\to\infty} \|x_n - Tx^*\| = \|x^* - Tx^*\|$. Thus by taking limit from both sides of (2.2), we get

$$\begin{aligned} \|x^* - Tx^*\|^2 &= \lim_{n\to\infty} \|Tx^* - Tx_n\|^2 \\ &\le \alpha \lim_{n\to\infty} \|x_n - Tx^*\|^2 + \alpha \lim_{n\to\infty} \|x^* - Tx_n\|^2 + \beta\|x^* - Tx^*\|^2 \\ &+ \beta \lim_{n\to\infty} \|x_n - Tx_n\|^2 + (1 - 2\alpha - 2\beta) \lim_{n\to\infty} \|x^* - x_n\|^2 \\ &\le (\alpha + \beta)\|x^* - Tx^*\|^2. \end{aligned} \tag{2.3}$$

Since $\alpha + \beta < 1$, then from (2.3), we obtain $Tx^* = x^*$.
Since

$$\{x^*\} \subseteq Fix(T) \subseteq \bigcap_{n\in\mathbb{N}} \overline{K_n} = \{x^*\},$$

then $Fix(T) = \{x^*\}$, that is, T has a unique fixed point.
Now we prove the statements (i), (ii) and (iii).
(i) Let $x \in C$ and let μ be a Banach limit. By the proof of Theorem 2.1 of [2], we have

$$\mu(\|T^{n+1}x - T^n x\|^2) = 0. \tag{2.4}$$

Also by the definition of Banach limit

$$\mu(\|T^{n+1}x - x^*\|^2) = \mu(\|T^n x - x^*\|^2). \tag{2.5}$$

Since T is (α, β)-nonexpansive and $Tx^* = x^*$, then for each $n \in \mathbb{N}$, we have

$$\|T^{n+1}x - x^*\|^2 \le \alpha\|T^n x - x^*\|^2 + \alpha\|T^{n+1}x - x^*\|^2 + \beta\|T^{n+1}x - T^n x\|^2 + (1 - 2\alpha - 2\beta)\|T^n x - x^*\|^2.$$

Hence

$$\|T^{n+1}x - x^*\|^2 \le \frac{1 - \alpha - 2\beta}{1 - \alpha}\|T^n x - x^*\|^2 + \frac{\beta}{1 - \alpha}\|T^{n+1}x - T^n x\|^2, \tag{2.6}$$

for each $n \in \mathbb{N}$. Thus by (2.4), (2.5) and (2.6), we obtain

$$\mu(\|T^n x - x^*\|^2) = \mu(\|T^{n+1}x - x^*\|^2) \le \frac{1 - \alpha - 2\beta}{1 - \alpha}\mu(\|T^n x - x^*\|^2),$$

which yields (note that $\beta > 0$)

$$\mu(\|T^n x - x^*\|^2) = 0.$$

Then

$$0 \le \liminf_{n\to\infty} \|T^n x - x^*\|^2 \le \mu(\|T^n x - x^*\|^2) = 0.$$

Thus there exists a subsequence of $(T^n x)$ which is convergent to x^*.
(ii) We first assume that T is asymptotically regular and let $x \in C$. To prove the claim, we show that each subsequence $(T^{k_n} x)$ of $(T^n x)$ has itself a subsequence which is convergent to x^*. Let $(T^{k_n} x)$ be a subsequence of $(T^n x)$, then we have (note that T is asymptotically regular)

$$\mu(\|T^{k_n+1}x - T^{k_n}x\|^2) = 0, \tag{2.7}$$

and so

$$\mu(\|T^{k_n+1}x - x^*\|^2) = \mu(\|T^{k_n}x - x^*\|^2). \tag{2.8}$$

From (2.6), we have

$$\|T^{k_n+1}x - x^*\|^2 \leq \frac{1-\alpha-2\beta}{1-\alpha}\|T^{k_n}x - x^*\|^2 + \frac{\beta}{1-\alpha}\|T^{k_n+1}x - T^{k_n}x\|^2, \tag{2.9}$$

for each $n \in \mathbb{N}$. Thus by (2.7), (2.8) and (2.9), we obtain

$$\mu(\|T^{k_n}x - x^*\|^2) = \mu(\|T^{k_n+1}x - x^*\|^2) \leq \frac{1-\alpha-2\beta}{1-\alpha}\mu(\|T^{k_n}x - x^*\|^2),$$

which yields

$$\mu(\|T^{k_n}x - x^*\|^2) = 0.$$

Then

$$0 \leq \liminf_{n\to\infty} \|T^{k_n}x - x^*\|^2 \leq \mu(\|T^{k_n}x - x^*\|^2) = 0.$$

Thus there exists a subsequence of $(T^{k_n}x)$ which is convergent to x^*. Therefore $T^n x \to x^*$ as $n \to \infty$, for each $x \in C$ and so T is a Picard operator.
Conversely, assume that T is a Picard operator, let $Fix(T) = \{x^*\}$ and let $x \in C$. Since $\lim_{n\to\infty} \|T^n x - x^*\| = 0$, then $\lim_{n\to\infty} \|T^{n+1}x - T^n x\| = 0$ and so T is asymptotically regular.
(iii) If we replace μ by the limit superior in the proof of Theorem 2.1 of [2], we obtain (note that $1 - 2\alpha - 2\beta \geq 0$)

$$\limsup_{n\to\infty} \|T^{n+1}x - T^n x\| = 0.$$

Thus $\|T^{n+1}x - T^n x\| \to 0$ as $n \to \infty$ and by (ii), T is a Picard operator. □

Now, the following problem naturally arises.
Problem. Let C be a nonempty closed bounded subset of a Banach space X and let $T : C \to C$ be an (α, β)-nonexpansive mapping with $\alpha > 0$, $\beta > 0$ and $\frac{1}{2} < \alpha + \beta < 1$. Is T a Picard operator? or equivalently, is T asymptotically regular?

From Proposition 2.3 and Theorem 2.4, we derive the following corollary.
Corollary 2.5. *Let C be a nonempty closed bounded subset of a Hilbert space H and let $T : C \to C$ be a (λ, μ)-hybrid mapping with $\lambda < 1$ and $\mu < 1$. Assume that $\lambda + \mu \geq 1$ or T be an asymptotically regular mapping. Then T is a Picard operator.*

To prove our next result, we need the following lemma.

Lemma 2.6. *Let C be a nonempty closed bounded subset of a Banach space X, and let $T : C \to C$ be a mapping. Assume that there exists $N \in \mathbb{N}$ such that T^N is a Picard operator. Then T is also a Picard operator.*

Proof. Let $p \in C$ be the unique fixed point of the Picard operator $T^N : C \to C$. We first show that p is also the unique fixed point of $T : C \to C$. Since $T^N(p) = p$ then $Tp = TT^N(p) = T^N(Tp)$ and so Tp is a fixed point of T^N. Since p is the unique fixed point of T^N, we get that $Tp = p$. To prove that p is the unique fixed point of T, assume that $Tq = q$, for some $q \in C$. Then $q = Tq = T^2q = ... = T^N(q)$ and so q is a fixed point of T^N, which yields $p = q$.
Now, we show that for each $x \in C$, $x_n = T^n x \to p$ as $n \to \infty$. To show the claim, it suffices to prove that $x_{Nn+j} \to p$ as $n \to \infty$, for each $j = 0, 1, ..., N-1$. Since T^N is a Picard operator and $p \in C$ is the unique fixed point of T^N, then

$$(T^N)^n(z) = T^{Nn}(z) \to p \text{ as } n \to \infty, \text{ for each } z \in C.$$

Let $z = T^j x$, then we have

$$x_{Nn+j} = T^{Nn+j}(x) = T^{Nn}(T^j x) \to p \text{ as } n \to \infty.$$

□

The following is a slight improvement of Theorem 2.4.

Theorem 2.7. *Let C be a nonempty closed bounded subset of a Banach space X and let $T : C \to C$ be a mapping. Assume that there exists $N \in \mathbb{N}$ such that T^N is (α, β)-nonexpansive with $\alpha > 0$, $\beta > 0$ and $\alpha + \beta < 1$ and T^N is asymptotically regular. Then T is a Picard operator.*

Proof. By Theorem 2.4, T^N is a Picard operator and by Lemma 2.6, we get that T is also a Picard operator. □

The following example shows that Theorem 2.7 is a real generalization of Theorem 2.4.

Example 2.8. Let $T : [0, 1] \to [0, 1]$ be defined as follows:

$$T(x) = \begin{cases} \frac{3}{4}, & x = 1, \\ \frac{1}{4}, & x = \frac{3}{4}, \\ \frac{7}{8}, & x \notin \{\frac{3}{4}, 1\}. \end{cases}$$

Assume first that T is (α, β)-nonexpansive, for some $\alpha, \beta > 0$, with $\alpha + \beta < 1$. Let $x = \frac{3}{4}$ and $y = \frac{7}{8}$. Then, we have

$$\begin{aligned} \|Tx - Ty\|^2 &\leq \alpha\|y - Tx\|^2 + \alpha\|x - Ty\|^2 \\ &\quad + \beta\|x - Tx\|^2 + \beta\|y - Ty\|^2 + (1 - 2\alpha - 2\beta)\|x - y\|^2, \end{aligned}$$

and so $24 \leq 24\alpha + 14\beta < 24(\alpha + \beta) < 24$, a contradiction. On the other hands

$$T^2(x) = \begin{cases} \frac{1}{4}, & x = 1, \\ \frac{7}{8}, & x \neq 1. \end{cases}$$

Then by Example 2.1 of [2], T^2 is $(\frac{1}{1000}, \frac{8}{9})$-nonexpansive and it is clear that T^2 is an asymptotically regular mapping. Thus, T satisfies the assumptions of Theorem 2.6, but we can not invoke Theorem 2.4 to prove that T is a Picard operator.

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