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EXISTENCE PROPERTY OF SOLUTIONS FOR MULTI-ORDER q-DIFFERENCE FBVPs BASED ON CONDENSING OPERATORS AND END-POINT TECHNIQUE

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Abstract. In this research study, we formulate two generalized nonlinear multi-order fractional boundary value problems with the help of quantum difference operators. To investigate the existence property for possible solutions of these q-difference FBVPs, we apply two separate methods motivated by some notions in relation to the measure of noncompactness and end-point technique. The condensing functions and multifunctions having the (AE)-property play an important role in our study. As well as, two examples corresponding to both techniques are provided to ensure the compatibility of the findings numerically.
Key Words and Phrases: q-difference FBVP, boundary value problem, endpoint, condensing operators, fixed point.
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1. Introduction

Because of the quick extensions in fractional calculus, many mathematicians turned to the theory of q-calculus which is an equivalent of traditional calculus without defining the concept of limit and also q refers to quantum. This theory was originally developed by [20, 19] and it includes many practical aspects in areas of hyper-geometric series, the theory of relativity, particle physics, discrete mathematics, quantum mechanics, combinatorics and complex analysis. For a fundamental introduction of the primitive notions of q-calculus, one can refer to [4, 12, 22]. In the early years, for finding positive solutions of given q-difference equations in the non-linear settings, we lead you to study a work published by both El-Shahed and Al-Askar [11] and also a manuscript by Graef and Kong [17].

So later, various mathematical q-difference fractional models of IVPs and BVPs have been abstracted out such as [1, 15, 5, 8, 9, 28, 21, 25, 26, 27, 14, 32, 31] in which different approaches like as the lower-upper solutions technique, fixed-point results and iterative methods have been implemented. Here, we apply another technique to discuss the existence property of solutions for given q-difference FBVP depending on the condensing operators and measure of noncompactness.

In 2014, Ahmad, Nieto, Alsaedi and Al-Hutami [3] turned to a q-sequential equation in the nonlinear case via four-point q-integral conditions displayed as

$$\begin{cases} {}^{C}_{q}\mathfrak{D}^{\varsigma_1}_{0^+}({}^{C}_{q}\mathfrak{D}^{\varsigma_2}_{0^+} + \theta)\mu(r) = \mathfrak{T}_*(r, \mu(r)), \ (r \in [0, 1], \ q \in (0, 1)), \\ \\ \mu(0) = c_1 {}^{R}_{q}\mathfrak{J}^{\sigma-1}_{0^+}\mu(\zeta_1), \ \mu(1) = c_2 {}^{R}_{q}\mathfrak{J}^{\sigma-1}_{0^+}\mu(\zeta_2) \end{cases}$$

so that $\varsigma_1, \varsigma_2 \in (0, 1)$, $\zeta_1, \zeta_2 \in (0, 1)$, $\sigma > 2$ and $\theta, c_1, c_2 \in \mathbb{R}$. As well as, $\mathfrak{T}_* : [0, 1] \times \mathbb{R} \to \mathbb{R}$ is continuous and ${}^{R}_{q}\mathfrak{J}^{\sigma-1}_{0^+}$ indicates the q-RL-integral. These mathematicians extracted different qualitative aspects of solutions for abive q-FBVP by means of the classical approaches which are available in the fixed-point theory. In 2015, Etemad et al. [13] focused on the new four-point three-term q-difference FBVP

$$\begin{cases} ({}^{C}_{q}\mathfrak{D}^{\varsigma}_{0^+}\mu)(r) = \mathfrak{T}(r, \mu(r), {}^{C}_{q}\mathfrak{D}^{1}_{0^+}\mu(r)), \ 0 < q < 1, \\ \\ a_1\mu(0) + \zeta_1 {}^{C}_{q}\mathfrak{D}^{1}_{0^+}\mu(0) = c_1 {}^{R}_{q}\mathfrak{J}^{\beta}_{0^+}\mu(\xi_1) = c_1 \displaystyle\int_0^{\xi_1} \frac{(\xi_1 - qv)^{(\beta-1)}}{\Gamma_q(\beta)}\mu(v)d_qv, \\ \\ a_2\mu(1) + \zeta_2 {}^{C}_{q}\mathfrak{D}^{1}_{0^+}\mu(1) = c_2 {}^{R}_{q}\mathfrak{J}^{\beta}_{0^+}\mu(\xi_2) = c_2 \displaystyle\int_0^{\xi_2} \frac{(\xi_2 - qv)^{(\beta-1)}}{\Gamma_q(\beta)}\mu(v)d_qv, \end{cases}$$

where $0 \le r \le 1$, $1 < \varsigma \le 2$, $\beta \in (0, 2]$, $a_1, a_2, \zeta_1, \zeta_2, c_1, c_2 \in \mathbb{R}$ and $\xi_1, \xi_2 \in (0, 1)$ via $\xi_1 < \xi_2$. In 2019, two mathematicians named Ntouyas and Samei [24] devoted their attention to investigating the existence property of solutions for a multi-term q-integro-difference FBVP

$$\begin{cases} {}^{C}_{q}\mathfrak{D}^{\varsigma}_{0^+}\mu(r) = \mathfrak{T}(r, \mu(r), (\psi_1\mu)(r), (\psi_2\mu)(r), {}^{C}_{q}\mathfrak{D}^{\varsigma_1}_{0^+}\mu(r), {}^{C}_{q}\mathfrak{D}^{\varsigma_2}_{0^+}\mu(r), \ldots, {}^{C}_{q}\mathfrak{D}^{\varsigma_k}_{0^+}\mu(r)) \\ \\ \mu(0) + c_1\mu(1) = 0, \ \mu'(0) + c_2\mu'(1) = 0, \end{cases}$$

where $r \in [0, 1]$, $q \in (0, 1)$, $1 < \varsigma < 2$, $\varsigma_j \in (0, 1)$ with $j = 1, 2, \ldots, k$, $c_1, c_2 \neq -1$, ψ_n are formulated as $(\psi_n\mu)(r) = \int_0^r z_n(r, v)\mu(v)\,\mathrm{d}_qv$ for $n = 1, 2$ and $\mathfrak{T} : [0, 1] \times \mathbb{R}^{k+3} \to \mathbb{R}$ is continuous with respect to all variables [24].

In 2020, Phuong, Sakar, Etemad and Rezapour [29] formulated a novel extended configuration of the Caputo q-multi-integro-difference equation via two nonlinearity via q-multi-order-integrals conditions

$$\begin{cases} \big(\xi {}^{C}_{q}\mathfrak{D}^{\varsigma}_{0^+} - (\xi + 1) {}^{R}_{q}\mathfrak{J}^{\delta_1}_{0^+} - (\xi + 2) {}^{R}_{q}\mathfrak{J}^{\delta_2}_{0^+}\big)\mu(r) = c_1 {}^{R}_{q}\mathfrak{J}^{\gamma_1}_{0^+}\mathfrak{T}_1(r, \mu(r)) + c_2 {}^{R}_{q}\mathfrak{J}^{\gamma_2}_{0^+}\mathfrak{T}_2(r, \mu(r)) \\ \\ \mu(0) = 0, \ \zeta {}^{R}_{q}\mathfrak{J}^{\theta_1}_{0^+}\mu(1) + (\zeta + 1) {}^{R}_{q}\mathfrak{J}^{\theta_2}_{0^+}\mu(1) + (\zeta + 2) {}^{R}_{q}\mathfrak{J}^{\theta_3}_{0^+}\mu(1) = 0, \end{cases}$$

in which $r \in [0,1]$, $\varsigma \in (1,2)$, $\delta_1, \delta_2, \gamma_1, \gamma_2 \in (0,1)$, $\theta_1, \theta_2, \theta_3, \xi, \zeta > 0$ and $c_1, c_2 \in \mathbb{R}^{\geq 0}$.

In this paper, stimulated by aforesaid q-FBVPs, we discuss a structure of the Caputo quantum difference FBVP (or Cap-q-difference FBVP) in the nonlinear settings via 3-point-sum q-integro-difference conditions

$$\begin{cases} {}^{C}_{q}\mathfrak{D}^{\varsigma}_{0^+}\mu(r) = \mathfrak{T}_*(r, \mu(r)), \ (\varsigma \in (2,3), \ q \in (0,1)), \\ \mu(0) + \mu(\zeta) = \sum_{j=1}^{k} \alpha_j \, {}^{R}_{q}\mathfrak{I}^{\sigma_j}_{0^+}\mu(1), \ (\alpha_j \in \mathbb{R}^{>0}), \\ {}^{C}_{q}\mathfrak{D}^{\varrho}_{0^+}\mu(0) + {}^{C}_{q}\mathfrak{D}^{\varrho}_{0^+}\mu(\zeta) = \sum_{j=1}^{k} \beta_j \, {}^{R}_{q}\mathfrak{I}^{\sigma_j}_{0^+}\mu(1), \ (\beta_j \in \mathbb{R}^{>0}), \\ {}^{C}_{q}\mathfrak{D}^{2}_{0^+}\mu(0) + {}^{C}_{q}\mathfrak{D}^{2}_{0^+}\mu(\zeta) = \sum_{j=1}^{k} \gamma_j \, {}^{R}_{q}\mathfrak{I}^{\sigma_j}_{0^+}\left[{}^{C}_{q}\mathfrak{D}^{2}_{0^+}\mu(1)\right], \ (\gamma_j \in \mathbb{R}^{>0}), \end{cases} \tag{1.1}$$

where $r \in \mathcal{O} = [0,1]$, $\zeta \in (0,1)$, $\varrho \in (1,2)$, and for $j = 1, 2, \ldots, k$, $\sigma_j > 0$. As the same way, the operators ${}^{C}_{q}\mathfrak{D}^{(\cdot)}_{0^+}$ ${}^{R}_{q}\mathfrak{I}^{(\cdot)}_{0^+}$ display the Cap-q-derivative and the RL-q-integral. The mapping $\mathfrak{T}_* : \mathcal{O} \times \mathbb{R} \to \mathbb{R}$ is continuous. Besides above problem, we consider the nonlinear Cap-q-difference inclusion FBVP with the same 3-point-sum-q-integro-difference conditions

$$\begin{cases} {}^{C}_{q}\mathfrak{D}^{\varsigma}_{0^+}\mu(r) \in \mathbb{T}_*(r, \mu(r)), \ (\varsigma \in (2,3), \ q \in (0,1)), \\ \mu(0) + \mu(\zeta) = \sum_{j=1}^{k} \alpha_j \, {}^{R}_{q}\mathfrak{I}^{\sigma_j}_{0^+}\mu(1), \ (\alpha_j \in \mathbb{R}^{>0}), \\ {}^{C}_{q}\mathfrak{D}^{\varrho}_{0^+}\mu(0) + {}^{C}_{q}\mathfrak{D}^{\varrho}_{0^+}\mu(\zeta) = \sum_{j=1}^{k} \beta_j \, {}^{R}_{q}\mathfrak{I}^{\sigma_j}_{0^+}\mu(1), \ (\beta_j \in \mathbb{R}^{>0}), \\ {}^{C}_{q}\mathfrak{D}^{2}_{0^+}\mu(0) + {}^{C}_{q}\mathfrak{D}^{2}_{0^+}\mu(\zeta) = \sum_{j=1}^{k} \gamma_j \, {}^{R}_{q}\mathfrak{I}^{\sigma_j}_{0^+}\left[{}^{C}_{q}\mathfrak{D}^{2}_{0^+}\mu(1)\right], \ (\gamma_j \in \mathbb{R}^{>0}), \end{cases} \tag{1.2}$$

so that $r \in \mathcal{O} = [0,1]$, $\zeta \in (0,1)$, $\varrho \in (1,2)$, and for $j = 1, 2, \ldots, k$, $\sigma_j > 0$ and multi-valued mapping $\mathbb{T}_* : \mathcal{O} \times \mathbb{R} \to \mathbb{P}(\mathbb{R})$ is regarded to be arbitrary via some required specifications. These two q-difference FBVPs (1.1) and (1.2) have general formulations with generalized boundary conditions which involve some simple cases studied before by other researchers. Indeed, it is an evident fact that if we take $k = 1$ and $\alpha_1 = \cdots = \alpha_k = \alpha$, $\beta_1 = \cdots = \beta_k = \beta$, $\gamma_1 = \cdots = \gamma_k = \gamma$, $\sigma_1 = \cdots = \sigma_k = \sigma$ and $q \to 1$, then the aforesaid Cap-q-difference FBVP (1.1) is transformed into the usual Caputo FBVP in the following format

$$\begin{cases} {}^{C}\mathfrak{D}_{0+}^{\varsigma}\mu(r) = \mathfrak{T}_{*}(r,\mu(r)), \ (\varsigma \in (2,3)), \\ \mu(0) + \mu(\zeta) = \alpha {}^{R}\mathfrak{J}_{0+}^{\sigma}\mu(1), \ (\alpha \in \mathbb{R}^{>0}), \\ {}^{C}\mathfrak{D}_{0+}^{\varrho}\mu(0) + {}^{C}\mathfrak{D}_{0+}^{\varrho}\mu(\zeta) = \beta {}^{R}\mathfrak{J}_{0+}^{\sigma}\mu(1), \ (\beta \in \mathbb{R}^{>0}), \\ \mu''(0) + \mu''(\zeta) = \gamma {}^{R}\mathfrak{J}_{0+}^{\sigma}\mu''(1), \ (\gamma \in \mathbb{R}^{>0}), \end{cases}$$

in which all q-difference operators are reduced to the usual Caputo and RL-ones. Despite the existence of different standard methods, we are going to get help the measure of noncompactness (KMNC) introduced by Kuratowski for the aims of this manuscript. For this reason, we consider a condensing operator depending on the KMNC and then prove the existence theorem with the help of a fixed-point criterion given by Sadovskii. The next step is devoted to discussing an inclusion formulation of the given Cap-q-difference FBVP as (1.2) in which the proof is done by terms of the approximate end-point property or (AE)-property for some special maps. Notice that these suggested nonlinear Cap-q-difference FBVPs (1.1) and (1.2) have novel generalized mixed q-integro-difference boundary conditions and so they are novel.

This manuscript is presented in such a format: In Sect. 2, some key concepts and theorems are assembled which are required in the rest of the manuscript. In Sect. 3, we investigate the existence property for the possible solutions of the Cap-q-difference FBVP (1.1) by means of the fixed-point result proved by Sadovskii. Next, the Cap-q-difference inclusion FBVP (1.2) is considered and the existence of end-points of the operator caused by the given inclusion FBVP is established by using inequalities and other properties of multi-valued functions which refers to the existence of solutions for the mentioned (1.2). Two examples are given in the same section to see the compatibility of findings in the context of the numerical views. We summarize the findings in Sect. 4.

2. Preliminaries

The primitive notions of q-calculus are collected in this part by assuming $q \in (0,1)$. The q-analogue of $(a_1 - a_2)^k$ is given by

$$(a_1 - a_2)^{(0)} = 1, \ (a_1 - a_2)^{(k)} = \prod_{j=0}^{k-1}(a_1 - a_2 q^j), \ \big(a_1, a_2 \in \mathbb{R}, \, k \in \mathbb{N}_0 := \{0,1,2,\dots\}\big)$$

[30]. Now, if $k = \varsigma \in \mathbb{R}$, then

$$(a_1 - a_2)^{(\varsigma)} = a_1^{\varsigma} \prod_{k=0}^{\infty} \frac{1 - (\frac{a_2}{a_1})q^k}{1 - (\frac{a_2}{a_1})q^{\varsigma+k}}, \ (a_1 \neq 0).$$

On the other side, by taking $a_2 = 0$, we have $a_1^{(\varsigma)} = a_1^{\varsigma}$ [30]. A q-number $[a_1]_q$ for $a_1 \in \mathbb{R}$ is represented by

$$[a_1]_q = \frac{1 - q^{a_1}}{1 - q} = q^{a_1 - 1} + \cdots + q + 1.$$

Accordingly, the Gamma function in the quantum settings is displayed by

$$\Gamma_q(r) = \frac{(1-q)^{(r-1)}}{(1-q)^{r-1}}, \ (r \in \mathbb{R} \backslash (\mathbb{Z}^- \cup \{0\})) \tag{2.1}$$

and $\Gamma_q(r+1) = [r]_q \Gamma_q(r)$ is valid [20, 30].

Definition 2.1. [2] The q-difference-derivative of the supposed function μ is constructed by

$$({}_q\mathfrak{D}_{0^+}\mu)(r) = \frac{\mu(r) - \mu(qr)}{(1-q)r} \tag{2.2}$$

in which $({}_q\mathfrak{D}_{0^+}\mu)(0) = \lim_{r\to 0}({}_q\mathfrak{D}_{0^+}\mu)(r)$.

Simply, we have $({}_q\mathfrak{D}^k_{0^+}\mu)(r) = {}_q\mathfrak{D}_{0^+}({}_q\mathfrak{D}^{k-1}_{0^+}\mu)(r)$ for all $k \in \mathbb{N}$ and $({}_q\mathfrak{D}^0_{0^+}\mu)(r) = \mu(r)$ [2].

Definition 2.2. [2] The q-integral of the supposed function $\mu \in C([0, m_2], \mathbb{R})$ is displayed as

$$({}_q\mathfrak{I}_{0^+}\mu)(r) = \int_0^r \mu(v)\, \mathrm{d}_q v = r(1-q)\sum_{j=0}^{\infty} \mu(rq^j)q^j, \tag{2.3}$$

if the series is absolutely convergent.

Similarly $({}_q\mathfrak{I}^k_{0^+}\mu)(r) = {}_q\mathfrak{I}_{0^+}({}_q\mathfrak{I}^{k-1}_{0^+}\mu)(r)$ for all $k \geq 1$ and $({}_q\mathfrak{I}^0_{0^+}\mu)(r) = \mu(r)$ [2].

Definition 2.3. [2] By letting $a_1 \in [0, a_2]$, the definite q-integral of the supposed function $\mu \in C([0, a_2], \mathbb{R})$ is formulated as

$$\begin{aligned}\int_{a_1}^{a_2} \mu(v)\, \mathrm{d}_q v &= {}_q\mathfrak{I}_{0^+}\mu(a_2) - {}_q\mathfrak{I}_{0^+}\mu(a_1) \\ &= \int_0^{a_2} \mu(v)\, \mathrm{d}_q v - \int_0^{a_1} \mu(v)\, \mathrm{d}_q v \\ &= (1-q)\sum_{j=0}^{\infty}[a_2\mu(a_2q^j) - a_1\mu(a_1q^j)]q^j\end{aligned}$$

if the series exists.

By considering μ as a continuous function at $r = 0$, then $({}_q\mathfrak{I}_{0^+}\, {}_q\mathfrak{D}_{0^+}\mu)(r) = \mu(r) - \mu(0)$ [2]. Furthermore, $({}_q\mathfrak{D}_{0^+}\, {}_q\mathfrak{I}_{0^+}\mu)(r) = \mu(r)$ for all r.

Definition 2.4. [16, 17] The ς^{th}-RL-q-integral of $\mu \in \mathcal{C}_{\mathbb{R}}([0, +\infty))$ is introduced by

$${}^R_q\mathfrak{I}^{\varsigma}_{0^+}\mu(r) = \begin{cases} \dfrac{1}{\Gamma_q(\varsigma)} \displaystyle\int_0^r (r - qv)^{(\varsigma-1)}\mu(v)\, \mathrm{d}_q v, & \varsigma > 0, \\ \mu(r), & \varsigma = 0, \end{cases}$$

if integral exists.

One can simply see that the q-semi-group property satisfies as ${}^{R}_{q}\mathfrak{I}^{\varsigma_1}_{0^+}\,({}^{R}_{q}\mathfrak{I}^{\varsigma_2}_{0^+}\mu)(r) = {}^{R}_{q}\mathfrak{I}^{\varsigma_1+\varsigma_2}_{0^+}\mu(r)$ for $\varsigma_1, \varsigma_2 \geq 0$ [16]. Also, for $\zeta > -1$, we have

$$ {}^{R}_{q}\mathfrak{I}^{\varsigma}_{0^+} r^{\zeta} = \frac{\Gamma_q(\zeta+1)}{\Gamma_q(\zeta+\varsigma+1)} r^{\zeta+\varsigma} \text{ and } {}^{R}_{q}\mathfrak{I}^{\varsigma}_{0^+} 1(r) = \frac{1}{\Gamma_q(\varsigma+1)} r^{\varsigma}, \ (r>0). $$

Definition 2.5. [16, 17] Let $\ell - 1 < \varsigma < \ell$, i.e. $\ell = [\varsigma]+1$. The ς^{th}-Caputo q-derivative of $\mu \in \mathcal{C}^{(\ell)}_{\mathbb{R}}([0,+\infty))$ is displayed as

$$ {}^{C}_{q}\mathfrak{D}^{\varsigma}_{0^+}\mu(r) = \frac{1}{\Gamma_q(\ell-\varsigma)} \int_0^r (r-qv)^{(\ell-\varsigma-1)} \ {}_{q}\mathfrak{D}^{\ell}_{0^+}\mu(v)\, \mathrm{d}_q v $$

if the integral exists.

Note that for $\zeta > -1$, we have

$$ {}^{C}_{q}\mathfrak{D}^{\varsigma}_{0^+} r^{\iota} = \frac{\Gamma_q(\iota+1)}{\Gamma_q(\iota-\varsigma+1)} r^{\iota-\varsigma} \text{ and } {}^{C}_{q}\mathfrak{D}^{\varsigma}_{0^+} 1(r) = 0, \ (r>0). $$

Lemma 2.6. [11] *Let $\ell - 1 < \varsigma < \ell$. Then,*

$$ ({}^{C}_{q}\mathfrak{I}^{\varsigma}_{0^+}\, {}^{C}_{q}\mathfrak{D}^{\varsigma}_{0^+}\mu)(r) = \mu(r) - \sum_{j=0}^{\ell-1} \frac{r^j}{\Gamma_q(j+1)} (\ {}_{q}\mathfrak{D}^{j}_{0^+}\mu)(0). $$

By Lemma 2.6, the general series solution of q-difference FDE ${}^{C}_{q}\mathfrak{D}^{\varsigma}_{0^+}\mu(r) = 0$ is computed as $\mu(r) = \tilde{c}_0 + \tilde{c}_1 r + \tilde{c}_2 r^2 + \cdots + \tilde{c}_{\ell-1} r^{\ell-1}$ via $\tilde{c}_0, \ldots, \tilde{c}_{\ell-1} \in \mathbb{R}$ and $\ell = [\varsigma]+1$ [11]. In this case, we get that

$$ ({}^{R}_{q}\mathfrak{I}^{\varsigma}_{0^+}\, {}^{C}_{q}\mathfrak{D}^{\varsigma}_{0^+}\mu)(r) = \mu(r) + \tilde{c}_0 + \tilde{c}_1 r + \tilde{c}_2 r^2 + \cdots + \tilde{c}_{\ell-1} r^{\ell-1}. $$

In the sequel, we take $\mathfrak{A}$ as a Banach space.

Definition 2.7. [18] Let the set $\mathfrak{O}$ be bounded in $\mathfrak{A}$. The measure of noncompactness Ω due to Kuratowski (KMNC) is presented as

$$ \Omega(\mathfrak{O}) := \operatorname{Inf}\{\epsilon > 0 : \ \mathfrak{O} = \bigcup_{j=1}^{k} \mathfrak{O}_j \text{ and } \operatorname{DIAM}(\mathfrak{O}_j) \leq \epsilon\}, $$

where

$$ \operatorname{DIAM}(\mathfrak{O}_j) = \sup\{|\mu - \mu'| : \mu, \mu' \in \mathfrak{O}_j\} $$

and $0 \leq \Omega(\mathfrak{O}) \leq \operatorname{DIAM}(\mathfrak{O}) \in [0, +\infty)$.

Lemma 2.8. [18] *Let $\mathfrak{O}, \mathfrak{O}_1, \mathfrak{O}_2 \subseteq \mathfrak{A}$ be bounded sets which belong to $\mathfrak{A}$. Then we have these assertions:*

- (Ω1) *if $\mathfrak{O}_1 \subseteq \mathfrak{O}_2$, then $\Omega(\mathfrak{O}_1) \leq \Omega(\mathfrak{O}_2)$;*
- (Ω2) *$\Omega(\ell + \mathfrak{O}) \leq \Omega(\mathfrak{O})$ and $\Omega(\ell\mathfrak{O}) = |\ell|\Omega(\mathfrak{O})$ for all $\ell \in \mathbb{R}$;*
- (Ω3) *$\Omega(\mathfrak{O}_1 + \mathfrak{O}_2) \leq \Omega(\mathfrak{O}_1) + \Omega(\mathfrak{O}_2)$ and $\Omega(\mathfrak{O}_1 \cup \mathfrak{O}_2) \leq \max\{\Omega(\mathfrak{O}_1), \Omega(\mathfrak{O}_2)\}$,*

where $\mathfrak{O}_1 + \mathfrak{O}_2 = \{\mu_1 + \mu_2; \mu_1 \in \mathfrak{O}_1, \mu_2 \in \mathfrak{O}_2\}$.

Lemma 2.9. [23] *For every bounded subset $\mathfrak{O}$ of $\mathfrak{A}$, it is found a countable set $\mathfrak{O}_0$ of $\mathfrak{O}$ such that $\Omega(\mathfrak{O}) \leq 2\,\Omega(\mathfrak{O}_0)$.*

Lemma 2.10. [18] *If $\mathfrak{O} \subseteq \mathcal{C}_{\mathfrak{A}}([a,b])$ is equi-continuous and bounded, then $\Omega(\mathfrak{O}(r))$ is continuous on $[a,b]$ and $\Omega(\mathfrak{O}) = \sup_{r\in[a,b]} \Omega(\mathfrak{O}(r))$.*

Lemma 2.11. [18] *If the set $\mathfrak{O} = \{\mu_n\}_{n\geq 1} \subseteq \mathcal{C}_{\mathfrak{A}}([a,b])$ is countable bounded, then $\Omega(\mathfrak{O}(r))$ is integrable on $[a,b]$ and*

$$\Omega\Big(\Big\{ \int_0^r \mu_n(v)\,\mathrm{d}v\Big\}_{n\geq 1}\Big) \leq 2\int_0^r \Omega(\{\mu_n(v)\}_{n\geq 1})\,\mathrm{d}v.$$

Definition 2.12. [18] The continuous bounded mapping $\mathfrak{T}_* : \mathbb{D} \subset \mathfrak{A} \to \mathfrak{A}$ is termed as condensing if $\Omega(\mathfrak{T}_*(\mathfrak{O})) < \Omega(\mathfrak{O})$ for each bounded closed set $\mathfrak{O} \subseteq \mathbb{D}$.

Theorem 2.13. ([18], *Sadovskii's fixed-point theorem) Let $\mathfrak{O} \subseteq \mathfrak{A}$ be convex bounded closed. Then there is a fixed-point in $\mathfrak{O}$ for the condensing map $\mathfrak{T}_* : \mathfrak{O} \to \mathfrak{O}$.*

Remark 2.14. We assume these notations for the convenience:

$$\mathcal{P}_{bnd}(\mathfrak{A}) := \big\{\mathcal{B} \in \mathfrak{A}|\ \mathcal{B} \text{ is bounded.}\big\},\ \mathcal{P}_{cls}(\mathfrak{A}) := \big\{\mathcal{B} \in \mathfrak{A}|\ \mathcal{B} \text{ is closed.}\big\},$$

$$\mathcal{P}_{cmp}(\mathfrak{A}) := \big\{\mathcal{B} \in \mathfrak{A}|\ \mathcal{B} \text{ is compact.}\big\},\ \mathcal{P}_{cvx}(\mathfrak{A}) := \big\{\mathcal{B} \in \mathfrak{A}|\ \mathcal{B} \text{ is convex.}\big\}.$$

Definition 2.15. [10] The Pompeiu-Hausdorff metric $H_{d_{\mathfrak{A}}} : \mathcal{P}(\mathfrak{A})\times\mathcal{P}(\mathfrak{A}) \to \mathbb{R}\cup\{\infty\}$ is presented by

$$H_{d_{\mathfrak{A}}}(\mathcal{B}_1, \mathcal{B}_2) = \mathrm{Max}\{\mathrm{Sup}_{b_1\in\mathcal{B}_1} d_{\mathfrak{A}}(b_1, \mathcal{B}_2), \mathrm{Sup}_{b_2\in\mathcal{B}_2} d_{\mathfrak{A}}(\mathcal{B}_1, b_2)\}$$

so that $d_{\mathfrak{A}}(\mathcal{B}_1, b_2) = \mathrm{Inf}_{b_1\in\mathcal{B}_1} d_{\mathfrak{A}}(b_1, b_2)$ and $d_{\mathfrak{A}}(b_1, \mathcal{B}_2) = \mathrm{Inf}_{b_2\in\mathcal{B}_2} d_{\mathfrak{A}}(b_1, b_2)$.

Definition 2.16. [10] The multifunction $\mathbb{T}_* : \mathfrak{A} \to \mathcal{P}(\mathfrak{A})$ is upper semi-continuous (u.s.c) if for each $\mu \in \mathfrak{A}$, $\mathbb{T}_*(\mu) \in \mathcal{P}_{cls}(\mathfrak{A})$ and for every open set $\mathbb{U}$ with $\mathbb{T}_*(\mu) \subset \mathbb{U}$, a neighborhood of μ like $\mathcal{G}_0^*$ exists such that $\mathbb{T}_*(\mathcal{G}_0^*) \subset \mathbb{U}$.

We display all selections of $\mathbb{T}_*$ at point $\mu \in \mathcal{C}_{\mathbb{R}}([0,1])$ by

$$S_{\mathbb{T}_*,\mu} := \{\mathfrak{N} \in \mathcal{L}^1_{\mathbb{R}}([0,1]) :\ \mathfrak{N}(r) \in \mathbb{T}_*(r, \mu(r))\},\ (a.e)\, r \in \mathcal{O} = [0,1].$$

As well as, $S_{\mathbb{T}_*,\mu} \neq \emptyset$ if $\mathrm{DIM}(\mathfrak{A}) < \infty$ in which DIM refers to the dimension of $\mathfrak{A}$ [7, 10].

Definition 2.17. [6] $\mu \in \mathfrak{A}$ is termed as an end-point for $\mathbb{T}_* : \mathfrak{A} \to \mathcal{P}(\mathfrak{A})$ if $\mathbb{T}_*(\mu) = \{\mu\}$.

Definition 2.18. [6] $\mathbb{T}_* : \mathfrak{A} \to \mathcal{P}(\mathfrak{A})$ includes an approximate end-point property or (AE)-property if $\mathrm{Inf}_{\mu_1\in\mathfrak{A}}\mathrm{Sup}_{\mu_2\in\mathbb{T}_*(\mu_1)} d_{\mathfrak{A}}(\mu_1, \mu_2) = 0$.

Theorem 2.19. [6] *Let $(\mathfrak{A}, d_{\mathfrak{A}})$ be a complete metric space, $\psi : [0,\infty) \to [0,\infty)$ be u.s.c via $\psi(r) < r$ and $\mathrm{limInf}_{r\to\infty}(r-\psi(r)) > 0$ for all $r > 0$ and $\mathfrak{Q}_* : \mathbb{V} \to \mathcal{P}_{cls,bnd}(\mathfrak{A})$ be such that*

$$H_{d_{\mathfrak{A}}}(\mathbb{T}_*\mu_1, \mathbb{T}_*\mu_2) \leq \psi(d_{\mathfrak{A}}(\mu_1, \mu_2))$$

for $\mu_1, \mu_2 \in \mathfrak{A}$. Then it is found an end-point for $\mathbb{T}_*$ uniquely iff $\mathbb{T}_*$ has (AE)-property.

3. On the Existence Property

Let $\mathfrak{A} = \mathcal{C}_{\mathbb{R}}(\mathcal{O})$ be the space of all real-valued continuous functions on $\mathcal{O} = [0,1]$. Simply $\mathfrak{A}$ is a Banach space subject to $\|\mu\|_{\mathfrak{A}} = \mathrm{Sup}_{r\in\mathcal{O}}|\mu(r)|$ for all members $\mu \in \mathfrak{A}$. In the first place, we present the following fundamental lemma which presents a characterization of the configuration of possible solutions for the proposed Cap-q-difference FBVP (1.1).

Remark 3.1. For convenience, we have nonzero constants:

$$
\begin{aligned}
&W_1 = 2 - \sum_{j=1}^{k} \frac{\alpha_j}{\Gamma_q(\sigma_j+1)},\ W_2 = \zeta - \sum_{j=1}^{k} \frac{\alpha_j}{\Gamma_q(\sigma_j+2)},\\
&W_3 = \zeta^2 - \sum_{j=1}^{k} \frac{\alpha_j(1+q)}{\Gamma_q(\sigma_j+3)},\ W_4 = -\sum_{j=1}^{k} \frac{\beta_j}{\Gamma_q(\sigma_j+1)},\\
&W_5 = -\sum_{j=1}^{k} \frac{\beta_j}{\Gamma_q(\sigma_j+2)},\ W_6 = \frac{2\zeta^{2-\varrho}}{\Gamma_q(3-\varrho)} - \sum_{j=1}^{k} \frac{\beta_j(1+q)}{\Gamma_q(\sigma_j+3)},\\
&W_7 = 2(1+q) - \sum_{j=1}^{k} \frac{\gamma_j(1+q)}{\Gamma_q(\sigma_j+1)},\ W_8 = W_2W_4 - W_1W_5,\\
&W_9 = W_3W_4 - W_1W_6,\ W_{10} = W_8 - W_2W_4,\ W_{11} = W_3W_8 - W_2W_9.
\end{aligned} \tag{3.1}
$$

Lemma 3.2. *Let $\phi_* \in \mathfrak{A}$, $\varsigma \in (2,3)$, $\varrho \in (1,2)$, $\zeta \in (0,1)$, $\alpha_j, \beta_j, \gamma_j \in \mathbb{R}^{>0}$ and $\sigma_j > 0$ for $j = 1, 2, \ldots, k$. The solution of the linear Cap-q-difference FBVP*

$$
\begin{cases}
{}^{C}_{q}\mathfrak{D}^{\varsigma}_{0^+}\mu(r) = \phi_*(r),\ (r \in \mathcal{O},\ q \in (0,1)),\\
\mu(0) + \mu(\zeta) = \displaystyle\sum_{j=1}^{k} \alpha_j\, {}^{R}_{q}\mathfrak{I}^{\sigma_j}_{0^+}\mu(1),\\
{}^{C}_{q}\mathfrak{D}^{\varrho}_{0^+}\mu(0) + {}^{C}_{q}\mathfrak{D}^{\varrho}_{0^+}\mu(\zeta) = \displaystyle\sum_{j=1}^{k} \beta_j\, {}^{R}_{q}\mathfrak{I}^{\sigma_j}_{0^+}\mu(1),\\
{}^{C}_{q}\mathfrak{D}^{2}_{0^+}\mu(0) + {}^{C}_{q}\mathfrak{D}^{2}_{0^+}\mu(\zeta) = \displaystyle\sum_{j=1}^{k} \gamma_j\, {}^{R}_{q}\mathfrak{I}^{\sigma_j}_{0^+}\left[{}^{C}_{q}\mathfrak{D}^{2}_{0^+}\mu(1)\right],
\end{cases} \tag{3.2}
$$

is displayed as

$$
\mu(r) = \int_0^r \frac{(r-qv)^{(\varsigma-1)}}{\Gamma_q(\varsigma)}\phi_*(v)\,\mathrm{d}_qv - \frac{\Theta_1(r)}{W_1W_8}\int_0^\zeta \frac{(\zeta-qv)^{(\varsigma-1)}}{\Gamma_q(\varsigma)}\phi_*(v)\,\mathrm{d}_qv
$$

$$+\frac{\Theta_2(r)}{W_8}\int_0^{\zeta}\frac{(\zeta-qv)^{(\varsigma-\varrho-1)}}{\Gamma_q(\varsigma-\varrho)}\phi_*(v)\,\mathrm{d}_qv-\frac{\Theta_3(r)}{W_1W_7W_8}\int_0^{\zeta}\frac{(\zeta-qv)^{(\varsigma-3)}}{\Gamma_q(\varsigma-2)}\phi_*(v)\,\mathrm{d}_qv$$

$$+\frac{\Theta_1(r)}{W_1W_8}\sum_{j=1}^{k}\alpha_j\int_0^1\frac{(1-qv)^{(\varsigma+\sigma_j-1)}}{\Gamma_q(\varsigma+\sigma_j)}\phi_*(v)\,\mathrm{d}_qv$$

$$-\frac{\Theta_2(r)}{W_8}\sum_{j=1}^{k}\beta_j\int_0^1\frac{(1-qv)^{(\varsigma+\sigma_j-1)}}{\Gamma_q(\varsigma+\sigma_j)}\phi_*(v)\,\mathrm{d}_qv$$

$$+\frac{\Theta_3(r)}{W_1W_7W_8}\sum_{j=1}^{k}\gamma_j\int_0^1\frac{(1-qv)^{(\varsigma+\sigma_j-3)}}{\Gamma_q(\varsigma+\sigma_j-2)}\phi_*(v)\,\mathrm{d}_qv, \tag{3.3}$$

where

$$\Theta_1(r)=rW_1W_4+W_{10},\ \Theta_2(r)=rW_1-W_2,\ \Theta_3(r)=r^2W_1W_8-rW_1W_9-W_{11} \tag{3.4}$$

and W_i *are characterized in* (3.1).

Proof. Let μ satisfies the linear Cap-q-difference FBVP (3.2). Then ${}^{C}_{q}\mathfrak{D}^{\varsigma}_{0^+}\mu(r)=\phi_*(r)$. By virtue of $\varsigma\in(2,3)$ and taking ς^{th}-RL-q-integral, we reach

$$\mu(r)=\frac{1}{\Gamma_q(\varsigma)}\int_0^r(r-qv)^{(\varsigma-1)}\phi_*(v)\,\mathrm{d}_qv+\tilde{c}_0+\tilde{c}_1r+\tilde{c}_2r^2, \tag{3.5}$$

in which $\tilde{c}_0,\tilde{c}_1,\tilde{c}_2\in\mathbb{R}$ are unknown coefficients that we have to explore them. It is immediately computed that

$${}^{C}_{q}\mathfrak{D}^{2}_{0^+}\mu(r)=\frac{1}{\Gamma_q(\varsigma-2)}\int_0^r(r-qv)^{(\varsigma-3)}\phi_*(v)\,\mathrm{d}_qv+\tilde{c}_2(1+q), \tag{3.6}$$

$${}^{C}_{q}\mathfrak{D}^{\varrho}_{0^+}\mu(r)=\frac{1}{\Gamma_q(\varsigma-\varrho)}\int_0^r(r-qv)^{(\varsigma-\varrho-1)}\phi_*(v)\,\mathrm{d}_qv+\tilde{c}_2\frac{2}{\Gamma_q(3-\varrho)}r^{2-\varrho}, \tag{3.7}$$

$$\begin{aligned}{}^{R}_{q}\mathfrak{I}^{\sigma_j}_{0^+}\mu(r)&=\frac{1}{\Gamma_q(\varsigma+\sigma_j)}\int_0^r(r-qv)^{(\varsigma+\sigma_j-1)}\phi_*(v)\,\mathrm{d}_qv+\tilde{c}_0\frac{1}{\Gamma_q(\sigma_j+1)}r^{\sigma_j}\\&\quad+\tilde{c}_1\frac{1}{\Gamma_q(\sigma_j+2)}r^{\sigma_j+1}+\tilde{c}_2\frac{1+q}{\Gamma_q(\sigma_j+3)}r^{\sigma_j+2},\end{aligned} \tag{3.8}$$

$${}^{R}_{q}\mathfrak{I}^{\sigma_j}_{0^+}\left[{}^{C}_{q}\mathfrak{D}^{2}_{0^+}\mu(r)\right]=\frac{1}{\Gamma_q(\varsigma+\sigma_j-2)}\int_0^r(r-qv)^{(\varsigma+\sigma_j-3)}\phi_*(v)\,\mathrm{d}_qv+\tilde{c}_2\frac{1+q}{\Gamma_q(\sigma_j+1)}r^{\sigma_j}. \tag{3.9}$$

By considering the constants $W_1, \dots, W_{11}$ given by (3.1) and by virtue the given boundary conditions implemented on (3.6)-(3.9) and by some straightforward computations, we get the coefficients

$$\begin{aligned}\tilde{c}_0 = {} & \frac{W_2}{W_8}\sum_{j=1}^{k}\beta_j\int_0^1\frac{(1-qv)^{(\varsigma+\sigma_j-1)}}{\Gamma_q(\varsigma+\sigma_j)}\phi_*(v)\,\mathrm{d}_qv \\ & -\frac{W_2}{W_8}\int_0^{\zeta}\frac{(\zeta-qv)^{(\varsigma-\varrho-1)}}{\Gamma_q(\varsigma-\varrho)}\phi_*(v)\,\mathrm{d}_qv \\ & +\frac{W_{10}}{W_1W_8}\sum_{j=1}^{k}\alpha_j\int_0^1\frac{(1-qv)^{(\varsigma+\sigma_j-1)}}{\Gamma_q(\varsigma+\sigma_j)}\phi_*(v)\,\mathrm{d}_qv \\ & -\frac{W_{10}}{W_1W_8}\int_0^{\zeta}\frac{(\zeta-qv)^{(\varsigma-1)}}{\Gamma_q(\varsigma)}\phi_*(v)\,\mathrm{d}_qv \\ & +\frac{W_{11}}{W_1W_7W_8}\int_0^{\zeta}\frac{(\zeta-qv)^{(\varsigma-3)}}{\Gamma_q(\varsigma-2)}\phi_*(v)\,\mathrm{d}_qv \\ & -\frac{W_{11}}{W_1W_7W_8}\sum_{j=1}^{k}\gamma_j\int_0^1\frac{(1-qv)^{(\varsigma+\sigma_j-3)}}{\Gamma_q(\varsigma+\sigma_j-2)}\phi_*(v)\,\mathrm{d}_qv\end{aligned} \tag{3.10}$$

and

$$\begin{aligned}\tilde{c}_1 = {} & \frac{W_4}{W_8}\sum_{j=1}^{k}\alpha_j\int_0^1\frac{(1-qv)^{(\varsigma+\sigma_j-1)}}{\Gamma_q(\varsigma+\sigma_j)}\phi_*(v)\,\mathrm{d}_qv \\ & -\frac{W_4}{W_8}\int_0^{\zeta}\frac{(\zeta-qv)^{(\varsigma-1)}}{\Gamma_q(\varsigma)}\phi_*(v)\,\mathrm{d}_qv \\ & +\frac{W_1}{W_8}\int_0^{\zeta}\frac{(\zeta-qv)^{(\varsigma-\varrho-1)}}{\Gamma_q(\varsigma-\varrho)}\phi_*(v)\,\mathrm{d}_qv \\ & -\frac{W_1}{W_8}\sum_{j=1}^{k}\beta_j\int_0^1\frac{(1-qv)^{(\varsigma+\sigma_j-1)}}{\Gamma_q(\varsigma+\sigma_j)}\phi_*(v)\,\mathrm{d}_qv \\ & +\frac{W_9}{W_7W_8}\int_0^{\zeta}\frac{(\zeta-qv)^{(\varsigma-3)}}{\Gamma_q(\varsigma-2)}\phi_*(v)\,\mathrm{d}_qv \\ & -\frac{W_9}{W_7W_8}\sum_{j=1}^{k}\gamma_j\int_0^1\frac{(1-qv)^{(\varsigma+\sigma_j-3)}}{\Gamma_q(\varsigma+\sigma_j-2)}\phi_*(v)\,\mathrm{d}_qv\end{aligned} \tag{3.11}$$

and

$$\tilde{c}_2 = \frac{1}{W_7}\sum_{j=1}^{k}\gamma_j \int_0^1 \frac{(1-qv)^{(\varsigma+\sigma_j-3)}}{\Gamma_q(\varsigma+\sigma_j-2)}\phi_*(v)\,\mathrm{d}_q v - \frac{1}{W_7}\int_0^{\zeta}\frac{(\zeta-qv)^{(\varsigma-3)}}{\Gamma_q(\varsigma-2)}\phi_*(v)\,\mathrm{d}_q v. \tag{3.12}$$

By inserting (3.10), (3.11) and (3.12) into (3.5), we derive equation (3.3) which is the same desired q-integral solution of the linear Cap-q-difference FBVP (3.2). □

In the first phase, we use the notion of KMNC to establish a novel criterion of the existence property for the Cap-q-difference FBVP (1.1) by terms of the aforesaid inequalities in the previous section. Before proceeding it, consider the following estimates:

$$\mathrm{Sup}_{r\in\mathcal{O}}|\Theta_1(r)| = \mathrm{Sup}_{r\in\mathcal{O}}\big(|rW_1W_4| + |W_{10}|\big) = |W_1W_4| + |W_{10}| := \Theta_1^* > 0,$$

$$\mathrm{Sup}_{r\in\mathcal{O}}|\Theta_2(r)| = \mathrm{Sup}_{r\in\mathcal{O}}\big(|rW_1| + |W_2|\big) = |W_1| + |W_2| := \Theta_2^* > 0,$$

$$\begin{aligned}\mathrm{Sup}_{r\in\mathcal{O}}|\Theta_3(r)| &= \mathrm{Sup}_{r\in\mathcal{O}}\big(|r^2W_1W_8| + |rW_1W_9| + |W_{11}|\big)\\ &= |W_1W_8| + |W_1W_9| + |W_{11}| := \Theta_3^* > 0.\end{aligned}$$

Theorem 3.3. *Consider the following assertions on the continuous mapping defined by $\mathfrak{T}_* : \mathcal{O}\times\mathfrak{A}\to\mathbb{R}$:*

(1) *$p \in C(\mathcal{O},\mathbb{R}^+)$ exists such that*

$$|\mathfrak{T}_*\big(r,\mu(r)\big)| \le p(r),\ \ (r\in\mathcal{O},\ \mu\in\mathfrak{A}); \tag{3.13}$$

(2) *$f_{\mathfrak{T}_*} : \mathcal{O}\to\mathbb{R}^+$ exists so that*

$$\Omega\left(\mathfrak{T}_*\big(r,\mathfrak{D}\big)\right) \le f_{\mathfrak{T}_*}(r)\Omega(\mathfrak{D}),\ \ (r\in\mathcal{O}), \tag{3.14}$$

for each bounded set $\mathfrak{D}\subset\mathfrak{A}$.

Then, the given Cap-q-difference FBVP (1.1) *includes a solution on $\mathcal{O}$ if*

$$\begin{aligned}&\frac{\hat{f}_{\mathfrak{T}_*}\big(|W_1W_8| + \Theta_1^*\zeta^{\varsigma}\big)}{|W_1W_8|\Gamma_q(\varsigma+1)} + \frac{\hat{f}_{\mathfrak{T}_*}\Theta_2^*\zeta^{\varsigma-\varrho}}{|W_8|\Gamma_q(\varsigma-\varrho+1)} + \frac{\hat{f}_{\mathfrak{T}_*}\Theta_3^*\zeta^{\varsigma-2}}{|W_1W_7W_8|\Gamma_q(\varsigma-1)}\\ &+ \frac{\hat{f}_{\mathfrak{T}_*}\Theta_1^*}{|W_1W_8|}\sum_{j=1}^{k}\frac{\alpha_j}{\Gamma_q(\varsigma+\sigma_j+1)} + \frac{\hat{f}_{\mathfrak{T}_*}\Theta_2^*}{|W_8|}\sum_{j=1}^{k}\frac{\beta_j}{\Gamma_q(\varsigma+\sigma_j+1)}\\ &+ \frac{\hat{f}_{\mathfrak{T}_*}\Theta_3^*}{|W_1W_7W_8|}\sum_{j=1}^{k}\frac{\gamma_j}{\Gamma_q(\varsigma+\sigma_j-1)} < \frac{1}{4},\end{aligned} \tag{3.15}$$

where $\hat{f}_{\mathfrak{T}_*} = \mathrm{Sup}_{r\in\mathcal{O}}|f_{\mathfrak{T}_*}(r)|$.

Proof. Regarding to the nonlinear multi-order Cap-q-difference FBVP (1.1) and by Lemma 3.2, we display $\mathfrak{B} : \overline{\mathbb{E}_\varepsilon} \to \overline{\mathbb{E}_\varepsilon}$ by

$$\begin{aligned}
\mathfrak{B}(\mu)(r) = & \int_0^r \frac{(r-qv)^{(\varsigma-1)}}{\Gamma_q(\varsigma)} \mathfrak{T}_*\big(v,\mu(v)\big)\,\mathrm{d}_q v \\
& - \frac{\Theta_1(r)}{W_1 W_8} \int_0^\zeta \frac{(\zeta-qv)^{(\varsigma-1)}}{\Gamma_q(\varsigma)} \mathfrak{T}_*\big(v,\mu(v)\big)\,\mathrm{d}_q v \\
& + \frac{\Theta_2(r)}{W_8} \int_0^\zeta \frac{(\zeta-qv)^{(\varsigma-\varrho-1)}}{\Gamma_q(\varsigma-\varrho)} \mathfrak{T}_*\big(v,\mu(v)\big)\,\mathrm{d}_q v \\
& - \frac{\Theta_3(r)}{W_1 W_7 W_8} \int_0^\zeta \frac{(\zeta-qv)^{(\varsigma-3)}}{\Gamma_q(\varsigma-2)} \mathfrak{T}_*\big(v,\mu(v)\big)\,\mathrm{d}_q v \\
& + \frac{\Theta_1(r)}{W_1 W_8} \sum_{j=1}^k \alpha_j \int_0^1 \frac{(1-qv)^{(\varsigma+\sigma_j-1)}}{\Gamma_q(\varsigma+\sigma_j)} \mathfrak{T}_*\big(v,\mu(v)\big)\,\mathrm{d}_q v \\
& - \frac{\Theta_2(r)}{W_8} \sum_{j=1}^k \beta_j \int_0^1 \frac{(1-qv)^{(\varsigma+\sigma_j-1)}}{\Gamma_q(\varsigma+\sigma_j)} \mathfrak{T}_*\big(v,\mu(v)\big)\,\mathrm{d}_q v \\
& + \frac{\Theta_3(r)}{W_1 W_7 W_8} \sum_{j=1}^k \gamma_j \int_0^1 \frac{(1-qv)^{(\varsigma+\sigma_j-3)}}{\Gamma_q(\varsigma+\sigma_j-2)} \mathfrak{T}_*\big(v,\mu(v)\big)\,\mathrm{d}_q v,
\end{aligned} \tag{3.16}$$

where $\overline{\mathbb{E}_\varepsilon} := \{\mu \in \mathfrak{A} : \ \|\mu\|_{\mathfrak{A}} \leq \varepsilon, \ \varepsilon \in \mathbb{R}_+\}$ is convex, closed and bounded and $\Theta_i(r)$ and W_i are displayed in (3.4) and (3.1). In such a situation, the supposed Cap-q-difference FBVP (1.1) is corresponding to the fixed-point problem $\mathfrak{B}\mu = \mu$ and we need to confirm that $\mathfrak{B}$ includes a fixed point, because the existence of fixed-point for $\mathfrak{B}$ will ensures the existence of solution for the supposed Cap-q-difference FBVP (1.1). To validate Theorem 2.13, we check the continuity of $\mathfrak{B}$ on $\overline{\mathbb{E}_\varepsilon}$. Let $\{\mu_n\}_{n\geq 1}$ be contained in $\overline{\mathbb{E}_\varepsilon}$ via $\mu_n \to \mu$ for $\mu \in \overline{\mathbb{E}_\varepsilon}$. Due to the continuity of $\mathfrak{T}_*$ on $\mathcal{O} \times \mathfrak{A}$, we have $\lim_{n\to\infty} \mathfrak{T}_*(r,\mu_n(r)) = \mathfrak{T}_*(r,\mu(r))$. So the dominated convergence theorem due to Lebesgue gives

$$\begin{aligned}
\lim_{n\to\infty} (\mathfrak{B}\mu_n)(r) = & \int_0^r \frac{(r-qv)^{(\varsigma-1)}}{\Gamma_q(\varsigma)} \lim_{n\to\infty} \mathfrak{T}_*\big(v,\mu_n(v)\big)\,\mathrm{d}_q v \\
& - \frac{\Theta_1(r)}{W_1 W_8} \int_0^\zeta \frac{(\zeta-qv)^{(\varsigma-1)}}{\Gamma_q(\varsigma)} \lim_{n\to\infty} \mathfrak{T}_*\big(v,\mu_n(v)\big)\,\mathrm{d}_q v \\
& + \frac{\Theta_2(r)}{W_8} \int_0^\zeta \frac{(\zeta-qv)^{(\varsigma-\varrho-1)}}{\Gamma_q(\varsigma-\varrho)} \lim_{n\to\infty} \mathfrak{T}_*\big(v,\mu_n(v)\big)\,\mathrm{d}_q v
\end{aligned}$$

$$
\begin{aligned}
&- \frac{\Theta_3(r)}{W_1 W_7 W_8} \int_0^\zeta \frac{(\zeta - qv)^{(\varsigma-3)}}{\Gamma_q(\varsigma-2)} \lim_{n\to\infty} \mathfrak{T}_*\big(v, \mu_n(v)\big)\, \mathrm{d}_q v \\
&+ \frac{\Theta_1(r)}{W_1 W_8} \sum_{j=1}^{k} \alpha_j \int_0^1 \frac{(1 - qv)^{(\varsigma+\sigma_j-1)}}{\Gamma_q(\varsigma+\sigma_j)} \lim_{n\to\infty} \mathfrak{T}_*\big(v, \mu_n(v)\big)\, \mathrm{d}_q v \\
&- \frac{\Theta_2(r)}{W_8} \sum_{j=1}^{k} \beta_j \int_0^1 \frac{(1 - qv)^{(\varsigma+\sigma_j-1)}}{\Gamma_q(\varsigma+\sigma_j)} \lim_{n\to\infty} \mathfrak{T}_*\big(v, \mu_n(v)\big)\, \mathrm{d}_q v \\
&+ \frac{\Theta_3(r)}{W_1 W_7 W_8} \sum_{j=1}^{k} \gamma_j \int_0^1 \frac{(1 - qv)^{(\varsigma+\sigma_j-3)}}{\Gamma_q(\varsigma+\sigma_j-2)} \lim_{n\to\infty} \mathfrak{T}_*\big(v, \mu_n(v)\big)\, \mathrm{d}_q v \\
&= (\mathfrak{B}\mu)(r)
\end{aligned}
$$

for any $r \in \mathcal{O}$. Hence, $\lim_{n\to\infty}(\mathfrak{B}\mu_n)(r) = (\mathfrak{B}\mu)(r)$ and thus $\mathfrak{B} \in C(\overline{\mathbb{E}_\varepsilon}, \overline{\mathbb{E}_\varepsilon})$. Now, to check the uniform boundedness of $\mathfrak{B}$ on $\overline{\mathbb{E}_\varepsilon}$, let $\mu \in \overline{\mathbb{E}_\varepsilon}$. By (3.13), we have the estimates

$$
\begin{aligned}
|(\mathfrak{B}\mu)(r)| \le& \int_0^r \frac{(r - qv)^{(\varsigma-1)}}{\Gamma_q(\varsigma)} \big|\mathfrak{T}_*\big(v, \mu(v)\big)\big|\, \mathrm{d}_q v \\
&+ \frac{|\Theta_1(r)|}{|W_1 W_8|} \int_0^\zeta \frac{(\zeta - qv)^{(\varsigma-1)}}{\Gamma_q(\varsigma)} \big|\mathfrak{T}_*\big(v, \mu(v)\big)\big|\, \mathrm{d}_q v \\
&+ \frac{|\Theta_2(r)|}{|W_8|} \int_0^\zeta \frac{(\zeta - qv)^{(\varsigma-\varrho-1)}}{\Gamma_q(\varsigma-\varrho)} \big|\mathfrak{T}_*\big(v, \mu(v)\big)\big|\, \mathrm{d}_q v \\
&+ \frac{|\Theta_3(r)|}{|W_1 W_7 W_8|} \int_0^\zeta \frac{(\zeta - qv)^{(\varsigma-3)}}{\Gamma_q(\varsigma-2)} \big|\mathfrak{T}_*\big(v, \mu(v)\big)\big|\, \mathrm{d}_q v \\
&+ \frac{|\Theta_1(r)|}{|W_1 W_8|} \sum_{j=1}^{k} \alpha_j \int_0^1 \frac{(1 - qv)^{(\varsigma+\sigma_j-1)}}{\Gamma_q(\varsigma+\sigma_j)} \big|\mathfrak{T}_*\big(v, \mu(v)\big)\big|\, \mathrm{d}_q v \\
&+ \frac{|\Theta_2(r)|}{|W_8|} \sum_{j=1}^{k} \beta_j \int_0^1 \frac{(1 - qv)^{(\varsigma+\sigma_j-1)}}{\Gamma_q(\varsigma+\sigma_j)} \big|\mathfrak{T}_*\big(v, \mu(v)\big)\big|\, \mathrm{d}_q v \\
&+ \frac{|\Theta_3(r)|}{|W_1 W_7 W_8|} \sum_{j=1}^{k} \gamma_j \int_0^1 \frac{(1 - qv)^{(\varsigma+\sigma_j-3)}}{\Gamma_q(\varsigma+\sigma_j-2)} \big|\mathfrak{T}_*\big(v, \mu(v)\big)\big|\, \mathrm{d}_q v
\end{aligned}
$$

$$\leq \frac{r^{\varsigma}}{\Gamma_q(\varsigma+1)}p(r) + \frac{\Theta_1^*\zeta^{\varsigma}}{|W_1W_8|\Gamma_q(\varsigma+1)}p(r) + \frac{\Theta_2^*\zeta^{\varsigma-\varrho}}{|W_8|\Gamma_q(\varsigma-\varrho+1)}p(r)$$

$$+ \frac{\Theta_3^*\zeta^{\varsigma-2}}{|W_1W_7W_8|\Gamma_q(\varsigma-1)}p(r) + \frac{\Theta_1^*}{|W_1W_8|}\sum_{j=1}^{k}\frac{\alpha_j}{\Gamma_q(\varsigma+\sigma_j+1)}p(r)$$

$$+ \frac{\Theta_2^*}{|W_8|}\sum_{j=1}^{k}\frac{\beta_j}{\Gamma_q(\varsigma+\sigma_j+1)}p(r) + \frac{\Theta_3^*}{|W_1W_7W_8|}\sum_{j=1}^{k}\frac{\gamma_j}{\Gamma_q(\varsigma+\sigma_j-1)}p(r)$$

for all $r \in \mathcal{O}$. Consequently, $\|\mathfrak{B}\mu\|_{\mathfrak{A}} \leq \mathbb{A}p^* < \infty$, where

$$\begin{aligned}\mathbb{A} = {} & \frac{|W_1W_8| + \Theta_1^*\zeta^{\varsigma}}{|W_1W_8|\Gamma_q(\varsigma+1)} + \frac{\Theta_2^*\zeta^{\varsigma-\varrho}}{|W_8|\Gamma_q(\varsigma-\varrho+1)} + \frac{\Theta_3^*\zeta^{\varsigma-2}}{|W_1W_7W_8|\Gamma_q(\varsigma-1)} \\ & + \frac{\Theta_1^*}{|W_1W_8|}\sum_{j=1}^{k}\frac{\alpha_j}{\Gamma_q(\varsigma+\sigma_j+1)} + \frac{\Theta_2^*}{|W_8|}\sum_{j=1}^{k}\frac{\beta_j}{\Gamma_q(\varsigma+\sigma_j+1)} \\ & + \frac{\Theta_3^*}{|W_1W_7W_8|}\sum_{j=1}^{k}\frac{\gamma_j}{\Gamma_q(\varsigma+\sigma_j-1)}. \end{aligned} \tag{3.17}$$

This guarantees the uniform boundedness of $\mathfrak{B}(\overline{\mathbb{E}_{\varepsilon}})$ in $\mathfrak{A}$. Next, we follow the proof by establishing the equi-continuity of $\mathfrak{B}$. Take $r_1, r_2 \in \mathcal{O}$ via $r_1 < r_2$ and $\mu \in \overline{\mathbb{E}_{\varepsilon}}$. Then, by letting $\sup_{(r,\mu)\in\mathcal{O}\times\overline{\mathbb{E}_{\varepsilon}}} |\mathfrak{T}_*(r,\mu)| = \widetilde{\mathfrak{T}}_* > 0$, we get that

$$\begin{aligned}|(\mathfrak{B}\mu)(r_2) - (\mathfrak{B}\mu)(r_1)| \leq {} & \frac{\widetilde{\mathfrak{T}}_*}{\Gamma_q(\varsigma+1)}\left(|r_2^{\varsigma} - r_1^{\varsigma}| + 2|r_1 - r_2|^{\varsigma}\right) \\ & + \frac{\widetilde{\mathfrak{T}}_*|\Theta_1(r_2) - \Theta_1(r_1)|\zeta^{\varsigma}}{|W_1W_8|\Gamma_q(\varsigma+1)} + \frac{\widetilde{\mathfrak{T}}_*|\Theta_2(r_2) - \Theta_2(r_1)|\zeta^{\varsigma-\varrho}}{|W_8|\Gamma_q(\varsigma-\varrho+1)} \\ & + \frac{\widetilde{\mathfrak{T}}_*|\Theta_3(r_2) - \Theta_3(r_1)|\zeta^{\varsigma-2}}{|W_1W_7W_8|\Gamma_q(\varsigma-1)} \\ & + \frac{\widetilde{\mathfrak{T}}_*|\Theta_1(r_2) - \Theta_1(r_1)|}{|W_1W_8|}\sum_{j=1}^{k}\frac{\alpha_j}{\Gamma_q(\varsigma+\sigma_j+1)} \\ & + \frac{\widetilde{\mathfrak{T}}_*|\Theta_2(r_2) - \Theta_2(r_1)|}{|W_8|}\sum_{j=1}^{k}\frac{\beta_j}{\Gamma_q(\varsigma+\sigma_j+1)} \\ & + \frac{\widetilde{\mathfrak{T}}_*|\Theta_3(r_2) - \Theta_3(r_1)|}{|W_1W_7W_8|}\sum_{j=1}^{k}\frac{\gamma_j}{\Gamma_q(\varsigma+\sigma_j-1)}. \end{aligned} \tag{3.18}$$

We figure out that as $r_1 \to r_2$, (3.18) goes to zero (not depending on $\mu \in \overline{\mathbb{E}_\varepsilon}$) and $\|(\mathfrak{B}\mu)(r_2) - (\mathfrak{B}\mu)(r_1)\|_{\mathfrak{A}} \to 0$ and $\mathfrak{B}$ is equi-continuous. Accordingly, the Arzela-Ascoli criterion gives the complete continuity of $\mathfrak{B}$ and thus the compactness of it on $\overline{\mathbb{E}_\varepsilon}$.

In the following, we check that $\mathfrak{B}$ is condensing on $\overline{\mathbb{E}_\varepsilon}$. In view of Lemma 2.9, for each bounded set $\mathfrak{O} \subset \overline{\mathbb{E}_\varepsilon}$, there is a countable set $\mathfrak{O}_0 = \{\mu_n\}_{n\geq 1} \subset \mathfrak{O}$ provided that $\Omega(\mathfrak{B}(\mathfrak{O})) \leq 2\,\Omega(\mathfrak{B}(\mathfrak{O}_0))$. By virtue of Lemmas 2.8, 2.10 and 2.11, the following inequalities are valid:

$$
\begin{aligned}
\Omega(\mathfrak{B}(\mathfrak{O}(r))) &\leq 2\Omega(\mathfrak{B}(\{\mu_n\}_{n\geq 1})) \\
&\leq 2\int_0^r \frac{(r-qv)^{(\varsigma-1)}}{\Gamma_q(\varsigma)}\Omega\left(\mathfrak{T}_*(v,\{\mu_n(v)\}_{n\geq 1})\right)\,\mathrm{d}_q v \\
&+ \frac{2|\Theta_1(r)|}{|W_1W_8|}\int_0^\zeta \frac{(\zeta-qv)^{(\varsigma-1)}}{\Gamma_q(\varsigma)}\Omega\left(\mathfrak{T}_*(v,\{\mu_n(v)\}_{n\geq 1})\right)\,\mathrm{d}_q v \\
&+ \frac{2|\Theta_2(r)|}{|W_8|}\int_0^\zeta \frac{(\zeta-qv)^{(\varsigma-\varrho-1)}}{\Gamma_q(\varsigma-\varrho)}\Omega\left(\mathfrak{T}_*(v,\{\mu_n(v)\}_{n\geq 1})\right)\,\mathrm{d}_q v \\
&+ \frac{2|\Theta_3(r)|}{|W_1W_7W_8|}\int_0^\zeta \frac{(\zeta-qv)^{(\varsigma-3)}}{\Gamma_q(\varsigma-2)}\Omega\left(\mathfrak{T}_*(v,\{\mu_n(v)\}_{n\geq 1})\right)\,\mathrm{d}_q v \\
&+ \frac{2|\Theta_1(r)|}{|W_1W_8|}\sum_{j=1}^k \alpha_j\int_0^1 \frac{(1-qv)^{(\varsigma+\sigma_j-1)}}{\Gamma_q(\varsigma+\sigma_j)}\Omega\left(\mathfrak{T}_*(v,\{\mu_n(v)\}_{n\geq 1})\right)\,\mathrm{d}_q v \\
&+ \frac{2|\Theta_2(r)|}{|W_8|}\sum_{j=1}^k \beta_j\int_0^1 \frac{(1-qv)^{(\varsigma+\sigma_j-1)}}{\Gamma_q(\varsigma+\sigma_j)}\Omega\left(\mathfrak{T}_*(v,\{\mu_n(v)\}_{n\geq 1})\right)\,\mathrm{d}_q v \\
&+ \frac{2|\Theta_3(r)|}{|W_1W_7W_8|}\sum_{j=1}^k \gamma_j\int_0^1 \frac{(1-qv)^{(\varsigma+\sigma_j-3)}}{\Gamma_q(\varsigma+\sigma_j-2)}\Omega\left(\mathfrak{T}_*(v,\{\mu_n(v)\}_{n\geq 1})\right)\,\mathrm{d}_q v \\
&\leq 4\int_0^r \frac{(r-qv)^{(\varsigma-1)}}{\Gamma_q(\varsigma)} f_{\mathfrak{T}_*}(v)\Omega(\{\mu_n(v)\}_{n\geq 1})\,\mathrm{d}_q v \\
&+ \frac{4|\Theta_1(r)|}{|W_1W_8|}\int_0^\zeta \frac{(\zeta-qv)^{(\varsigma-1)}}{\Gamma_q(\varsigma)} f_{\mathfrak{T}_*}(v)\Omega(\{\mu_n(v)\}_{n\geq 1})\,\mathrm{d}_q v \\
&+ \frac{4|\Theta_2(r)|}{|W_8|}\int_0^\zeta \frac{(\zeta-qv)^{(\varsigma-\varrho-1)}}{\Gamma_q(\varsigma-\varrho)} f_{\mathfrak{T}_*}(v)\Omega(\{\mu_n(v)\}_{n\geq 1})\,\mathrm{d}_q v
\end{aligned}
$$

$$+\frac{4|\Theta_3(r)|}{|W_1W_7W_8|}\int_0^{\zeta}\frac{(\zeta-qv)^{(\varsigma-3)}}{\Gamma_q(\varsigma-2)}f_{\mathfrak{T}_*}(v)\Omega(\{\mu_n(v)\}_{n\geq 1})\,\mathrm{d}_qv$$

$$+\frac{4|\Theta_1(r)|}{|W_1W_8|}\sum_{j=1}^{k}\alpha_j\int_0^{1}\frac{(1-qv)^{(\varsigma+\sigma_j-1)}}{\Gamma_q(\varsigma+\sigma_j)}f_{\mathfrak{T}_*}(v)\Omega(\{\mu_n(v)\}_{n\geq 1})\,\mathrm{d}_qv$$

$$+\frac{4|\Theta_2(r)|}{|W_8|}\sum_{j=1}^{k}\beta_j\int_0^{1}\frac{(1-qv)^{(\varsigma+\sigma_j-1)}}{\Gamma_q(\varsigma+\sigma_j)}f_{\mathfrak{T}_*}(v)\Omega(\{\mu_n(v)\}_{n\geq 1})\,\mathrm{d}_qv$$

$$+\frac{4|\Theta_3(r)|}{|W_1W_7W_8|}\sum_{j=1}^{k}\gamma_j\int_0^{1}\frac{(1-qv)^{(\varsigma+\sigma_j-3)}}{\Gamma_q(\varsigma+\sigma_j-2)}f_{\mathfrak{T}_*}(v)\Omega(\{\mu_n(v)\}_{n\geq 1})\,\mathrm{d}_qv$$

$$\leq\frac{4\hat{f}_{\mathfrak{T}_*}\big(r^{\varsigma}|W_1W_8|+\Theta_1^*\zeta^{\varsigma}\big)\Omega(\mathfrak{O})}{|W_1W_8|\Gamma_q(\varsigma+1)}+\frac{4\hat{f}_{\mathfrak{T}_*}\Theta_2^*\zeta^{\varsigma-\varrho}\Omega(\mathfrak{O})}{|W_8|\Gamma_q(\varsigma-\varrho+1)}$$

$$+\frac{4\hat{f}_{\mathfrak{T}_*}\Theta_3^*\zeta^{\varsigma-2}\Omega(\mathfrak{O})}{|W_1W_7W_8|\Gamma_q(\varsigma-1)}$$

$$+\frac{4\hat{f}_{\mathfrak{T}_*}\Theta_1^*\Omega(\mathfrak{O})}{|W_1W_8|}\sum_{j=1}^{k}\frac{\alpha_j}{\Gamma_q(\varsigma+\sigma_j+1)}+\frac{4\hat{f}_{\mathfrak{T}_*}\Theta_2^*\Omega(\mathfrak{O})}{|W_8|}\sum_{j=1}^{k}\frac{\beta_j}{\Gamma_q(\varsigma+\sigma_j+1)}$$

$$+\frac{4\hat{f}_{\mathfrak{T}_*}\Theta_3^*\Omega(\mathfrak{O})}{|W_1W_7W_8|}\sum_{j=1}^{k}\frac{\gamma_j}{\Gamma_q(\varsigma+\sigma_j-1)}.$$

Hence,

$$\Omega(\mathfrak{B}(\mathfrak{O}))\leq 4\Bigg[\frac{\hat{f}_{\mathfrak{T}_*}\big(|W_1W_8|+\Theta_1^*\zeta^{\varsigma}\big)}{|W_1W_8|\Gamma_q(\varsigma+1)}+\frac{\hat{f}_{\mathfrak{T}_*}\Theta_2^*\zeta^{\varsigma-\varrho}}{|W_8|\Gamma_q(\varsigma-\varrho+1)}+\frac{\hat{f}_{\mathfrak{T}_*}\Theta_3^*\zeta^{\varsigma-2}}{|W_1W_7W_8|\Gamma_q(\varsigma-1)}$$

$$+\frac{\hat{f}_{\mathfrak{T}_*}\Theta_1^*}{|W_1W_8|}\sum_{j=1}^{k}\frac{\alpha_j}{\Gamma_q(\varsigma+\sigma_j+1)}+\frac{\hat{f}_{\mathfrak{T}_*}\Theta_2^*}{|W_8|}\sum_{j=1}^{k}\frac{\beta_j}{\Gamma_q(\varsigma+\sigma_j+1)}$$

$$+\frac{\hat{f}_{\mathfrak{T}_*}\Theta_3^*}{|W_1W_7W_8|}\sum_{j=1}^{k}\frac{\gamma_j}{\Gamma_q(\varsigma+\sigma_j-1)}\Bigg]\Omega(\mathfrak{O}).$$

Then, the condition (3.15) yields $\Omega(\mathfrak{B}(\mathfrak{O}))<\Omega(\mathfrak{O})$ and therefore $\mathfrak{B}$ is condensing on $\overline{\mathbb{E}_{\varepsilon}}$. By resorting to Theorem 2.13, it is figured out that the operator $\mathfrak{B}$ includes a fixed-point belonging to $\overline{\mathbb{E}_{\varepsilon}}$ which is referred to a solution for the nonlinear multi-order Cap-q-difference FBVP (1.1). □

Example 3.4. Based on the supposed multi-order Cap-q-difference FBVP (1.1), consider

$$\begin{cases} {}^{C}_{0.4}\mathfrak{D}^{2.5}_{0^+}\mu(r) = \dfrac{e^{-r}}{3000}\sin(\mu(r)), \\ \mu(0)+\mu(0.01) = \displaystyle\sum_{j=1}^{3}\alpha_j\, {}^{R}_{0.4}\mathfrak{I}^{\sigma_j}_{0^+}\mu(1), \\ {}^{C}_{0.4}\mathfrak{D}^{1.5}_{0^+}\mu(0) + {}^{C}_{0.4}\mathfrak{D}^{1.5}_{0^+}\mu(0.01) = \displaystyle\sum_{j=1}^{3}\beta_j\, {}^{R}_{0.4}\mathfrak{I}^{\sigma_j}_{0^+}\mu(1), \\ {}^{C}_{0.4}\mathfrak{D}^{2}_{0^+}\mu(0) + {}^{C}_{0.4}\mathfrak{D}^{2}_{0^+}\mu(0.01) = \displaystyle\sum_{j=1}^{3}\gamma_j\, {}^{R}_{0.4}\mathfrak{I}^{\sigma_j}_{0^+}\big[{}^{C}_{0.4}\mathfrak{D}^{2}_{0^+}\mu(1)\big], \end{cases} \tag{3.19}$$

where $\varsigma = 2.5$, $q = 0.4$, $\varrho = 1.5$, $\zeta = 0.01$, $k = 3$, $\sigma_1 = 0.2$, $\sigma_2 = 0.4$, $\sigma_3 = 0.6$, $\alpha_1 = 0.02$, $\alpha_2 = 0.04$, $\alpha_3 = 0.06$, $\beta_1 = 0.01$, $\beta_2 = 0.03$, $\beta_3 = 0.05$, $\gamma_1 = 0.07$, $\gamma_2 = 0.08$, $\gamma_3 = 0.09$ and $r \in \mathcal{O} = [0,1]$. Also,

$$W_1 = 1.8718,\ W_2 = -0.0946,\ W_3 = -0.0981,\ W_4 = -0.0961,$$

$$W_5 = -0.0773,\ W_6 = 0.1421,\ W_7 = 2.44198,\ W_8 = 0.1536,$$

$$W_9 = -0.2565,\ W_{10} = 0.1446,\ W_{11} = -0.0392,$$

$$\Theta_1^* = 0.3244,\ \Theta_2^* = 1.9664,\ \Theta_3^* = 0.8068.$$

Define $\mathfrak{T}_* : \mathcal{O}\times\mathbb{R}\to\mathbb{R}$ by

$$\mathfrak{T}_*\big(r,\mu(r)\big) = \frac{e^{-r}}{3000}\sin(\mu(r)).$$

For every $\mu \in \mathbb{R}$, we have

$$\big|\mathfrak{T}_*\big(r,\mu(r)\big)\big| \le \frac{e^{-r}}{3000}\big|\sin(\mu(r))\big| \le \frac{e^{-r}}{3000} = p(r),$$

where $p \in C(\mathcal{O},\mathbb{R}^+)$ is displayed by $p(r) = \dfrac{e^{-r}}{3000}$. As well as, for each $\mu_1,\mu_2 \in \mathbb{R}$, we have

$$\begin{aligned} \big|\mathfrak{T}_*\big(r,\mu_1(r)\big) - \mathfrak{T}_*\big(r,\mu_2(r)\big)\big| &\le \frac{e^{-r}}{3000}\,\big|\sin(\mu_1(r)) - \sin(\mu_2(r))\big| \\ &\le \frac{e^{-r}}{3000}\,\big|\mu_1(z) - \mu_2(z)\big|. \end{aligned}$$

Accordingly, for every bounded set $\mathfrak{O} \subset \mathbb{R}$, we get

$$\Omega\big(\mathfrak{T}_*(r,\mathfrak{O})\big) \le \frac{e^{-r}}{3000}\,\Omega\big(\mathfrak{O}\big) := f_{\mathfrak{T}_*}(r)\Omega\big(\mathfrak{O}\big)$$

such that $\hat{f}_{\mathfrak{T}_*} = \sup_{r\in\mathcal{O}} |f_{\mathfrak{T}_*}(r)| \simeq 0.0003333$. In view of above data, we obtain

$$\frac{\hat{f}_{\mathfrak{T}_*}\big(|W_1W_8| + \Theta_1^*\zeta^{\varsigma}\big)}{|W_1W_8|\Gamma_q(\varsigma+1)} + \frac{\hat{f}_{\mathfrak{T}_*}\Theta_2^*\zeta^{\varsigma-\varrho}}{|W_8|\Gamma_q(\varsigma-\varrho+1)} + \frac{\hat{f}_{\mathfrak{T}_*}\Theta_3^*\zeta^{\varsigma-2}}{|W_1W_7W_8|\Gamma_q(\varsigma-1)}$$

$$+ \frac{\hat{f}_{\mathfrak{T}_*}\Theta_1^*}{|W_1W_8|}\sum_{j=1}^{k}\frac{\alpha_j}{\Gamma_q(\varsigma+\sigma_j+1)} + \frac{\hat{f}_{\mathfrak{T}_*}\Theta_2^*}{|W_8|}\sum_{j=1}^{k}\frac{\beta_j}{\Gamma_q(\varsigma+\sigma_j+1)}$$

$$+ \frac{\hat{f}_{\mathfrak{T}_*}\Theta_3^*}{|W_1W_7W_8|}\sum_{j=1}^{k}\frac{\gamma_j}{\Gamma_q(\varsigma+\sigma_j-1)} \simeq 0.00056634 < \frac{1}{4}.$$

As the condition (3.15) occurs, so Theorem 3.3 is fulfilled which implies that it is found a solution for the multi-order Cap-q-difference FBVP (3.19).

In the current position, we continue our investigation to derive the existence property for the generalized nonlinear multi-order Cap-q-difference inclusion FBVP (1.2) displayed by

$$\begin{cases} {}^{C}_{q}\mathfrak{D}^{\varsigma}_{0^+}\mu(r) \in \mathbb{T}_*(r,\mu(r)), \ (\varsigma\in(2,3), \ q\in(0,1)), \\ \mu(0)+\mu(\zeta) = \displaystyle\sum_{j=1}^{k}\alpha_j\, {}^{R}_{q}\mathfrak{I}^{\sigma_j}_{0^+}\mu(1), \ (\alpha_j\in\mathbb{R}^{>0}), \\ {}^{C}_{q}\mathfrak{D}^{\varrho}_{0^+}\mu(0) + {}^{C}_{q}\mathfrak{D}^{\varrho}_{0^+}\mu(\zeta) = \displaystyle\sum_{j=1}^{k}\beta_j\, {}^{R}_{q}\mathfrak{I}^{\sigma_j}_{0^+}\mu(1), \ (\beta_j\in\mathbb{R}^{>0}), \\ {}^{C}_{q}\mathfrak{D}^{2}_{0^+}\mu(0) + {}^{C}_{q}\mathfrak{D}^{2}_{0^+}\mu(\zeta) = \displaystyle\sum_{j=1}^{k}\gamma_j\, {}^{R}_{q}\mathfrak{I}^{\sigma_j}_{0^+}\big[{}^{C}_{q}\mathfrak{D}^{2}_{0^+}\mu(1)\big], \ (\gamma_j\in\mathbb{R}^{>0}), \end{cases} \tag{3.20}$$

so that $r\in\mathcal{O}=[0,1]$, $\zeta\in(0,1)$, $\varrho\in(1,2)$, and for $j=1,2,\dots,k$, $\sigma_j>0$ and $\mathbb{T}_* : \mathcal{O}\times\mathbb{R}\to\mathbb{P}(\mathbb{R})$ is a multifunction. The notions of the (AE)-property and end-points are key tools in this step.

Definition 3.5. The function $\mu\in AC(\mathcal{O},\mathbb{R})$ is termed as a solution for the multi-order Cap-q-difference inclusion FBVP (3.20) if there is $\mathfrak{N}\in\mathcal{L}^1(\mathcal{O},\mathbb{R})$ subject to $\mathfrak{N}(r)\in\mathbb{T}_*\big(r,\mu(r)\big)$ for almost all $r\in\mathcal{O}$ satisfying the q-boundary conditions in (3.20) and

$$\mu(r) = \int_0^r \frac{(r-qv)^{(\varsigma-1)}}{\Gamma_q(\varsigma)}\mathfrak{N}(v)\,\mathrm{d}_qv - \frac{\Theta_1(r)}{W_1W_8}\int_0^{\zeta}\frac{(\zeta-qv)^{(\varsigma-1)}}{\Gamma_q(\varsigma)}\mathfrak{N}(v)\,\mathrm{d}_qv$$

$$+ \frac{\Theta_2(r)}{W_8}\int_0^{\zeta}\frac{(\zeta-qv)^{(\varsigma-\varrho-1)}}{\Gamma_q(\varsigma-\varrho)}\mathfrak{N}(v)\,\mathrm{d}_qv - \frac{\Theta_3(r)}{W_1W_7W_8}\int_0^{\zeta}\frac{(\zeta-qv)^{(\varsigma-3)}}{\Gamma_q(\varsigma-2)}\mathfrak{N}(v)\,\mathrm{d}_qv$$

$$+\frac{\Theta_1(r)}{W_1W_8}\sum_{j=1}^{k}\alpha_j\int_0^1\frac{(1-qv)^{(\varsigma+\sigma_j-1)}}{\Gamma_q(\varsigma+\sigma_j)}\mathfrak{N}(v)\,\mathrm{d}_qv$$

$$-\frac{\Theta_2(r)}{W_8}\sum_{j=1}^{k}\beta_j\int_0^1\frac{(1-qv)^{(\varsigma+\sigma_j-1)}}{\Gamma_q(\varsigma+\sigma_j)}\mathfrak{N}(v)\,\mathrm{d}_qv$$

$$+\frac{\Theta_3(r)}{W_1W_7W_8}\sum_{j=1}^{k}\gamma_j\int_0^1\frac{(1-qv)^{(\varsigma+\sigma_j-3)}}{\Gamma_q(\varsigma+\sigma_j-2)}\mathfrak{N}(v)\,\mathrm{d}_qv$$

for all $r\in\mathcal{O}$.

We introduce all selections of $\mathbb{T}_*$ for each $\mu\in\mathfrak{A}$ by

$$S_{\mathbb{T}_*,\mu}=\{\mathfrak{N}\in\mathcal{L}^1(\mathcal{O}):\mathfrak{N}(r)\in\mathbb{T}_*(r,\mu(r))\}$$

for almost all $r\in\mathcal{O}$. Also consider the multifunction $\mathbb{X}:\mathfrak{A}\to\mathcal{P}(\mathfrak{A})$ as

$$\mathbb{X}(\mu)=\{\kappa\in\mathfrak{A}:\kappa(r)=\ell(r)\}\tag{3.21}$$

in which

$$\ell(r)=\int_0^r\frac{(r-qv)^{(\varsigma-1)}}{\Gamma_q(\varsigma)}\mathfrak{N}(v)\,\mathrm{d}_qv-\frac{\Theta_1(r)}{W_1W_8}\int_0^\zeta\frac{(\zeta-qv)^{(\varsigma-1)}}{\Gamma_q(\varsigma)}\mathfrak{N}(v)\,\mathrm{d}_qv$$

$$+\frac{\Theta_2(r)}{W_8}\int_0^\zeta\frac{(\zeta-qv)^{(\varsigma-\varrho-1)}}{\Gamma_q(\varsigma-\varrho)}\mathfrak{N}(v)\,\mathrm{d}_qv-\frac{\Theta_3(r)}{W_1W_7W_8}\int_0^\zeta\frac{(\zeta-qv)^{(\varsigma-3)}}{\Gamma_q(\varsigma-2)}\mathfrak{N}(v)\,\mathrm{d}_qv$$

$$+\frac{\Theta_1(r)}{W_1W_8}\sum_{j=1}^{k}\alpha_j\int_0^1\frac{(1-qv)^{(\varsigma+\sigma_j-1)}}{\Gamma_q(\varsigma+\sigma_j)}\mathfrak{N}(v)\,\mathrm{d}_qv$$

$$-\frac{\Theta_2(r)}{W_8}\sum_{j=1}^{k}\beta_j\int_0^1\frac{(1-qv)^{(\varsigma+\sigma_j-1)}}{\Gamma_q(\varsigma+\sigma_j)}\mathfrak{N}(v)\,\mathrm{d}_qv$$

$$+\frac{\Theta_3(r)}{W_1W_7W_8}\sum_{j=1}^{k}\gamma_j\int_0^1\frac{(1-qv)^{(\varsigma+\sigma_j-3)}}{\Gamma_q(\varsigma+\sigma_j-2)}\mathfrak{N}(v)\,\mathrm{d}_qv,\ \mathfrak{N}\in S_{\mathbb{T}_*,\mu}.$$

Theorem 3.6. *Let $\mathbb{T}_*:\mathcal{O}\times\mathfrak{A}\to\mathcal{P}_{cmp}(\mathfrak{A})$ be a multifunction and:*

- ($\mathfrak{C}_1$) *$\psi:[0,\infty)\to[0,\infty)$ be u.s.c increasing via* $\mathrm{LimInf}_{r\to\infty}(r-\psi(r))>0$ *and $\psi(r)<r$ for any $r>0$;*
- ($\mathfrak{C}_2$) *$\mathbb{T}_*$ be integrable bounded subject to $\mathbb{T}_*(\cdot,\mu):\mathcal{O}\to\mathcal{P}_{cmp}(\mathfrak{A})$ is measurable for all $\mu\in\mathfrak{A}$;*
- ($\mathfrak{C}_3$) *$b\in\mathcal{C}(\mathcal{O},[0,\infty))$ exists so that*

$$H_{d_{\mathfrak{A}}}(\mathbb{T}_*(r,\mu_1(r)),\mathbb{T}_*(r,\mu_2(r)))\le b(r)\psi(|\mu_1(r)-\mu_2(r)|)\frac{1}{\widehat{\mathbb{L}}}$$

for all $r \in \mathcal{O}$ *and* $\mu_1, \mu_2 \in \mathfrak{A}$*, where* $\mathrm{Sup}_{r\in\mathcal{O}}|b(r)| = \|b\|$ *and*

$$\widehat{\mathbb{L}} = \Bigg[\frac{|W_1W_8| + \Theta_1^*\zeta^{\varsigma}}{|W_1W_8|\Gamma_q(\varsigma+1)} + \frac{\Theta_2^*\zeta^{\varsigma-\varrho}}{|W_8|\Gamma_q(\varsigma-\varrho+1)} + \frac{\Theta_3^*\zeta^{\varsigma-2}}{|W_1W_7W_8|\Gamma_q(\varsigma-1)}$$
$$+ \frac{\Theta_1^*}{|W_1W_8|}\sum_{j=1}^{k}\frac{\alpha_j}{\Gamma_q(\varsigma+\sigma_j+1)} + \frac{\Theta_2^*}{|W_8|}\sum_{j=1}^{k}\frac{\beta_j}{\Gamma_q(\varsigma+\sigma_j+1)}$$
$$+ \frac{\Theta_3^*}{|W_1W_7W_8|}\sum_{j=1}^{k}\frac{\gamma_j}{\Gamma_q(\varsigma+\sigma_j-1)}\Bigg]\|b\|; \tag{3.22}$$

($\mathfrak{C}_4$) $\mathbb{X}$ *displayed by* (3.21) *involves (AE)-property.*

Then a solution is found for the multi-order Cap-q-difference inclusion FBVP (3.20).

Proof. Our approach in this proof relies on the existence of end-point for the multifunction $\mathbb{X} : \mathfrak{A} \to \mathcal{P}(\mathfrak{A})$ introduced by (3.21). At first, we verify that $\mathbb{X}(\mu)$ is closed for every $\mu \in \mathfrak{A}$. With due attention to ($\mathfrak{C}_2$), $r \mapsto \mathbb{T}_*(r, \mu(r))$ is a measurable closed multifunction for each $\mu \in \mathfrak{A}$. Accordingly, $\mathbb{T}_*$ includes a measurable selection due to $S_{\mathbb{T}_*,\mu} \neq \emptyset$. We claim that $\mathbb{X}(\mu) \subseteq \mathfrak{A}$ is closed for every $\mu \in \mathfrak{A}$. Let $(\mu_n)_{n\geq 1} \subset \mathbb{X}(\mu)$ be provided that $\mu_n \to \mu^*$. For all n, it is found $\mathfrak{N}_n \in S_{\mathbb{T}_*,\mu}$ such that

$$\mu_n(r) = \int_0^r \frac{(r-qv)^{(\varsigma-1)}}{\Gamma_q(\varsigma)}\mathfrak{N}_n(v)\,\mathrm{d}_qv - \frac{\Theta_1(r)}{W_1W_8}\int_0^\zeta \frac{(\zeta-qv)^{(\varsigma-1)}}{\Gamma_q(\varsigma)}\mathfrak{N}_n(v)\,\mathrm{d}_qv$$
$$+ \frac{\Theta_2(r)}{W_8}\int_0^\zeta \frac{(\zeta-qv)^{(\varsigma-\varrho-1)}}{\Gamma_q(\varsigma-\varrho)}\mathfrak{N}_n(v)\,\mathrm{d}_qv - \frac{\Theta_3(r)}{W_1W_7W_8}\int_0^\zeta \frac{(\zeta-qv)^{(\varsigma-3)}}{\Gamma_q(\varsigma-2)}\mathfrak{N}_n(v)\,\mathrm{d}_qv$$
$$+ \frac{\Theta_1(r)}{W_1W_8}\sum_{j=1}^{k}\alpha_j\int_0^1 \frac{(1-qv)^{(\varsigma+\sigma_j-1)}}{\Gamma_q(\varsigma+\sigma_j)}\mathfrak{N}_n(v)\,\mathrm{d}_qv$$
$$- \frac{\Theta_2(r)}{W_8}\sum_{j=1}^{k}\beta_j\int_0^1 \frac{(1-qv)^{(\varsigma+\sigma_j-1)}}{\Gamma_q(\varsigma+\sigma_j)}\mathfrak{N}_n(v)\,\mathrm{d}_qv$$
$$+ \frac{\Theta_3(r)}{W_1W_7W_8}\sum_{j=1}^{k}\gamma_j\int_0^1 \frac{(1-qv)^{(\varsigma+\sigma_j-3)}}{\Gamma_q(\varsigma+\sigma_j-2)}\mathfrak{N}_n(v)\,\mathrm{d}_qv,\ \ \mathfrak{N} \in S_{\mathbb{T}_*,\mu}$$

for almost all $r \in \mathcal{O}$. By the compactness of $\mathbb{T}_*$, we acquire a subsequence $\{\mathfrak{N}_n\}_{n\geq 1}$ approaching to $\mathfrak{N} \in \mathcal{L}^1(\mathcal{O})$.

We take $\mathfrak{N} \in S_{\mathbb{T}_*,\mu}$ and thus for any $r \in \mathcal{O}$,

$$\begin{aligned}
\lim_{n\to\infty} \mu_n(r) &= \int_0^r \frac{(r-qv)^{(\varsigma-1)}}{\Gamma_q(\varsigma)}\mathfrak{N}(v)\,\mathrm{d}_q v - \frac{\Theta_1(r)}{W_1W_8}\int_0^\zeta \frac{(\zeta-qv)^{(\varsigma-1)}}{\Gamma_q(\varsigma)}\mathfrak{N}(v)\,\mathrm{d}_q v \\
&+ \frac{\Theta_2(r)}{W_8}\int_0^\zeta \frac{(\zeta-qv)^{(\varsigma-\varrho-1)}}{\Gamma_q(\varsigma-\varrho)}\mathfrak{N}(v)\,\mathrm{d}_q v \\
&- \frac{\Theta_3(r)}{W_1W_7W_8}\int_0^\zeta \frac{(\zeta-qv)^{(\varsigma-3)}}{\Gamma_q(\varsigma-2)}\mathfrak{N}(v)\,\mathrm{d}_q v \\
&+ \frac{\Theta_1(r)}{W_1W_8}\sum_{j=1}^k \alpha_j \int_0^1 \frac{(1-qv)^{(\varsigma+\sigma_j-1)}}{\Gamma_q(\varsigma+\sigma_j)}\mathfrak{N}(v)\,\mathrm{d}_q v \\
&- \frac{\Theta_2(r)}{W_8}\sum_{j=1}^k \beta_j \int_0^1 \frac{(1-qv)^{(\varsigma+\sigma_j-1)}}{\Gamma_q(\varsigma+\sigma_j)}\mathfrak{N}(v)\,\mathrm{d}_q v \\
&+ \frac{\Theta_3(r)}{W_1W_7W_8}\sum_{j=1}^k \gamma_j \int_0^1 \frac{(1-qv)^{(\varsigma+\sigma_j-3)}}{\Gamma_q(\varsigma+\sigma_j-2)}\mathfrak{N}(v)\,\mathrm{d}_q v,\ \mathfrak{N} \in S_{\mathbb{T}_*,\mu} \\
&= \mu(r)
\end{aligned}$$

So $\mu \in \mathbb{X}$ and $\mathbb{X}$ is closed-valued. In the next phase, it is obvious that $\mathbb{X}(\mu)$ is bounded for each $\mu \in \mathfrak{A}$ due to the compactness of $\mathbb{T}_*$. At last, we investigate the inequality

$$H_{d_{\mathfrak{A}}}(\mathbb{X}(\mu_1), \mathbb{X}(\mu_2)) \le \psi(\|\mu_1-\mu_2\|).$$

Let $\mu_1, \mu_2 \in \mathfrak{A}$ and $y_1 \in \mathbb{X}(\mu_2)$. Take $\mathfrak{N}_1 \in S_{\mathbb{T}_*,\mu_2}$ such that

$$\begin{aligned}
y_1(r) &= \int_0^r \frac{(r-qv)^{(\varsigma-1)}}{\Gamma_q(\varsigma)}\mathfrak{N}_1(v)\,\mathrm{d}_q v - \frac{\Theta_1(r)}{W_1W_8}\int_0^\zeta \frac{(\zeta-qv)^{(\varsigma-1)}}{\Gamma_q(\varsigma)}\mathfrak{N}_1(v)\,\mathrm{d}_q v \\
&+ \frac{\Theta_2(r)}{W_8}\int_0^\zeta \frac{(\zeta-qv)^{(\varsigma-\varrho-1)}}{\Gamma_q(\varsigma-\varrho)}\mathfrak{N}_1(v)\,\mathrm{d}_q v - \frac{\Theta_3(r)}{W_1W_7W_8}\int_0^\zeta \frac{(\zeta-qv)^{(\varsigma-3)}}{\Gamma_q(\varsigma-2)}\mathfrak{N}_1(v)\,\mathrm{d}_q v \\
&+ \frac{\Theta_1(r)}{W_1W_8}\sum_{j=1}^k \alpha_j \int_0^1 \frac{(1-qv)^{(\varsigma+\sigma_j-1)}}{\Gamma_q(\varsigma+\sigma_j)}\mathfrak{N}_1(v)\,\mathrm{d}_q v \\
&- \frac{\Theta_2(r)}{W_8}\sum_{j=1}^k \beta_j \int_0^1 \frac{(1-qv)^{(\varsigma+\sigma_j-1)}}{\Gamma_q(\varsigma+\sigma_j)}\mathfrak{N}_1(v)\,\mathrm{d}_q v
\end{aligned}$$

$$+\frac{\Theta_3(r)}{W_1W_7W_8}\sum_{j=1}^{k}\gamma_j\int_0^1\frac{(1-qv)^{(\varsigma+\sigma_j-3)}}{\Gamma_q(\varsigma+\sigma_j-2)}\mathfrak{N}_1(v)\,\mathrm{d}_qv$$

for almost all $r\in\mathcal{O}$. Due to

$$H_{d_{\mathfrak{A}}}\big(\mathbb{T}_*(z,\mu_1(r)),\mathbb{T}_*(r,\mu_2(r))\big)\le b(r)(\psi\big(\mu_1(r)-\mu_2(r)\big))\frac{1}{\widehat{\mathbb{L}}}$$

for any $r\in\mathcal{O}$, there is $\mathfrak{N}^*\in\mathbb{T}_*(r,\mu_1(r))$ provided

$$|\mathfrak{N}_1(r)-\mathfrak{N}^*|\le b(r)(\psi\big(\mu_1(r)-\mu_2(r)\big))\frac{1}{\widehat{\mathbb{L}}}$$

for a.e. $r\in\mathcal{O}$. Further, consider $\mathfrak{F}:\mathcal{O}\to\mathcal{P}(\mathfrak{A})$ displayed by

$$\mathfrak{F}(r)=\big\{\mathfrak{N}^*\in\mathfrak{A}:|\mathfrak{N}_1(r)-\mathfrak{N}^*|\le b(r)(\psi\big(\mu_1(r)-\mu_2(r)\big))\frac{1}{\widehat{\mathbb{L}}}\big\}.$$

As $\mathfrak{N}_1$ and $w=b(\psi\big(\mu_1-\mu_2\big))\frac{1}{\widehat{\mathbb{L}}}$ are measurable, take $\mathfrak{N}_2(r)\in\mathbb{T}_*(r,\mu_1(r))$ so that

$$|\mathfrak{N}_1(r)-\mathfrak{N}_2(r)|\le b(r)(\psi\big(\mu_1(r)-\mu_2(r)\big))\frac{1}{\widehat{\mathbb{L}}}$$

for a.e. $r\in\mathcal{O}$. Select $y_2\in\mathbb{X}(\mu_1)$ such that

$$\begin{aligned}
y_2(r)={}&\int_0^r\frac{(r-qv)^{(\varsigma-1)}}{\Gamma_q(\varsigma)}\mathfrak{N}_2(v)\,\mathrm{d}_qv-\frac{\Theta_1(r)}{W_1W_8}\int_0^\zeta\frac{(\zeta-qv)^{(\varsigma-1)}}{\Gamma_q(\varsigma)}\mathfrak{N}_2(v)\,\mathrm{d}_qv\\
&+\frac{\Theta_2(r)}{W_8}\int_0^\zeta\frac{(\zeta-qv)^{(\varsigma-\varrho-1)}}{\Gamma_q(\varsigma-\varrho)}\mathfrak{N}_2(v)\,\mathrm{d}_qv-\frac{\Theta_3(r)}{W_1W_7W_8}\int_0^\zeta\frac{(\zeta-qv)^{(\varsigma-3)}}{\Gamma_q(\varsigma-2)}\mathfrak{N}_2(v)\,\mathrm{d}_qv\\
&+\frac{\Theta_1(r)}{W_1W_8}\sum_{j=1}^{k}\alpha_j\int_0^1\frac{(1-qv)^{(\varsigma+\sigma_j-1)}}{\Gamma_q(\varsigma+\sigma_j)}\mathfrak{N}_2(v)\,\mathrm{d}_qv\\
&-\frac{\Theta_2(r)}{W_8}\sum_{j=1}^{k}\beta_j\int_0^1\frac{(1-qv)^{(\varsigma+\sigma_j-1)}}{\Gamma_q(\varsigma+\sigma_j)}\mathfrak{N}_2(v)\,\mathrm{d}_qv\\
&+\frac{\Theta_3(r)}{W_1W_7W_8}\sum_{j=1}^{k}\gamma_j\int_0^1\frac{(1-qv)^{(\varsigma+\sigma_j-3)}}{\Gamma_q(\varsigma+\sigma_j-2)}\mathfrak{N}_2(v)\,\mathrm{d}_qv
\end{aligned}$$

for a.e. $r \in \mathcal{O}$. Therefore, we reach

$$\begin{aligned}
|y_1(r) - y_2(r)| &\leq \int_0^r \frac{(r-qv)^{(\varsigma-1)}}{\Gamma_q(\varsigma)} \big|\mathfrak{N}_1(v) - \mathfrak{N}_2(v)\big| \, \mathrm{d}_q v \\
&+ \frac{|\Theta_1(r)|}{|W_1 W_8|} \int_0^\zeta \frac{(\zeta-qv)^{(\varsigma-1)}}{\Gamma_q(\varsigma)} \big|\mathfrak{N}_1(v) - \mathfrak{N}_2(v)\big| \, \mathrm{d}_q v \\
&+ \frac{|\Theta_2(r)|}{|W_8|} \int_0^\zeta \frac{(\zeta-qv)^{(\varsigma-\varrho-1)}}{\Gamma_q(\varsigma-\varrho)} \big|\mathfrak{N}_1(v) - \mathfrak{N}_2(v)\big| \, \mathrm{d}_q v \\
&+ \frac{|\Theta_3(r)|}{|W_1 W_7 W_8|} \int_0^\zeta \frac{(\zeta-qv)^{(\varsigma-3)}}{\Gamma_q(\varsigma-2)} \big|\mathfrak{N}_1(v) - \mathfrak{N}_2(v)\big| \, \mathrm{d}_q v \\
&+ \frac{|\Theta_1(r)|}{|W_1 W_8|} \sum_{j=1}^k \alpha_j \int_0^1 \frac{(1-qv)^{(\varsigma+\sigma_j-1)}}{\Gamma_q(\varsigma+\sigma_j)} \big|\mathfrak{N}_1(v) - \mathfrak{N}_2(v)\big| \, \mathrm{d}_q v \\
&+ \frac{|\Theta_2(r)|}{|W_8|} \sum_{j=1}^k \beta_j \int_0^1 \frac{(1-qv)^{(\varsigma+\sigma_j-1)}}{\Gamma_q(\varsigma+\sigma_j)} \big|\mathfrak{N}_1(v) - \mathfrak{N}_2(v)\big| \, \mathrm{d}_q v \\
&+ \frac{|\Theta_3(r)|}{|W_1 W_7 W_8|} \sum_{j=1}^k \gamma_j \int_0^1 \frac{(1-qv)^{(\varsigma+\sigma_j-3)}}{\Gamma_q(\varsigma+\sigma_j-2)} \big|\mathfrak{N}_1(v) - \mathfrak{N}_2(v)\big| \, \mathrm{d}_q v \\
&\leq \Bigg[\frac{|W_1 W_8| + \Theta_1^* \zeta^\varsigma}{|W_1 W_8| \Gamma_q(\varsigma+1)} + \frac{\Theta_2^* \zeta^{\varsigma-\varrho}}{|W_8| \Gamma_q(\varsigma-\varrho+1)} + \frac{\Theta_3^* \zeta^{\varsigma-2}}{|W_1 W_7 W_8| \Gamma_q(\varsigma-1)} \\
&+ \frac{\Theta_1^*}{|W_1 W_8|} \sum_{j=1}^k \frac{\alpha_j}{\Gamma_q(\varsigma+\sigma_j+1)} + \frac{\Theta_2^*}{|W_8|} \sum_{j=1}^k \frac{\beta_j}{\Gamma_q(\varsigma+\sigma_j+1)} \\
&+ \frac{\Theta_3^*}{|W_1 W_7 W_8|} \sum_{j=1}^k \frac{\gamma_j}{\Gamma_q(\varsigma+\sigma_j-1)} \Bigg] \|b\| \psi\big(\|\mu_1 - \mu_2\|\big) \frac{1}{\widehat{\mathbb{L}}} \\
&= \widehat{\mathbb{L}} \psi\big(\|\mu_1 - \mu_2\|\big) \frac{1}{\widehat{\mathbb{L}}} = \psi\big(\|\mu_1 - \mu_2\|\big).
\end{aligned}$$

This yields $\|y_1 - y_2\| \leq \psi\big(\|\mu_1 - \mu_2\|\big)$ and thus

$$H_{d_{\mathfrak{A}}}(\mathbb{X}(\mu_1), \mathbb{X}(\mu_2)) \leq \psi\big(\|\mu_1 - \mu_2\|\big), \ \forall \, \mu_1, \mu_2 \in \mathfrak{A}.$$

From $(\mathfrak{C}_4)$, we know that $\mathbb{X}$ has the (AE)-property. By Theorem 2.19, $\mathbb{X}$ possesses an end-point uniquely; that is

$$\exists \, \mu^* \in \mathfrak{A} \text{ s.t. } \mathbb{X}(\mu^*) = \{\mu^*\}.$$

Consequently, it is found a solution like μ^* for the multi-order Cap-q-difference inclusion FBVP (3.20). □

Example 3.7. Based on (3.20) and by using the same data given in Example 3.4, consider

$$\begin{cases} {}^{C}_{0.4}\mathfrak{D}^{2.5}_{0^+}\mu(r) \in \left[0, \dfrac{5e^{-3r}|\sin(\mu(r))|}{2(e^r+10000)(|\sin(\mu(r))|+1)}\right], \\ \mu(0)+\mu(0.01) = \displaystyle\sum_{j=1}^{3} \alpha_j \, {}^{R}_{0.4}\mathfrak{I}^{\sigma_j}_{0^+}\mu(1), \\ {}^{C}_{0.4}\mathfrak{D}^{1.5}_{0^+}\mu(0) + {}^{C}_{0.4}\mathfrak{D}^{1.5}_{0^+}\mu(0.01) = \displaystyle\sum_{j=1}^{3} \beta_j \, {}^{R}_{0.4}\mathfrak{I}^{\sigma_j}_{0^+}\mu(1), \\ {}^{C}_{0.4}\mathfrak{D}^{2}_{0^+}\mu(0) + {}^{C}_{0.4}\mathfrak{D}^{2}_{0^+}\mu(0.01) = \displaystyle\sum_{j=1}^{3} \gamma_j \, {}^{R}_{0.4}\mathfrak{I}^{\sigma_j}_{0^+}\left[{}^{C}_{0.4}\mathfrak{D}^{2}_{0^+}\mu(1)\right], \end{cases} \tag{3.23}$$

where $\varsigma = 2.5$, $q = 0.4$, $\varrho = 1.5$, $\zeta = 0.01$, $k = 3$, $\sigma_1 = 0.2$, $\sigma_2 = 0.4$, $\sigma_3 = 0.6$, $\alpha_1 = 0.02$, $\alpha_2 = 0.04$, $\alpha_3 = 0.06$, $\beta_1 = 0.01$, $\beta_2 = 0.03$, $\beta_3 = 0.05$, $\gamma_1 = 0.07$, $\gamma_2 = 0.08$, $\gamma_3 = 0.09$ and $r \in \mathcal{O} = [0,1]$. Firstly, construct the Banach space $\mathfrak{A} = \{\mu(r) : \mu(r) \in \mathcal{C}_{\mathbb{R}}(\mathcal{O})\}$ via $\|\mu\|_{\mathfrak{A}} = \mathrm{Sup}_{r\in\mathcal{O}}|\mu(r)|$. Further, define $\mathbb{T}_* : \mathcal{O} \times \mathfrak{A} \to \mathcal{P}(\mathfrak{A})$ as

$$\mathbb{T}_*\big(r, \mu(r)\big) = \left[0, \frac{5e^{-3r}|\sin(\mu(r))|}{2(e^r+10000)(|\sin(\mu(r))|+1)}\right]$$

for all $r \in \mathcal{O}$. Next, consider the u.s.c increasing mapping $\psi : [0,\infty) \to [0,\infty)$ by $\psi(r) = \dfrac{r}{2}$ for all $r > 0$ such that $\liminf_{r\to\infty}(r - \psi(r)) > 0$ and $\psi(r) < r$ for all $r > 0$. Now, for any $\mu_1, \mu_2 \in \mathfrak{A}$, we have

$$\begin{aligned} H_{d_{\mathfrak{A}}}\Big(\mathbb{T}_*\big(r,\mu_1(r)\big), \mathbb{T}_*\big(r,\mu_2(r)\big)\Big) &\le \frac{5e^{-3r}}{2(e^r+10000)}\left(|\sin(\mu_1(r)) - \sin(\mu_2(r))|\right) \\ &\le \frac{5e^{-3r}}{2(e^r+10000)}\left(|\mu_1(r) - \mu_2(r)|\right) \\ &= \frac{5e^{-3r}}{e^r+10000}\psi\left(|\mu_1(r) - \mu_2(r)|\right) \\ &\le b(r)\psi\left(|\mu_1(z) - \mu_2(z)|\right)\frac{1}{\widehat{\mathbb{L}}}, \end{aligned}$$

where

$$\widehat{\mathbb{L}} = \Big[\frac{|W_1 W_8| + \Theta_1^* \zeta^{\varsigma}}{|W_1 W_8| \Gamma_q(\varsigma+1)} + \frac{\Theta_2^* \zeta^{\varsigma-\varrho}}{|W_8| \Gamma_q(\varsigma-\varrho+1)} + \frac{\Theta_3^* \zeta^{\varsigma-2}}{|W_1 W_7 W_8| \Gamma_q(\varsigma-1)}$$

$$+ \frac{\Theta_1^*}{|W_1 W_8|} \sum_{j=1}^{k} \frac{\alpha_j}{\Gamma_q(\varsigma+\sigma_j+1)} + \frac{\Theta_2^*}{|W_8|} \sum_{j=1}^{k} \frac{\beta_j}{\Gamma_q(\varsigma+\sigma_j+1)}$$

$$+ \frac{\Theta_3^*}{|W_1 W_7 W_8|} \sum_{j=1}^{k} \frac{\gamma_j}{\Gamma_q(\varsigma+\sigma_j-1)}\Big] \|b\| \simeq 0.0008478958,$$

and we explore $b \in \mathcal{C}(\mathcal{O}, [0, \infty))$ defined by $b(r) = \dfrac{5e^{-3r}}{e^r + 10000}$ for every r. Thus, evidently $\|b\| = \operatorname{Sup}_{r \in \mathcal{O}} |b(r)| \simeq 0.0004999$. At last, we introduce $\mathbb{X} : \mathfrak{A} \to \mathcal{P}(\mathfrak{A})$ by

$$\mathbb{X}(\mu) = \{\kappa \in \mathfrak{A} : \text{there exists } \mathfrak{N} \in S_{\mathbb{T}_*, \mu} \text{ s.t. } \kappa(r) = \ell(r), \forall r \in \mathcal{O}\},$$

where

$$\ell(r) = \int_0^r \frac{(r-0.4v)^{(2.5-1)}}{\Gamma_{0.4}(2.5)} \mathfrak{N}(v)\, \mathrm{d}_{0.4}v - \frac{\Theta_1(r)}{0.2875} \int_0^{0.01} \frac{(0.01-0.4v)^{(2.5-1)}}{\Gamma_{0.4}(2.5)} \mathfrak{N}(v)\, \mathrm{d}_{0.4}v$$

$$+ \frac{\Theta_2(r)}{0.1536} \int_0^{0.01} \frac{(0.01-0.4v)^{(2.5-1.5-1)}}{\Gamma_{0.4}(2.5-1.5)} \mathfrak{N}(v)\, \mathrm{d}_{0.4}v$$

$$- \frac{\Theta_3(r)}{0.70206} \int_0^{0.01} \frac{(0.01-0.4v)^{(2.5-3)}}{\Gamma_{0.4}(2.5-2)} \mathfrak{N}(v)\, \mathrm{d}_{0.4}v$$

$$+ \frac{\Theta_1(r)}{0.2875} \sum_{j=1}^{3} \alpha_j \int_0^1 \frac{(1-0.4v)^{(2.5+\sigma_j-1)}}{\Gamma_{0.4}(2.5+\sigma_j)} \mathfrak{N}(v)\, \mathrm{d}_{0.4}v$$

$$- \frac{\Theta_2(r)}{0.1536} \sum_{j=1}^{3} \beta_j \int_0^1 \frac{(1-0.4v)^{(2.5+\sigma_j-1)}}{\Gamma_{0.4}(2.5+\sigma_j)} \mathfrak{N}(v)\, \mathrm{d}_{0.4}v$$

$$+ \frac{\Theta_3(r)}{0.70206} \sum_{j=1}^{3} \gamma_j \int_0^1 \frac{(1-0.4v)^{(2.5+\sigma_j-3)}}{\Gamma_{0.4}(2.5+\sigma_j-2)} \mathfrak{N}(v)\, \mathrm{d}_{0.4}v,$$

and

$$\Theta_1(r) = -0.1798r + 0.1446, \ \Theta_2(r) = 1.8718r + 0.0946,$$

$$\Theta_3(r) = 0.2875r^2 + 0.4801r + 0.0392.$$

Thus Theorem 3.6 is fulfilled and the multi-order Cap-q-difference inclusion FBVP (3.23) possesses a solution.

4. Conclusion

In this research work, we considered two generalized structures of the nonlinear multi-order Caputo q-difference FBVPs which involve some special cases as mentioned before. By recalling the measure of noncompactness and condensing maps, we discussed the existence property for solutions of the given Cap-q-difference FBVP (1.1). After that, we implemented a new method on the Cap-q-difference inclusion FBVP (1.2) to prove the existence of end-points. In fact, by applying the notion of end-points corresponding to the solutions of the fractional q-system (1.2), we ensured the existence property by means of (AE)-property of the relevant multifunction. As well as, to see the compatibility of our findings, we provided two separate examples for both of methods. As q-calculus and existing modelings based on it are applicable in physics and mechanics particularly, so one can extend different models and numerical techniques by means of the generalized q-operators for various physical processes in the next research studies.

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