Fixed Point Theory, 25(2024), No. 1, 143-162
DOI: 10.24193/fpt-ro.2024.1.09
http://www.math.ubbcluj.ro/~nodeacj/sfptcj.html

TWO GENERALIZED STRONG CONVERGENCE ALGORITHMS FOR VARIATIONAL INEQUALITY PROBLEMS IN BANACH SPACES

MOSTAFA GHADAMPOUR*, EBRAHIM SOORI**,[1], RAVI P. AGARWAL****
AND DONAL O'REGAN****

*,**Department of Mathematics, Lorestan University
P.O. Box 465, Khoramabad, Lorestan, Iran
E-mail: sori.e@lu.ac.ir, sori.ebrahim@yahoo.com (E. Soori),
m.ghadampour@gmail.com (M. Ghadampour)

[1]Corresponding author

***Department of Mathematics, Texas A and M University-Kingsville
Texas 78363, USA
E-mail: Ravi.Agarwal@tamuk.edu

****School of Mathematics, Statistics and Applied Mathematics
National University of Ireland, Galway, Ireland
E-mail: donal.oregan@nuigalway.ie

Abstract. In this paper, two generalized algorithms for solving the variational inequality problem in Banach spaces are proposed. Then the strong convergence of the sequences generated by these algorithms will be proved under suitable conditions. Finally, using MATLAB software, we provide some numerical examples to illustrate our results.
Key Words and Phrases: Variational inequality, relatively nonexpansive mapping, monotone mapping, asymptotical fixed point.
2020 Mathematics Subject Classification: 47H09, 47H10.

1. Introduction

Let C be a nonempty closed convex subset of a Banach space E with norm $\|.\|$ and let E^* denotes the dual of E. The variational inequality problem (VIP) is to find a point $x \in C$ such that

$$\langle Ax, y - x\rangle \geq 0 \quad \forall \; y \in C, \tag{1.1}$$

where A is a mapping of C into E^* and $\langle ., .\rangle$ denotes the pairing between E and E^*. The solution set of (1.1) is denoted by $VI(C, A)$.
It is well known that variational inequalities cover a variety of fields in optimal control, optimization, mathematical programming, operational research, partial differential

equations, engineering, and equilibrium models and hence, it have been studied by many authors; see the recent papers [16, 24, 26, 6, 28, 11, 12, 18].

The operator A of C to E^* is said to be
(i) monotone if

$$\langle x-y, Ax-Ay\rangle \geq 0, \ \ \forall x,y \in C;$$

(ii) $\alpha-$inverse strongly monotone if there exists a constant $\alpha > 0$ such that

$$\langle x-y, Ax-Ay\rangle \geq \alpha\|Ax-Ay\|^2 \ \ \forall x,y \in C;$$

(iii) L-Lipchitz continuous if there exists $L > 0$ such that

$$\|Ax-Ay\| \leq L\|x-y\|, \ \ \forall x,y \in C.$$

Let $f : C \times C \to \mathbb{R}$ be a bifunction. The equilibrium problem (GEP) is as follows: Find $x \in C$ such that

$$f(x,y) + \langle Ax, y-x\rangle \geq 0, \ \ \forall\, y \in C. \tag{1.2}$$

The set of solutions of (1.2) is denoted by $GEP(f,A)$. Clearly, problem (1.2) is equivalent to (VIP) if $f \equiv 0$.

Korpelevich[15] proposed the following algorithm for solving the problem (VIP) that is known as the extragradient method in (1.3). Let x_1 be an arbitrarily element in H and

$$\begin{cases} y_n = P_C(x_n - \lambda Ax_n), \\ x_{n+1} = P_C(x_n - \lambda Ay_n), \end{cases} \tag{1.3}$$

Tseng [25] proposed the following algorithm which was introduced using the modified front-to-back (F-B) method:

$$\begin{cases} y_n = P_C(x_n - \lambda Ax_n), \\ x_{n+1} = P_X(y_n - \lambda(Ay_n - Ax_n)), \end{cases} \tag{1.4}$$

where $X = C$ and $X = H$ if A is Lipschitz continuous. Thong et al [23] proposed the following convergent algorithm based on the Tseng algorithm:

$$\begin{cases} y_n = P_C(x_n - \lambda_n Ax_n), \\ z_n = y_n - \lambda_n(Ay_n - Ax_n), \\ x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)z_n, \end{cases} \tag{1.5}$$

where the operator A is monotone and Lipschitz continuous, $\gamma > 0, \ l \in (0,1), \ \mu \in (0,1)$ and λ_n is chosen to be the largest $\lambda \in \{\gamma, \gamma l, \gamma l^2, ...\}$ satisfying

$$\lambda\|Ax_n - Ay_n\| \leq \mu\|x_n - y_n\|. \tag{1.6}$$

In this paper, we present our algorithms in Banach spaces motivated by the Thong algorithm and prove the strong convergence of the sequences generated by these algorithms. Finally, using MATLAB software, we provide some numerical examples to illustrate our claims.

2. Preliminaries

Let E be a real Banach space with norm $\|.\|$ and let E^* be the dual space of E. The strong convergence and the weak convergence of the sequence $\{x_n\}$ to x in E are denoted by $x_n \to x$ and $x_n \rightharpoonup x$ throughout the paper, respectively. The modulus δ of convexity of E is defined by

$$\delta(\epsilon) = \inf\{1 - \frac{\|x+y\|}{2} : \|x\| \leq 1, \|y\| \leq 1, \|x-y\| \geq \epsilon\}$$

for every $\epsilon \in [0,2]$. *A* Banach space E is said to be uniformly convex if $\delta(0) = 0$ and $\delta(\epsilon) > 0$ for every $\epsilon > 0$. It is known that a Banach space E is uniformly convex if and only if for any two sequences $\{x_n\}$ and $\{y_n\}$ in E such that

$$\lim_{n\to\infty} \|x_n\| = \lim_{n\to\infty} \|y_n\| = 1 \text{ and } \lim_{n\to\infty} \|x_n + y_n\| = 2,$$

$\lim_{n\to\infty} \|x_n - y_n\| = 0$ holds. Suppose that p is a fixed real number with $p \geq 2$. A Banach space E is said to be p-uniformly convex[22], if there exists a constant $c > 0$ such that $\delta \geq c\epsilon^p$ for all $\epsilon \in [0,2]$. It is also known that a uniformly convex Banach space has the Kadec-Klee property, that is, $x_n \rightharpoonup u$ and $\|x_n\| \to \|u\|$ imply that $x_n \to u$(see [10, 19]).
The normalized duality mapping $J : E \to E^*$ is defined by

$$J(x) = \{f \in E^* : \langle x, f\rangle = \|x\|^2 = \|f\|^2\},$$

for each $x \in E$. Suppose that $S(E) = \{x \in E : \|x\| = 1\}$. A Banach space E is called smooth if for all $x \in S(E)$, there exists a unique functional $j_x \in E^*$ such that $\langle x, j_x\rangle = \|x\|$ and $\|j_x\| = 1$(see [1]).

The norm of E is said to be *Gâteaux* differentiable if for each $x, y \in S(E)$, the limit

$$\lim_{t\to 0} \frac{\|x+ty\| - \|x\|}{t} \tag{2.1}$$

exists. In this case, E is called smooth and E is said to be uniformly smooth if the limit (2.1) is attained uniformly for all $x, y \in S(E)$[21]. If a Banach space E is uniformly convex, then E is reflexive and strictly convex, and E^* is uniformly smooth[1]. It is well known that if E is a reflexive, strictly convex and smooth Banach space and $J^* : E^* \to E$ is the duality mapping on E^*, then $J^{-1} = J^*$, also, if E is a uniformly smooth Banach space, then J is uniformly norm to norm continuous on bounded sets of E and $J^{-1} = J^*$ is also uniformly norm to norm continuous on bounded sets of E^*. Let E be a smooth Banach space and let J be the duality mapping on E. The function $\phi : E \times E \to \mathbb{R}$ is defined by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2, \quad \forall x, y \in E. \tag{2.2}$$

Clearly, from (2.2), we can conclude that

$$(\|x\| - \|y\|)^2 \leq \phi(x, y) \leq (\|x\| + \|y\|)^2. \tag{2.3}$$

If E is a reflexive, strictly convex and smooth Banach space, then for all $x, y \in E$

$$\phi(x, y) = 0 \Leftrightarrow x = y. \tag{2.4}$$

Also, it is clear from the definition of the function ϕ that the following conditions hold for all $x, y, z, w \in E$,

$$\phi(x,y) = \phi(x,z) + \phi(z,y) + 2\langle x - z, Jz - Jy\rangle, \tag{2.5}$$

$$2\langle x - y, Jz - Jw\rangle = \phi(x,w) + \phi(y,z) - \phi(x,z) - \phi(y,w). \tag{2.6}$$

$$\phi(x,y) = \langle x, Jx - Jy\rangle + \langle y - x, Jy\rangle \leq \|x\|\|Jx - Jy\| + \|y - x\|\|y\|. \tag{2.7}$$

Now, the function $V : E \times E^* \to \mathbb{R}$ is defined as follows

$$V(x, x^*) = \|x\|^2 - 2\langle x, x^*\rangle + \|x^*\|^2,$$

for all $x \in E$ and $x^* \in E^*$. Moreover, $V(x, x^*) = \phi(x, J^{-1}x^*)$ for all $x \in E$ and $x^* \in E^*$. If E is a reflexive strictly convex and smooth Banach space with E^* as its dual, we can conclude that

$$V(x, x^*) + 2\langle J^{-1}x^* - x, y^*\rangle \leq V(x, x^* + y^*), \tag{2.8}$$

for all $x \in E$ and all $x^*, y^* \in E^*$[14].

An operator $A : C \to E^*$ is hemicontinuous at $x_0 \in C$, if for any sequence $\{x_n\}$ converging to x_0 along a line implies that $Tx_n \rightharpoonup Tx_0$, i.e., $Tx_n = T(x_0 + t_n x) \rightharpoonup Tx_0$ as $t_n \to 0$ for all $x \in C$.

The generalized projection $\Pi_C : E \to C$ is a mapping that assigns to an arbitrary point $x \in E$, the minimum point of the functional $\phi(y, x)$; that is, $\Pi_C x = x_0$, where x_0 is the solution of the minimization problem

$$\phi(x_0, x) = \min_{y \in C} \phi(y, x). \tag{2.9}$$

The existence and uniqueness of the operator Π_C follows from the properties of the functional $\phi(x, y)$ and strict monotonicity of the mapping J[2]. Suppose that C is a nonempty closed convex subset of E, and T is a mapping from C into itself. A point $p \in C$ is called an asymptotically fixed point of T if C contains a sequence $\{x_n\}$ which converges weakly to p such that $Tx_n - x_n \to 0$[1]. The set of asymptotical fixed points of T will be denoted by $\hat{F}(T)$. A mapping T from C into itself is said to be relatively nonexpansive if $\hat{F}(T) = F(T)$ and $\phi(p, Tx) \leq \phi(p, x)$ for all $x \in C$ and $p \in F(T)$. The asymptotic behavior of a relatively nonexpansive mapping was studied in [4, 5, 7].

We need the following lemmas for the proof of our main results.

Lemma 2.1. ([13]) *Let E be a smooth and uniformly convex Banach space and let $\{x_n\}$ and $\{y_n\}$ be two sequences of E. If $\phi(x_n, y_n) \to 0$ and either $\{x_n\}$ or $\{y_n\}$ is bounded, then $x_n - y_n \to 0$.*

Lemma 2.2. ([2]) *Let C be a nonempty closed convex subset of a smooth, strictly convex and reflexive Banach space E, let $x \in E$ and let $z \in C$. Then*

$$z = \Pi_C x \Leftrightarrow \langle y - z, Jx - Jz\rangle \leq 0, \text{ for all } y \in C.$$

Lemma 2.3. ([2]) *Let C be a nonempty closed convex subset of a smooth, strictly convex and reflexive Banach space E and let $y \in E$. Then*

$$\phi(x, \Pi_C y) + \phi(\Pi_C y, y) \leq \phi(x, y), \quad \forall x \in C.$$

Lemma 2.4. ([3, 27]) *Let E be a 2-uniformly convex and smooth Banach space. Then, for all $x,\ y \in E$, we have that*

$$\|x - y\| \leq \frac{2}{c^2}\|Jx - Jy\|,$$

where $\frac{1}{c}(0 \leq c \leq 1)$is the 2-uniformly convex constant of E.

Lemma 2.5. ([27]) *Let E be a uniformly convex Banach space and $r > 0$. Then there exists a continuous strictly increasing convex function $g : [0, 2r] \to [0, \infty)$ such that $g(0) = 0$ and*

$$\|tx + (1-t)y\|^2 \leq t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)g(\|x - y\|),$$

for all $x, y \in B_r(0) = \{z \in E : \|z\| \leq r\}$ and $t \in [0, 1]$.

Lemma 2.6. ([13]) *Let E be a uniformly convex Banach space and $r > 0$. Then there exists a continuous strictly increasing convex function $g : [0, 2r] \to [0, \infty)$ such that $g(0) = 0$ and*

$$g(\|x - y\|) \leq \phi(x, y),$$

for all $x, y \in B_r(0) = \{z \in E : \|z\| \leq r\}$.

Throughout this paper, we assume that $f : C \times C \to \mathbb{R}$ is a bifunction satisfying the following conditions

(A1) $f(x, x) = 0$ for all $x \in C$,
(A2) f is monotone, i.e. $f(x, y) + f(y, x) \leq 0$, for all $x, y \in C$,
(A3) $\lim\limits_{t \downarrow 0} f(tz + (1-t)x, y) \leq f(x, y)$, for all $x, y, z \in C$,
(A4) for each $x \in C, y \mapsto f(x, y)$ is convex and lower semicontinuous.

Lemma 2.7. ([17]) *Let C be a nonempty closed convex subset of a smooth, strictly convex and reflexive Banach space E. Let $A : C \longrightarrow E^*$ be an $\alpha-$inverse-strongly monotone operator and f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying $(A_1) - (A_4)$. Then for all $r > 0$ the following hold*

(i) *for $x \in E$, there exists $u \in C$ such that*

$$f(u, x) + \langle Au, y - u \rangle + \frac{1}{r}\langle y - u, Ju - Jx \rangle \geq 0, \quad \forall y \in C,$$

(ii) *if E is additionally uniformly smooth and $K_r : E \longrightarrow C$ is defined as*

$$K_r(x) = \{u \in C \quad : \quad f(u, y) + \langle Au, y - u \rangle + \frac{1}{r}\langle y - u, Ju - Jx \rangle \geq 0, \quad \forall y \in C\},$$

then, the following conditions hold:

(1) *K_r is single-valued,*
(2) *K_r is firmly nonexpansive, i.e., for all $x, y \in E$,*

$$\langle K_r x - K_r y, JK_r x - JK_r y \rangle \leq \langle K_r x - K_r y, Jx - Jy \rangle,$$

(3) *$F(K_r) = F(\hat{K}_r) = GEP(f, A)$,*
(4) *GEP is a closed convex subset of C,*
(5) *$\phi(p, K_r x) + \phi(K_r x, x) \leq \phi(p, x), \ \forall \ p \in F(K_r)$.*

The normal cone for C at a point $v \in C$ is denoted by $N_C(v)$, that is

$$N_C(v) := \{x^* \in E^* : \langle v - y, x^* \rangle \geq 0, \forall y \in C\}.$$

Lemma 2.8. ([20]) *Let C be a nonempty closed convex subset of a Banach space E and let T be monotone and hemicontinuous operator of C into E^* with $C = D(T)$. Let $B \subset E \times E^*$ be an operator defined as follows:*

$$Bv = \begin{cases} Tv + N_C v, & v \in C, \\ \emptyset, & v \notin C. \end{cases}$$

Then B is maximal monotone and $B^{-1}(0) = SOL(T, C)$.

3. Main results

In this section, we introduce new iterative algorithms for solving monotone variational inequality problems which are based on Tseng's intergradient method. We prove strong convergence theorems for generated sequences by presenting intergradient algorithms, under suitable conditions.

Throughout this section, we assume that C is a nonempty closed convex subset of a real 2-uniformly convex and uniformly smooth Banach space E and E^* is the dual space of E, and $A : C \to E^*$ is a α-inverse strongly monotone operator. Assume that $\{\lambda_n\}$ is a sequence of real numbers such that $0 < \lambda_n < \frac{c^2\alpha}{2}$ for all $n \in \mathbb{N}$, where $\frac{1}{c}$ is the 2-uniformly convexity constant of E.

Theorem 3.1 *Let $x_0 \in C$, $\Gamma := VI(C, A) \cap F(f) \neq \emptyset$ and*

$$\begin{cases} y_n = \Pi_C J^{-1}(Jx_n - \lambda_n Ax_n), \\ z_n = J^{-1}(Jy_n - \lambda_n Ay_n), \\ x_{n+1} = \Pi_C J^{-1}(\alpha_{n,1} Jx_n + \alpha_{n,2} Jf(x_n) + \alpha_{n,3} Jz_n), \end{cases} \tag{3.1}$$

where $\{\lambda_n\} \subseteq [0, 1]$ such that $\lim_{n\to\infty} \lambda_n = 0$. Let $\{\alpha_{n,i}\} \subset (0, 1)$ for $i = 1, 2, 3$, $\alpha_{n,1} + \alpha_{n,2} + \alpha_{n,3} = 1$ and $\liminf_{n\to\infty} \alpha_{n,2}\alpha_{n,3} > 0$. Let f be a relatively nonexpansive self-mapping on C and $\|Ax\| \leq \|Ax - Au\|$ for all $x \in C$ and $u \in \Gamma$. Consider the sequence $\{x_n\}$ generated by the algorithm (3.1). Then the sequence $\{x_n\}$ converges strongly to $q = \Pi_{VI(C,A)} \circ f(q)$, where $P_{VI(C,A)} \circ f : H \to VI(C, A)$ is the mapping defined by $P_{VI(C,A)} \circ f(x) = P_{VI(C,A)}(f(x))$ for each $x \in H$.

Proof. Let $\hat{u} \in \Gamma$. From the definition of the function V and the inequality (2.8), we conclude that

$$\begin{aligned} \phi(\hat{u}, z_n) =& \phi(\hat{u}, J^{-1}(Jy_n - \lambda_n Ay_n)) \\ =& V(\hat{u}, Jy_n - \lambda_n Ay_n) \\ \leq& V(\hat{u}, Jy_n) - 2\langle J^{-1}(Jy_n - \lambda_n Ay_n) - \hat{u}, \lambda_n Ay_n \rangle \\ =& \phi(\hat{u}, y_n) + 2\langle J^{-1}(Jy_n - \lambda_n Ay_n) - J^{-1}(Jy_n), -\lambda_n Ay_n \rangle \\ & - 2\langle y_n - \hat{u}, \lambda_n Ay_n \rangle, \end{aligned} \tag{3.2}$$

then from Lemma 2.4 and the condition $\|Ax\| \leq \|Ax - A\hat{u}\|$ for all $x \in C$, it follows that

$$
\begin{aligned}
2\langle J^{-1}(Jy_n-&\lambda_n Ay_n) - J^{-1}(Jy_n), -\lambda_n Ay_n\rangle \\
&\leq 2\|J^{-1}(Jy_n - \lambda_n Ay_n) - J^{-1}(Jy_n)\|\| - \lambda_n Ay_n\| \\
&\leq \frac{4\lambda_n^2}{c^2}\|Ay_n\|^2 \\
&\leq \frac{4\lambda_n^2}{c^2}\|Ay_n - A\hat{u}\|^2. \qquad (3.3)
\end{aligned}
$$

Since A is α-inverse strongly monotone and the fact that $\hat{u} \in VI(C, A)$, we have

$$
\begin{aligned}
-2\langle y_n - &\hat{u}, \lambda_n Ay_n\rangle \\
&= -2\lambda_n\langle y_n - \hat{u}, Ay_n - A\hat{u}\rangle - 2\lambda_n\langle y_n - \hat{u}, A\hat{u}\rangle \\
&\leq -2\lambda_n\langle y_n - \hat{u}, Ay_n - A\hat{u}\rangle \\
&\leq -2\lambda_n\alpha\|Ay_n - A\hat{u}\|^2, \qquad (3.4)
\end{aligned}
$$

substituting (3.3) and (3.4) in (3.2) and using our assumptions, we obtain

$$
\begin{aligned}
\phi(\hat{u}, z_n) &\leq \phi(\hat{u}, y_n) + (\frac{4\lambda_n^2}{c^2} - 2\lambda_n\alpha)\|Ay_n - Au\|^2 \\
&= \phi(\hat{u}, y_n) + 2\lambda_n(\frac{2\lambda_n}{c^2} - \alpha)\|Ay_n - A\hat{u}\|^2 \\
&\leq \phi(\hat{u}, y_n),
\end{aligned}
$$

hence,

$$\phi(\hat{u}, z_n) \leq \phi(\hat{u}, y_n). \qquad (3.5)$$

From Lemma 2.3 and the inequality (2.8), we have

$$
\begin{aligned}
\phi(\hat{u}, y_n) &= \phi(\hat{u}, \Pi_C J^{-1}(Jx_n - \lambda_n Ax_n)) \\
&\leq \phi(\hat{u}, J^{-1}(Jx_n - \lambda_n Ax_n)) = V(\hat{u}, Jx_n - \lambda_n Ax_n) \\
&\leq V(\hat{u}, Jx_n) - 2\langle J^{-1}(Jx_n - \lambda_n Ax_n) - \hat{u}, \lambda_n Ax_n\rangle \\
&= \phi(\hat{u}, x_n) - 2\lambda_n\langle x_n - \hat{u}, Ax_n\rangle \\
&\quad + 2\langle J^{-1}(Jx_n - \lambda_n Ax_n) - J^{-1}(Jx_n), -\lambda_n Ax_n\rangle, \qquad (3.6)
\end{aligned}
$$

since A is $\alpha-$inverse strongly monotone and $\hat{u} \in VI(C, A)$, it follows that

$$
\begin{aligned}
-2\lambda_n\langle x_n - &\hat{u}, Ax_n\rangle \\
&= -2\lambda_n\langle x_n - \hat{u}, Ax_n - A\hat{u}\rangle - 2\lambda_n\langle x_n - \hat{u}, A\hat{u}\rangle \\
&\leq -2\lambda_n\langle x_n - \hat{u}, Ax_n - A\hat{u}\rangle \\
&\leq -2\lambda_n\alpha\|Ax_n - A\hat{u}\|^2. \qquad (3.7)
\end{aligned}
$$

From Lemma 2.4 and our assumptions, we can conclude that

$$\begin{aligned}
2\langle J^{-1}(Jx_n-\lambda_nAx_n)-J^{-1}(Jx_n),-\lambda_nAx_n\rangle
&\leq 2\|J^{-1}(Jx_n-\lambda_nAx_n)-J^{-1}(Jx_n)\|\|-\lambda_nAx_n\|\\
&\leq \frac{4\lambda_n^2}{c^2}\|Ax_n\|^2\\
&\leq \frac{4\lambda_n^2}{c^2}\|Ax_n-A\hat{u}\|^2. \qquad (3.8)
\end{aligned}$$

By applying (3.7) and (3.8) in (3.6) and our assumptions, we have that

$$\phi(\hat{u},y_n)\leq \phi(\hat{u},x_n)+2\lambda_n(\frac{2\lambda_n}{c^2}-\alpha)\|Ay_n-Ax_n\|^2\leq \phi(\hat{u},x_n). \qquad (3.9)$$

Hence, from (3.5) and (3.9), we have

$$\phi(\hat{u},z_n)\leq \phi(\hat{u},x_n). \qquad (3.10)$$

Next, we will show that the sequence $\{\phi(\hat{u},x_n)\}$ is decreasing. From the relatively nonexpansiveness condition of f, the convexity of $\|.\|^2$, Lemma 2.3 and the inequality (3.10), we have that

$$\begin{aligned}
\phi(\hat{u},x_{n+1}) &\leq \phi(\hat{u},J^{-1}(\alpha_{n,1}Jx_n+\alpha_{n,2}Jf(x_n)+\alpha_{n,3}Jz_n)\\
&=\|\hat{u}\|^2-2\langle\hat{u},\alpha_{n,1}Jx_n+\alpha_{n,2}Jf(x_n)+\alpha_{n,3}Jz_n\rangle\\
&\quad+\|\alpha_{n,1}Jx_n+\alpha_{n,2}Jf(x_n)+\alpha_{n,3}Jz_n\|^2\\
&\leq\|\hat{u}\|^2-2\alpha_{n,1}\langle\hat{u},Jx_n\rangle-2\alpha_{n,2}\langle\hat{u},Jf(x_n)\rangle-2\alpha_{n,3}\langle\hat{u},Jz_n\rangle\\
&\quad+\alpha_{n,1}\|x_n\|^2+\alpha_{n,2}\|f(x_n)\|^2+\alpha_{n,3}\|z_n\|^2\\
&=\alpha_{n,1}\phi(\hat{u},x_n)+\alpha_{n,2}\phi(\hat{u},f(x_n))+\alpha_{n,3}\phi(\hat{u},z_n)\\
&\leq\alpha_{n,1}\phi(\hat{u},x_n)+\alpha_{n,2}\phi(\hat{u},x_n)+\alpha_{n,3}\phi(\hat{u},x_n)\\
&=\phi(\hat{u},x_n), \qquad (3.11)
\end{aligned}$$

so $\{\phi(\hat{u},x_n)\}$ is decreasing. Then $\{\phi(\hat{u},x_n)\}$ is bounded, hence $\lim_{n\to\infty}\phi(\hat{u},x_n)$ exists. Then from (2.3), $\{x_n\}$ is bounded. It follows from the relatively nonexpansiveness condition of f, (3.9) and (3.10) that $\{f(x_n)\}$, $\{y_n\}$ and $\{z_n\}$ are bounded. From Lemmas 2.3, 2.4, the inequality (2.8) and the condition $\lim_{n\to\infty}\lambda_n=0$, we have

$$\begin{aligned}
\phi(x_n,y_n) &\leq \phi(x_n,J^{-1}(Jx_n-\lambda_nAx_n))\\
&=V(x_n,Jx_n-\lambda_nAx_n)\\
&\leq V(x_n,Jx_n)-2\langle J^{-1}(Jx_n-\lambda_nAx_n)-x_n,\lambda_nAx_n)\\
&=\phi(x_n,x_n)-2\langle J^{-1}(Jx_n-\lambda_nAx_n)-J^{-1}(Jx_n),\lambda_nAx_n)\rangle\\
&\leq 2\|J^{-1}(Jx_n-\lambda_nAx_n)-J^{-1}(Jx_n)\|\|\lambda_nAx_n\|\\
&\leq\frac{4\lambda_n^2}{c^2}\|Ax_n\|^2\to 0 \quad as\ n\to\infty. \qquad (3.12)
\end{aligned}$$

From Lemma 2.1, we have that

$$\lim_{n\to\infty} \|x_n - y_n\| = 0. \tag{3.13}$$

Next, from (2.7), (3.13), the boundedness of the sequences $\{x_n\}$ and $\{y_n\}$, and using uniformly norm-to-norm continuity of J on bounded sets, it is clear that

$$\phi(y_n, x_n) \leq \|y_n\|\|Jy_n - Jx_n\| + \|x_n - y_n\|\|x_n\| \to 0 \quad \text{as} \;\; n \to \infty. \tag{3.14}$$

From Lemma 2.4, the inequality (2.8) and the condition $\lim_{n\to\infty} \lambda_n = 0$, we have

$$\begin{aligned}\phi(y_n, z_n) = \phi(y_n, &J^{-1}(Jy_n - \lambda_n Ay_n)) \\ =&V(y_n, Jy_n - \lambda_n Ay_n) \\ \leq&V(y_n, Jy_n) - 2\langle J^{-1}(Jy_n - \lambda_n Ay_n) - y_n, \lambda_n Ay_n) \\ =&\phi(y_n, y_n) - 2\langle J^{-1}(Jy_n - \lambda_n Ay_n) - J^{-1}(Jy_n), \lambda_n Ay_n)\rangle \\ \leq&2\|J^{-1}(Jy_n - \lambda_n Ay_n) - J^{-1}(Jy_n)\|\|\lambda_n Ay_n\| \\ \leq&\frac{4\lambda_n^2}{c^2}\|Ay_n\|^2 \to 0 \quad as\; n \to \infty. \end{aligned} \tag{3.15}$$

From Lemma 2.1, we have that

$$\lim_{n\to\infty} \|y_n - z_n\| = 0. \tag{3.16}$$

Since $\{f(x_n)\}$ and $\{z_n\}$ are bounded, now, setting $r_1 = sup\{\|f(x_n)\|, \|z_n\|\}$, from Lemma 2.5 there exists a continuous strictly increasing and convex function $g_1 : [0, 2r_1] \longrightarrow [0, \infty]$ with $g_1(0) = 0$. From (3.10), Lemmas 2.3, 2.5 and the condition relatively nonexpansiveness of f, we conclude for each $\hat{u} \in \Gamma$ that

$$\begin{aligned}\phi(\hat{u}, x_{n+1}) \leq&\phi(\hat{u}, J^{-1}(\alpha_{n,1}Jx_n + \alpha_{n,2}Jf(x_n) + \alpha_{n,3}Jz_n) \\ =&\|\hat{u}\|^2 - 2\langle \hat{u}, \alpha_{n,1}Jx_n + \alpha_{n,2}Jf(x_n) + \alpha_{n,3}Jz_n\rangle \\ &+ \|\alpha_{n,1}Jx_n + \alpha_{n,2}Jf(x_n) + \alpha_{n,3}Jz_n\|^2 \\ \leq&\|\hat{u}\|^2 - 2\alpha_{n,1}\langle \hat{u}, Jx_n\rangle - 2\alpha_{n,2}\langle \hat{u}, Jf(x_n)\rangle - 2\alpha_{n,3}\langle \hat{u}, Jz_n\rangle \\ &+ \alpha_{n,1}\|x_n\|^2 + \alpha_{n,2}\|f(x_n)\|^2 + \alpha_{n,3}\|z_n\|^2 \\ &- \alpha_{n,2}\alpha_{n,3}g_1(\|Jf(x_n) - Jz_n\|) \\ =&\alpha_{n,1}\phi(\hat{u}, x_n) + \alpha_{n,2}\phi(\hat{u}, f(x_n)) + \alpha_{n,3}\phi(\hat{u}, z_n) \\ &- \alpha_{n,2}\alpha_{n,3}g_1(\|Jf(x_n) - Jz_n\|) \\ \leq&\alpha_{n,1}\phi(\hat{u}, x_n) + \alpha_{n,2}\phi(\hat{u}, x_n) + \alpha_{n,3}\phi(\hat{u}, x_n) \\ &- \alpha_{n,2}\alpha_{n,3}g_1(\|Jf(x_n) - Jz_n\|) \\ =&\phi(\hat{u}, x_n) - \alpha_{n,2}\alpha_{n,3}g_1(\|Jf(x_n) - Jz_n\|),\end{aligned}$$

therefore

$$\alpha_{n,2}\alpha_{n,3}g_1(\|Jf(x_n) - Jz_n\|) \leq \phi(\hat{u}, x_n) - \phi(\hat{u}, x_{n+1}).$$

Since $\liminf_{n\to\infty} \alpha_{n,2}\alpha_{n,3} > 0$, we have

$$\lim_{n\to\infty} g_1(\|Jf(x_n) - Jz_n\|) = 0, \tag{3.17}$$

because $\{\phi(\hat{u}, x_n)\}$ is Cauchy and $\liminf_{n\to\infty} \alpha_{n,2}\alpha_{n,3} > 0$. Since g_1 is a continuous function, so

$$g_1(\lim_{n\to\infty} \|Jf(x_n) - Jz_n\|) = \lim_{n\to\infty} g_1(\|Jf(x_n) - Jz_n\|) = 0 = g_1(0), \tag{3.18}$$

and also g_1 is strictly increasing, hence

$$\lim_{n\to\infty} \|Jf(x_n) - Jz_n\| = 0. \tag{3.19}$$

On the other hand, since J^{-1} is uniformly norm-to-norm continuous on bounded sets, we obtain that

$$\lim_{n\to\infty} \|f(x_n) - z_n\| = \lim_{n\to\infty} \|J^{-1}(Jf(x_n)) - J^{-1}(Jz_n)\| = 0. \tag{3.20}$$

Next, from (2.7) and (3.20), we have

$$\lim_{n\to\infty} \phi(z_n, f(x_n)) = 0. \tag{3.21}$$

Similarly, from (2.7), (3.13) and (3.16), we obtain

$$\lim_{n\to\infty} \phi(z_n, x_n) = 0. \tag{3.22}$$

Moreover, from Lemma 2.3, the inequalities (3.21), (3.22) and the convexity of $\|.\|^2$, we conclude that

$$\begin{aligned}
\phi(z_n, x_{n+1}) \le& \phi(z_n, J^{-1}(\alpha_{n,1}Jx_n + \alpha_{n,2}Jf(x_n) + \alpha_{n,3}Jz_n)) \\
=& \|z_n\|^2 - 2\langle z_n, \alpha_{n,1}Jx_n + \alpha_{n,2}Jf(x_n) + \alpha_{n,3}Jz_n\rangle \\
&+ \|\alpha_{n,1}Jx_n + \alpha_{n,2}Jf(x_n) + \alpha_{n,3}Jz_n\|^2 \\
\le& \|z_n\|^2 - 2\alpha_{n,1}\langle z_n, Jx_n\rangle - 2\alpha_{n,2}\langle z_n, Jf(x_n)\rangle - 2\alpha_{n,3}\langle z_n, Jz_n\rangle \\
&+ \alpha_{n,1}\|x_n\|^2 + \alpha_{n,2}\|f(x_n)\|^2 + \alpha_{n,3}\|z_n\|^2 \\
=& \alpha_{n,1}\phi(z_n, x_n) + \alpha_{n,2}\phi(z_n, f(x_n)) + \alpha_{n,3}\phi(z_n, z_n) \\
=& \alpha_{n,1}\phi(z_n, x_n) + \alpha_{n,2}\phi(z_n, f(x_n)) \to 0 \quad as\ n \to \infty,
\end{aligned}$$

then using Lemma 2.1, we get

$$\lim_{n\to\infty} \|z_n - x_{n+1}\| = 0. \tag{3.23}$$

It follows from (3.13), (3.16) and (3.23) that

$$\|x_{n+1} - x_n\| \le \|x_{n+1} - z_n\| + \|z_n - y_n\| + \|y_n - x_n\| \to 0 \quad \text{as} \ \ n \to \infty. \tag{3.24}$$

Thus $\{x_n\}$ is a Cauchy sequence, so $\{x_n\}$ converges strongly to a point $q \in C$. It follows from (3.13) and (3.16) that the sequences $\{y_n\}$ and $\{z_n\}$ are convergent to q. Next, we show that $q \in VI(C, A)$. Let $B \subset E \times E^*$ be an operator defined as follows:

$$Bv = \begin{cases} \lambda_n Av + N_C v, & v \in C, \\ \emptyset, & v \notin C. \end{cases} \tag{3.25}$$

Since $\lambda_n A$ is $\lambda_n\alpha$-inverse strongly monotone, it follows that $\lambda_n A$ is $\frac{1}{\lambda_n\alpha}$-Lipschitz continuous, hence $\lambda_n A$ is hemicontinuous. Therefore, by Lemma 2.8 B is maximal monotone and $B^{-1}(0) = VI(C, \lambda_n A) = VI(C, A)$. Let $(\upsilon, w) \in G(B)$ with $w \in B\upsilon = \lambda_n A\upsilon + N_C(\upsilon)$. Then $w - \lambda_n A\upsilon \in N_C(\upsilon)$, hence

$$\langle \upsilon - y_n, w - \lambda_n A\upsilon \rangle \geq 0, \tag{3.26}$$

because $y_n \in C$. On the other hand from Lemma 2.2, we conclude that

$$\langle \upsilon - y_n, J(J^{-1}(Jx_n - \lambda_n Ax_n)) - Jy_n \rangle \leq 0,$$

so

$$\langle \upsilon - y_n, \lambda_n Ax_n + Jy_n - Jx_n \rangle \geq 0. \tag{3.27}$$

From (3.26), (3.27) and using the definition A, we get

$$\begin{aligned}
&\langle \upsilon - y_n, w \rangle \\
&\quad \geq \lambda_n \langle \upsilon - y_n, A\upsilon \rangle - \langle \upsilon - y_n, \lambda_n Ax_n + Jy_n - Jx_n \rangle \\
&\quad = \lambda_n \langle \upsilon - y_n, A\upsilon - Ay_n \rangle + \lambda_n \langle \upsilon - y_n, Ay_n \rangle \\
&\qquad - \langle \upsilon - y_n, \lambda_n Ax_n + Jy_n - Jx_n \rangle \\
&\quad \geq \lambda_n \langle \upsilon - y_n, Ay_n - Ax_n \rangle - \langle \upsilon - y_n, Jy_n - Jx_n \rangle \\
&\quad \geq -\lambda_n \|\upsilon - y_n\| \|Ax_n - Ay_n\| - \|\upsilon - y_n\| \|Jx_n - Jy_n\|.
\end{aligned} \tag{3.28}$$

Hence, using uniformly norm-to-norm continuity of J on bounded sets and (3.13), $\langle \upsilon - y_n, w \rangle \geq 0$ as $n \to \infty$, i.e. $\langle \upsilon - q, w \rangle \geq 0$. Therefore $\langle q - \upsilon, 0 - w \rangle \geq 0$, and we conclude from Lemma 2.8 that $q \in B^{-1}(0) = VI(C, A)$, because B is a maximal monotone operator.

Next, we show that $q \in F(f)$. From (3.13), (3.16) and (3.20), we have

$$\|f(x_n) - x_n\| \leq \|f(x_n) - z_n\| + \|z_n - y_n\| + \|y_n - x_n\| \to 0 \quad \text{as} \quad n \to \infty, \tag{3.29}$$

and since $x_n \rightharpoonup q$, then q is an asymptotic fixed point of f. Moreover, $\hat{F}(f) = F(f)$, because f is a relatively nonexpansive mapping, hence $q \in F(f)$. Therefore, $\Pi_{VI(C,A)} o f(q) = \Pi_{VI(C,A)}(q) = q$.

Theorem 3.2. *Suppose that $\tilde{F}$ is a bifunction from $C \times C$ to $\mathbb{R}$ which satisfies the conditions $(A_1) - (A_4)$. Let f be a relatively nonexpansive self-mapping on C and $\|Ax\| \leq \|Ax - Au\|$ for all $x \in C$ and $u \in \Omega := VI(C, A) \cap GEP(\tilde{F}, A) \cap F(f)$. Let x_0 be an arbitrary point in C and $\{x_n\}$ be a sequence generated by*

$$\begin{cases}
u_n \in C \ \ s.t \ \ \tilde{F}(u_n, y) + \langle Au_n, y - u_n \rangle + \frac{1}{r_n} \langle y - u_n, Ju_n - Jx_n \rangle \geq 0, \\
w_n = \Pi_C J^{-1}(Ju_n - \lambda_n Au_n), \\
y_n = \Pi_C J^{-1}(Jx_n - \lambda_n Ax_n), \\
C_n = \{\upsilon \in C : \phi(\upsilon, w_n) \leq \phi(\upsilon, x_n)\}, \\
z_n = \Pi_{C_n} J^{-1}(Jy_n - \lambda_n Ay_n), \\
x_{n+1} = \Pi_C J^{-1}(\alpha_{n,1} Jx_n + \alpha_{n,2} Jf(x_n) + \alpha_{n,3} Jz_n + \alpha_{n,4} Jw_n).
\end{cases} \tag{3.30}$$

where $r_n \in [a, \infty)$ for some $a > 0$, $\{\lambda_n\} \subseteq [0, 1]$ such that $\lim_{n\to\infty} \lambda_n = 0$, and $\{r_n\} \subset [a, \infty)$ for some $a > 0$. If $\{\alpha_{n,i}\} \subset [0, 1]$ for $i = 1, 2, 3, 4$ such that $\sum_{i=1}^{4} \alpha_{n,i} = 1$ and $\liminf_{n\to\infty} \alpha_{n,2}\alpha_{n,3} > 0$ and $\liminf_{n\to\infty} \alpha_{n,2}\alpha_{n,4} > 0$ then the sequence $\{x_n\}$ generated by (3.30) *converges strongly to $q = \Pi_{VI(C,A)\cap GEP(\tilde{F},A)} \circ f(q)$.*

Proof. Clearly, by part (i) of Lemma 2.7, the sequence $\{u_n\}$ exists. Now, we check that C_n is closed and convex for each $n \geq 1$. Obviously, by the definition of C_n, it is clear that C_n is closed. Applying the definition of ϕ, the inequality $\phi(v, w_n) \leq \phi(v, x_n)$ is equivalent to

$$2\langle v, Jx_n - Jw_n\rangle \leq \|x_n\|^2 - \|w_n\|^2. \tag{3.31}$$

It is clear from (3.31) that C_n is convex for all $n \geq 1$.

Now, we verify that $\{x_n\}$ is well defined. Suppose that $p \in \Omega$. By Lemma 2.7, we may put $u_n = K_{r_n}x_n$. So, by condition (5) of Lemma 2.7, we conclude that

$$\phi(p, u_n) = \phi(p, K_{r_n}x_n) \leq \phi(p, x_n). \tag{3.32}$$

Moreover, from Lemma 2.3 and the inequality (2.8), it follows that

$$\begin{aligned}\phi(p, w_n) =&\phi(p, \Pi_C J^{-1}(Ju_n - \lambda_n Au_n))\\ \leq&\phi(p, J^{-1}(Ju_n - \lambda_n Au_n))\\ \leq&V(p, Ju_n - \lambda_n Au_n)\\ \leq&V(p, Ju_n) - 2\langle J^{-1}(Ju_n - \lambda_n Au_n) - p, \lambda_n Au_n\rangle\\ =&\phi(p, u_n) - 2\lambda_n\langle u_n - p, Au_n\rangle\\ &+ 2\langle J^{-1}(Ju_n - \lambda_n Au_n) - J^{-1}(Ju_n), -\lambda_n Au_n\rangle,\end{aligned} \tag{3.33}$$

since A is an $\alpha-$inverse strongly monotone operator, and we have that

$$\begin{aligned}-2\lambda_n\langle u_n - p,&Au_n\rangle\\ &= - 2\lambda_n\langle u_n - p, Au_n - Ap\rangle - 2\lambda_n\langle u_n - p, Ap\rangle\\ &\leq - 2\lambda_n\alpha\|Au_n - Ap\|^2.\end{aligned} \tag{3.34}$$

From Lemma 2.4 and the condition $\|Ax\| \leq \|Ax - Ap\|$ for all $x \in C$, it follows that

$$\begin{aligned}2\langle J^{-1}(Ju_n - \lambda_n Au_n) - J^{-1}(Ju_n), -\lambda_n Au_n\rangle&\\ \leq&2\|J^{-1}(Ju_n - \lambda_n Au_n) - J^{-1}(Ju_n)\|\|\lambda_n Au_n\|\\ =&\frac{4\lambda_n^2}{c^2}\|Au_n\|^2\\ \leq&\frac{4\lambda_n^2}{c^2}\|Au_n - Ap\|^2.\end{aligned} \tag{3.35}$$

By substituting (3.34) and (3.35) in (3.33) and the assumption $0 < \lambda_n < \frac{c^2\alpha}{2}$, we have that

$$\phi(p, w_n) \leq \phi(p, u_n) + 2\lambda_n(\frac{2}{c^2}\lambda_n - \alpha)\|Au_n - Ap\|^2 \leq \phi(p, u_n). \tag{3.36}$$

From (3.32) and (3.36), it is clear that

$$\phi(p, w_n) \leq \phi(p, x_n). \tag{3.37}$$

Then $p \in C_n$ and hence $\{x_n\}$ is well defined.

Let $\Omega \neq \emptyset$ and $\hat{u} \in \Omega$. From Lemma 2.3, the convexity of $\|.\|^2$ and the relatively nonexpansiveness of f, it follows that

$$\begin{aligned}
\phi(\hat{u}, x_{n+1}) &\leq \phi(\hat{u}, J^{-1}(\alpha_{n,1}Jx_n + \alpha_{n,2}Jf(x_n) + \alpha_{n,3}Jz_n + \alpha_{n,4}Jw_n)) \\
&= \|\hat{u}\|^2 - 2\langle \hat{u}, \alpha_{n,1}Jx_n + \alpha_{n,2}Jf(x_n) + \alpha_{n,3}Jz_n + \alpha_{n,4}Jw_n\rangle \\
&+ \|\alpha_{n,1}Jx_n + \alpha_{n,2}Jf(x_n) + \alpha_{n,3}Jz_n + \alpha_{n,4}Jw_n\|^2 \\
&\leq \|\hat{u}\|^2 - 2\alpha_{n,1}\langle \hat{u}, Jx_n\rangle - 2\alpha_{n,2}\langle \hat{u}, Jf(x_n)\rangle - 2\alpha_{n,3}\langle \hat{u}, Jz_n\rangle - 2\alpha_{n,4}\langle \hat{u}, Jw_n\rangle \\
&+ \alpha_{n,1}\|x_n\|^2 + \alpha_{n,2}\|f(x_n)\|^2 + \alpha_{n,3}\|z_n\|^2 + \alpha_{n,4}\|w_n\|^2 \\
&= \alpha_{n,1}\phi(\hat{u}, x_n) + \alpha_{n,2}\phi(\hat{u}, f(x_n)) + \alpha_{n,3}\phi(\hat{u}, z_n) + \alpha_{n,4}\phi(\hat{u}, w_n) \\
&\leq \alpha_{n,1}\phi(\hat{u}, x_n) + \alpha_{n,2}\phi(\hat{u}, x_n) + \alpha_{n,3}\phi(\hat{u}, z_n) + \alpha_{n,4}\phi(\hat{u}, w_n) \\
&= (\alpha_{n,1} + \alpha_{n,2})\phi(\hat{u}, x_n) + \alpha_{n,3}\phi(\hat{u}, z_n) + \alpha_{n,4}\phi(\hat{u}, w_n).
\end{aligned}$$

Similarly, using Lemma 2.3, the inequality (3.10) holds for the algorithm (3.30), too. Hence, from (3.10) and (3.37), we have that

$$\phi(\hat{u}, x_{n+1}) \leq \phi(\hat{u}, x_n). \tag{3.38}$$

We conclude that $\{\phi(\hat{u}, x_n)\}$ is decreasing, so from the boundedness of the sequence $\{\phi(\hat{u}, x_n)\}$, $\lim\limits_{n\to\infty} \phi(\hat{u}, x_n)$ exists. Also from (2.3), $\{x_n\}$ is bounded and hence from (3.32) and the relatively nonexpansiveness of f, $\{u_n\}$ and $\{f(x_n)\}$ are bounded. Similarly, using Lemma 2.3, the inequalities (3.13) and (3.16) hold for the algorithm (3.30). Hence, we conclude from (3.13) and (3.16) that the sequences $\{y_n\}$ and $\{z_n\}$ are bounded, now, let $r_1 = \sup\{\|z_n\|, \|f(x_n)\|\}$, from Lemma 2.5, there exists a continuous strictly increasing and convex function $g_1 : [0, 2r_1] \longrightarrow [0, \infty)$ with $g_1(0) = 0$. We get

$$\begin{aligned}
\phi(\hat{u}, x_{n+1}) &\leq \phi(\hat{u}, J^{-1}(\alpha_{n,1}Jx_n + \alpha_{n,2}Jf(x_n) + \alpha_{n,3}Jz_n + \alpha_{n,4}Jw_n)) \\
&= \|\hat{u}\|^2 - 2\langle \hat{u}, \alpha_{n,1}Jx_n + \alpha_{n,2}Jf(x_n) + \alpha_{n,3}Jz_n + \alpha_{n,4}Jw_n\rangle \\
&+ \|\alpha_{n,1}Jx_n + \alpha_{n,2}Jf(x_n) + \alpha_{n,3}Jz_n + \alpha_{n,4}Jw_n\|^2 \\
&\leq \|\hat{u}\|^2 - 2\alpha_{n,1}\langle \hat{u}, Jx_n\rangle - 2\alpha_{n,2}\langle \hat{u}, Jf(x_n)\rangle - 2\alpha_{n,3}\langle \hat{u}, Jz_n\rangle \\
&- 2\alpha_{n,4}\langle \hat{u}, Jw_n\rangle + \alpha_{n,1}\|x_n\|^2 + \alpha_{n,2}\|f(x_n)\|^2 + \alpha_{n,3}\|z_n\|^2 \\
&+ \alpha_{n,4}\|w_n\|^2 - \alpha_{n,2}\alpha_{n,3}g_1(\|Jf(x_n) - Jz_n\|) \\
&= \alpha_{n,1}\phi(\hat{u}, x_n) + \alpha_{n,2}\phi(\hat{u}, f(x_n)) + \alpha_{n,3}\phi(\hat{u}, z_n) + \alpha_{n,4}\phi(\hat{u}, w_n) \\
&- \alpha_{n,2}\alpha_{n,3}g_1(\|Jf(x_n) - Jz_n\|) \\
&\leq \alpha_{n,1}\phi(\hat{u}, x_n) + \alpha_{n,2}\phi(\hat{u}, x_n) + \alpha_{n,3}\phi(\hat{u}, z_n) + \alpha_{n,4}\phi(\hat{u}, w_n) \\
&- \alpha_{n,2}\alpha_{n,3}g_1(\|Jf(x_n) - Jz_n\|) \\
&= (\alpha_{n,1} + \alpha_{n,2})\phi(\hat{u}, x_n) + \alpha_{n,3}\phi(\hat{u}, z_n) + \alpha_{n,4}\phi(\hat{u}, w_n) \\
&- \alpha_{n,2}\alpha_{n,3}g_1(\|Jf(x_n) - Jz_n\|).
\end{aligned}$$

Now from (3.10) and (3.37), we have

$$\phi(\hat{u}, x_{n+1}) \leq \phi(\hat{u}, x_n) - \alpha_{n,2}\alpha_{n,3}g_1(\|Jf(x_n) - Jz_n\|), \tag{3.39}$$

so

$$\alpha_{n,2}\alpha_{n,3}g_1(\|Jf(x_n) - Jz_n\|) \leq \phi(\hat{u}, x_n) - \phi(\hat{u}, x_{n+1}).$$

Since $\liminf_{n\to\infty} \alpha_{n,2}\alpha_{n,3} > 0$, using the reasoning as in the proof of Theorem 3.1, we conclude that the inequality (3.21) and (3.22) hold.

By Lemma 2.3 and convexity of $\|.\|^2$, we obtain that

$$\begin{aligned}
\phi(z_n, x_{n+1}) &\leq \phi(z_n, J^{-1}(\alpha_{n,1}Jx_n + \alpha_{n,2}Jf(x_n) + \alpha_{n,3}Jz_n + \alpha_{n,4}Jw_n)) \\
&= \|z_n\|^2 - 2\langle z_n, \alpha_{n,1}Jx_n + \alpha_{n,2}Jf(x_n) + \alpha_{n,3}Jz_n + \alpha_{n,4}Jw_n\rangle \\
&+ \|\alpha_{n,1}Jx_n + \alpha_{n,2}Jf(x_n) + \alpha_{n,3}Jz_n + \alpha_{n,4}Jw_n\|^2 \\
&\leq \|z_n\|^2 - 2\alpha_{n,1}\langle z_n, Jx_n\rangle - 2\alpha_{n,2}\langle z_n, Jf(x_n)\rangle - 2\alpha_{n,3}\langle z_n, Jz_n\rangle \\
&- 2\alpha_{n,4}\langle z_n, Jw_n\rangle + \alpha_{n,1}\|x_n\|^2 + \alpha_{n,2}\|f(x_n)\|^2 + \alpha_{n,3}\|z_n\|^2 \\
&+ \alpha_{n,4}\|w_n\|^2 \\
&= \alpha_{n,1}\phi(z_n, x_n) + \alpha_{n,2}\phi(z_n, f(x_n)) + \alpha_{n,3}\phi(z_n, z_n) + \alpha_{n,4}\phi(z_n, w_n) \\
&\leq (\alpha_{n,1} + \alpha_{n,4})\phi(z_n, x_n) + \alpha_{n,2}\phi(z_n, f(x_n)),
\end{aligned}$$

because $z_n \in C_n$. Using (3.21), (3.22) and taking the limit in the above as $n \to \infty$, we deduce that

$$\phi(z_n, x_{n+1}) \to 0.$$

Then, from Lemma 2.1, we have

$$\lim_{n\to\infty} \|x_{n+1} - z_n\| = 0,$$

therefore, it follows from (3.13), (3.16) that

$$\|x_{n+1} - x_n\| \leq \|x_{n+1} - z_n\| + \|z_n - y_n\| + \|y_n - x_n\| \to 0 \;\; as \;\; n \to \infty,$$

hence, $\{x_n\}$ is a Cauchy sequence. Thus, $\{x_n\}$ converges strongly to a point $q \in C$. Obviously, the relations (3.25), (3.26), (3.27) and (3.28) are valid for the algorithm (3.30). Hence, as in the proof of Theorem 3.1, we see that $q \in VI(C, A)$.

Now, we prove that $q \in GEP(\tilde{F}, A)$. From (3.22) and the fact that $z_n \in C_n$, we have that $\phi(z_n, w_n) \to 0$ as $n \to \infty$. Therefore, by Lemma 2.1, we have

$$\lim_{n\to\infty} \|z_n - w_n\| = 0. \tag{3.40}$$

From (3.13), (3.16) and (3.40), it is clear that

$$\lim_{n\to\infty} \|x_n - w_n\| = 0. \tag{3.41}$$

Assume that $r_2 = \sup\{\|u_n\|, \|x_n\|\}$. From Lemma 2.6, there exists a continuous, convex and strictly increasing function $g_2 : [0, 2r_2] \longrightarrow [0, \infty)$ such that $g_2(0) = 0$ and

$$g_2(\|u_n - x_n\|) \leq \phi(u_n, x_n). \tag{3.42}$$

Since $u_n = K_{r_n}(x_n)$ and by using (3.36), (3.42) and condition (5) of Lemma 2.7, we have that

$$\begin{aligned} g_2(\|u_n - x_n\|) \leq& \phi(u_n, x_n) \\ \leq& \phi(u, x_n) - \phi(u, u_n) \\ \leq& \phi(u, x_n) - \phi(u, w_n) \\ =& \|u\|^2 - 2\langle u, Jx_n\rangle + \|x_n\|^2 - \|u\|^2 + 2\langle u, Jw_n\rangle - \|w_n\|^2 \\ =& \|x_n\|^2 - \|w_n\|^2 + 2\langle u, Jw_n - Jx_n\rangle \\ \leq& \|x_n\|^2 - \|w_n\|^2 + 2\|u\|\|Jw_n - Jx_n\| \\ \leq& (\|x_n - w_n\| + \|w_n\|)^2 - \|w_n\|^2 + 2\|u\|\|Jw_n - Jx_n\| \\ \leq& \|x_n - w_n\|^2 + 2\|w_n\|\|x_n - w_n\| + 2\|u\|\|Jw_n - Jx_n\|, \end{aligned}$$

from (3.41) and the condition uniformly norm-to-norm continuity of J on bounded sets, we have $\lim_{n\to\infty} g_2(\|u_n - x_n\|) = 0$. Then it is followed from the conditions that g_2 is a strictly increasing and continuous function that $\|u_n - x_n\| \to 0$ as $n \to \infty$. Then

$$\lim_{n\to\infty} \|Ju_n - Jx_n\| \to 0. \tag{3.43}$$

Since $u_n = K_{r_n}x_n$, we conclude that

$$\tilde{F}(u_n, y) + \langle Au_n, y - u_n\rangle + \frac{1}{r_n}\langle y - u_n, Ju_n - Jx_n\rangle \geq 0, \tag{3.44}$$

for all $y \in C$. From the condition (A_2), we have

$$\tilde{F}(y, u_n) \leq -\tilde{F}(u_n, y) \text{ for all } y \in C. \tag{3.45}$$

From (3.44) and (3.45), we have that

$$\tilde{F}(y, u_n) \leq -\tilde{F}(u_n, y) \leq \langle Au_n, y - u_n\rangle + \frac{1}{r_n}\langle y - u_n, Ju_n - Jx_n\rangle,$$

for all $y \in C$. Letting $n \to \infty$, using condition (A_4) and by (3.43), we conclude that

$$\tilde{F}(y, q) \leq \langle Aq, y - q\rangle \quad \text{for all } y \in C. \tag{3.46}$$

Put $y_\lambda = \lambda y + (1-\lambda)q$ for all $y \in C$ and $\lambda \in (0, 1)$. Now from the conditions (A_1), (A_4), the inequality (3.46), the monotonicity of A and the convexity of $\tilde{F}$, we have

$$\begin{aligned} 0 =& \tilde{F}(y_\lambda, y_\lambda) + \langle Ay_\lambda, y_\lambda - y_\lambda\rangle \\ &\leq \lambda\tilde{F}(y_\lambda, y) + (1-\lambda)\tilde{F}(y_\lambda, q) + \langle Ay_\lambda, \lambda y + (1-\lambda)q - y_\lambda\rangle \\ &= \lambda\tilde{F}(y_\lambda, y) + (1-\lambda)\tilde{F}(y_\lambda, q) + \lambda\langle Ay_\lambda, y - y_\lambda\rangle + (1-\lambda)\langle Ay_\lambda, q - y_\lambda\rangle \\ &= \lambda\tilde{F}(y_\lambda, y) + (1-\lambda)\tilde{F}(y_\lambda, q) + \lambda\langle Ay_\lambda, y - y_\lambda\rangle + (1-\lambda)\langle Ay_\lambda - Aq, q - y_\lambda\rangle \\ &+ (1-\lambda)\langle Aq, q - y_\lambda\rangle \\ &\leq \lambda\tilde{F}(y_\lambda, y) + \lambda\langle Ay_\lambda, y - y_\lambda\rangle, \end{aligned}$$

for all $y \in C$. So $0 \leq \tilde{F}(y_\lambda, y) + \langle Ay_\lambda, y - y_\lambda \rangle$. Now by taking the limit as $\lambda \to 0$ and by using the condition (A_3), it follows that $0 \leq \tilde{F}(q, y) + \langle Aq, y - q \rangle$ for all $y \in C$. Therefore $q \in GEP(\tilde{F}, A)$.

Now, we show that $q \in F(f)$. Let $r_3 = \sup\{\|w_n\|, \|f(x_n)\|\}$, hence, in a similar way with (3.39), there exists a continuous, convex and strictly increasing function $g_3 : [0, 2r_3] \longrightarrow [0, \infty)$ with $g_3(0) = 0$, such that

$$\phi(\hat{u}, x_{n+1}) \leq \phi(\hat{u}, x_n) - \alpha_{n,2}\alpha_{n,4} g_3(\|Jf(x_n) - Jw_n\|),$$

hence

$$\alpha_{n,2}\alpha_{n,4} g_3(\|Jf(x_n) - Jw_n\|) \leq \phi(\hat{u}, x_{n+1}) - \phi(\hat{u}, x_n).$$

Let $n \longrightarrow \infty$ and using our assumptions, we obtain

$$\lim_{n\to\infty} g_3(\|Jf(x_n) - Jw_n\|) = 0,$$

since g_3 is a continuous function, it is easy to see that

$$\lim_{n\to\infty} \|Jf(x_n) - Jw_n\| = 0. \tag{3.47}$$

Therefore

$$\lim_{n\to\infty} \|f(x_n) - w_n\| = \lim_{n\to\infty} \|J^{-1}(Jf(x_n)) - J^{-1}(Jw_n)\| = 0, \tag{3.48}$$

because J^{-1} is uniformly norm-to-norm continuous on bounded sets. From (3.41) and (3.48), we conclude that

$$\|f(x_n) - x_n\| \leq \|f(x_n) - w_n\| + \|w_n - x_n\| \to 0 \quad as\ n \to \infty,$$

and since $x_n \rightharpoonup q$, then $q \in \hat{F(f)} = F(f)$. Hence $\{x_n\}$ is strongly convergent to a point $q \in \Omega$, and also we have $q = \Pi_{VI(C,A)\cap GEP(\tilde{F},A)} \circ f(q)$.

4. Numerical example

Now, some examples are given to illustrate Theorem 3.2. Then the behavior of the sequences $\{x_n\}, \{y_n\}, \{z_n\}$ and $\{w_n\}$ are investigated which were generated by the algorithm (3.30).

Example. Let $E = \mathbb{R}$, $C = [-5, 5]$, $A = I$, $\lambda_n = \frac{1}{n}$, $c = 1$, $\alpha = 1$ and f be a self-mapping on C defined by $f(x) = \frac{x}{3}$ for all $x \in C$. Consider the function $\tilde{F} : C \times C \to \mathbb{R}$ defined by

$$\tilde{F}(u, y) := 16y^2 + 9uy - 25u^2,$$

for all u, $y \in C$. We see that $\tilde{F}$ satisfies the conditions (A1) - (A4) as follows:

(A1) $\tilde{F}(u, u) = 16u^2 + 9u^2 - 25u^2 = 0$ for all $u \in [-5, 5]$,

(A2) $\tilde{F}$ is monotone, because $\tilde{F}(u, y) + \tilde{F}(y, u) = -9(u - y)^2 \leq 0$ for all $y, u \in [-5, 5]$,

(A3) for each $u, y, z \in [-5, 5]$,

$$\begin{aligned}
\lim_{\lambda\to 0} \tilde{F}(\lambda z + (1-\lambda)u, y) &= \lim_{\lambda\to 0} (16y^2 + 9(\lambda z + (1-\lambda)u)y - 25(\lambda z + (1-\lambda)u)^2) \\
&= 16y^2 + 9uy - 25u^2 \\
&= \tilde{F}(u, y).
\end{aligned}$$

(A4) For each $u \in [-5, 5]$, $y \to (16y^2+9uy-25u^2)$ is convex and lower semicontinuous. Let $u \in K_r x$, hence, we conclude from Lemma 2.7 that

$$\tilde{F}(u, y) + \langle Au, y - u\rangle + \frac{1}{r}\langle y - u, Ju - Jx\rangle \geq 0,$$

for all $y \in [-5, 5]$ and $r > 0$, i.e.,

$$\begin{aligned} 0 \leq 16ry^2 + 9ruy - 25ru^2 &+ ruy - ru^2 + uy - u^2 + ux - xy \\ &= 16ry^2 + (10ru + u - x)y - 26ru^2 - u^2 + ux. \end{aligned}$$

Let $a = 16r$, $b = 10ru + u - x$ and $c = -26ru^2 - u^2 + ux$. Then, we have that $\triangle = b^2 - 4ac \leq 0$, i.e.,

$$\begin{aligned} 0 \geq (10ru + u - x)^2 &- 64r(-26ru^2 - u^2 + ux) \\ &= 1764r^2u^2 + 84ru^2 + u^2 - 84rux - 2ux + x^2 \\ &= ((42r + 1)u - x)^2. \end{aligned}$$

It follows that $u = \frac{x}{42r+1}$. We conclude from Lemma 2.7 that K_r is single valued. Hence, $K_r x = \frac{x}{42r+1}$. Now by applying it in Theorem 3.2, we have that $u_n = \frac{x_n}{42r_n+1}$ where $\{x_n\}$ is a sequence generated by the algorithm (3.30). Since $F(K_{r_n}) = \{0\}$, from condition (3) of Lemma 2.7, we have $GEP(\tilde{F}, I) = \{0\}$.
Obviously, $F(f) = \{0\}$ and $\phi(0, f(x)) \leq \phi(0, x)$, for all $x \in C$. Now, let $x_n \rightharpoonup q$ and also $\lim_{n\to\infty}(f(x_n) - x_n) = 0$, hence $q = 0$ and $\hat{F(f)} = \{0\} = F(f)$. Therefore, f is a relatively nonexpansive mapping. Moreover, it is obvious that $0 \in VI(C, I)$. Therefore, $0 = \Pi_{\{0\}} of(0) = \Pi_{VI(C,I)\cap GEP(\tilde{F},I)} of(0)$.

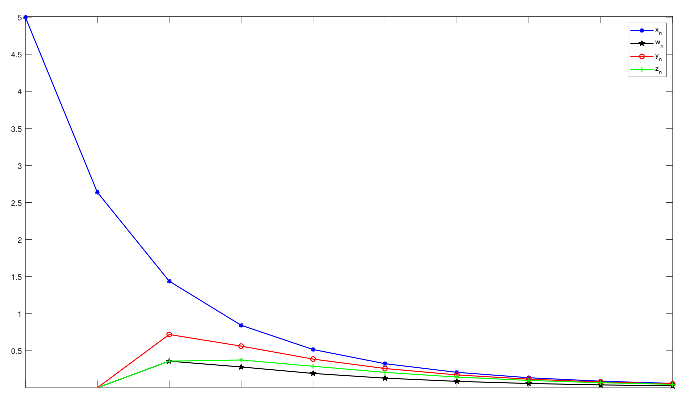

Figure 1. Convergence behavior of generated sequences by Example 4

Next, assume that

$$\alpha_{n,1} = \frac{1}{4} + \frac{1}{4n},\ \alpha_{n,2} = \frac{1}{4} - \frac{1}{6n},\ \alpha_{n,3} = \frac{1}{4} + \frac{1}{12n},\ \alpha_{n,4} = \frac{1}{4} - \frac{1}{6n},\ r_n = \frac{1}{42},$$

for all $n \in \mathbb{N}$ and $u_0 = 0$. So clearly $\{\alpha_{n,i}\}_{i=1}^{4}$ satisfy in the conditions of Theorem 3.2. Since $x_n \in C$, we have

$$\begin{cases} w_n = \Pi_C J^{-1}(u_n - \frac{1}{n}u_n) = \frac{n-1}{n}u_n = \frac{n-1}{2n}x_n, \\ y_n = \Pi_C J^{-1}(x_n - \frac{1}{n}x_n) = \Pi_C \frac{n-1}{n}x_n = \frac{n-1}{n}x_n, \\ C_n = \{v \in C : |v - w_n| \le |v - x_n|\}, \\ z_n = \Pi_{C_n} J^{-1}(y_n - \frac{1}{n}y_n) = \frac{n-1}{n}y_n = (\frac{n-1}{n})^2 x_n, \\ x_{n+1} = \Pi_C J^{-1}((\frac{1}{4} + \frac{1}{4n})x_n + (\frac{1}{4} - \frac{1}{6n})\frac{1}{3}x_n + (\frac{1}{4} + \frac{1}{12n})(\frac{n-1}{n})^2 x_n \\ \qquad +(\frac{1}{4} - \frac{1}{6n})\frac{n-1}{2n}x_n). \end{cases}$$

See Table 1 and Figure 1 with the initial point $x_1 = 5$ of the sequence $\{x_n\}$.

Table 1. Numerical results of convergence for x_1=5 in Example 4

n	x_n	y_n	z_n	w_n
1	5.0000	0.0000	0.0000	0.0000
2	2.6389	0.0000	0.0000	0.0000
3	1.4386	0.3596	0.5547	0.3596
.	.	.	.	.
.	.	.	.	.
.	.	.	.	.
13	0.0178	0.0163	0.0150	0.0082
14	0.0120	0.0111	0.0103	0.0056
15	0.0082	0.0076	0.0070	0.0038
.	.	.	.	.
.	.	.	.	.
.	.	.	.	.
28	0.0001	0.0001	0.0001	0.0000
29	0.0000	0.0000	0.0000	0.0000
30	0.0000	0.0000	0.0000	0.0000

References

[1] R.P. Agarwal, D. O′Regan, D.R. Sahu, *Fixed Point Theory for Lipschitzian-Type Mappings With Applications*, vol. 6, Springer, New York, 2009.

[2] Ya.I. Alber, *Metric and generalized projection operators in Banach spaces: Properties and applications*, In: Kartsatos AG (ed.), Theory Appl Nonlinear Operat Accretive Monotone Type, Marcel Dekker, New York, **178**(1996), 15-50.

[3] B. Beauzamy, *Introduction to Banach Spaces and Their Geometry*, In: North-Holland Mathematics Studies, North-Holland Publishing Co., Amsterdam, **68**(1985).

[4] D. Butnariu, S. Reich, A.J. Zaslavski, *Asymptotic behavior of relatively nonexpansive operators in Banach spaces*, J. Appl. Anal., **7**(2001), 151-174.

[5] D. Butnariu, S. Reich, A.J. Zaslavaski, *Weak convergence of orbits of nonlinear operators in reflexive Banach spaces*, Numer Funct. Anal. Optim, **24**(2003), 489-508.

[6] G. Cai, A. Gibali, O.S. Iyiola, *A new double-projection method for solving variational inequalities in Banach spaces*, J. Optim. Theory Appl., **178**(2018), 219-239.

[7] Y. Cens, S. Reich, *Iterations of paracontractions and firmly nonexpansive operators with applications to feasibility and optimization*, Optimization, **37**(1996), 323-339, doi:10.1080/02331939608844225.
[8] Y. Censor, A. Gibali, S. Reich, *Two extensions of Korpelevich's extragradient method for solving the variational inequality problem in Euclidean space*, Technical Report, 2010.
[9] Y. Censor, A. Gibali, S. Reich, *The subgradient extragradient method for solving variational inequalities in Hilbert space*, J. Optim. Theory Appl., **148**(2011), 318-335.
[10] I. Cioranescu, *Geometry of Banach Spaces, Duality Mappings and Nonlinear Problems*, Kluwer, Dordrecht, 1990.
[11] M. Ghadampour, D. O′Regan, E. Soori, R.P. Agarwal, *A generalized strong convergence algorithm in the presence of errors for variational inequality problems in Hilbert spaces*, J. Funct. Spaces, (2021).
[12] N. Hussain, M. Lashkarizadeh Bami, E. Soori, *An implicit method for finding a common fixed point of a representation of nonexpansive mappings in Banach spaces*, Fixed Point Theory Appl, (2014).
[13] S. Kamimura, W. Takahashi, *Strong convergence of a proximal-type algorithm in a Banach space*, SIAM J. Optim., **13**(2002), 938-945.
[14] F. Kohsaka, W. Takahashi, *Strong convergence of an iterative sequence for maximal monotone operators in a Banach space*, Abstr. Appl. Anal., **3**(2004), 239-249.
[15] G.M. Korpelevich, *The extragradient method for finding saddle points and other problems*, Ekonomikai Matematicheskie Metody, **12**(1976), 747-756.
[16] L.O. Jolaoso, A. Taiwo, T.O. Alakoya, *A strong convergence theorem for solving pseudo-monotone variational inequalities using projection methods*, J. Optim. Theory. Appl., **185**(2020), 744-766, doi:10.1007/s10957-020-01672-3.
[17] Y.C. Liou, *Shrinking projection method of proximal-type for a generalized equilibrium problem, a maximal monotone operator and a pair of relatively nonexpansive mappings*, Taiwanese J. Math., **14**(2010), 517-540.
[18] B. Orouji, R.P. Agarwal, E. Soori, D. O′Regan, *The split common null point problem for generalized resolvents and nonexpansive mappings in Banach spaces*, J. Nonlinear Convex Anal., **23**(2022), 165-183.
[19] S. Reich, *Book review: Geometry of Banach spaces, duality mappings and nonlinear problems*, Bull. Amer. Math. Soc, **26**(1992), 367-370.
[20] R.T. Rockafellar, *Monotone operators and the proximal point algorithm*, SIAM J. Control. Optim., **14**(1976), 877-898, doi:10.1137/0314056.
[21] W. Takahashi, *Nonlinear Functional Analysis: Fixed Point Theory and its Applications*, Yokohama Publishers, Yokohama, 2000.
[22] Y. Takahashi, K. Hashimoto, M. Kato, *On sharp uniform convexity, smoothness, and strong type, cotype inegualities*, J. Nonlinear Convex Anal, **3**(2002), 267-281.
[23] D.V. Thong, D.V. Hieu, *Weak and strong convergence theorems for variational inequality problems*, Numerical Algorithms, **78**(2018), 1045-1060.
[24] D.V. Thong, D.V. Hieu, *Some extragradient-viscosity algorithms for solving variational inequality problems and fixed point problems*, Numer. Algor., **82**(2019), 761-789.
[25] P. Tseng, *A modified forward-backward splitting method for maximal monotone mappings*, SIAM J. Control Optim., **38**(2000), 431-446.
[26] P.T. Vuong, Y. Shehu, *Convergence of an extragradient-type method for variational inequality with applications to optimal control problems*, Numer. Algor., **81**(2019), 269-291.
[27] H.K. Xu, *Inequalities in Banach spaces with applications*, Nonlinear Anal, **16**(2009), 1127-1138.
[28] A.J. Zaslavski, *Numerical Optimization with Computational Errors*, Springer International Publishing, **308**(2016).

Received: July 12, 2021; Accepted: March 7, 2022.