Fixed Point Theory, 25(2024), No. 1, 111-114
DOI: 10.24193/fpt-ro.2024.1.07
http://www.math.ubbcluj.ro/~nodeacj/sfptcj.html

FURTHER COMMENTS ON A FIXED POINT THEOREM IN THE FRACTAL SPACE

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Abstract. In this paper, we give some further comments to the counterexample and the results of Bisht in [R.K. Bisht, *Comment on: A new fixed point theorem in the fractal space*, Indagationes Mathematicae, **29**(2018), no.2, 819-823.].
Key Words and Phrases: Fixed point, fractal space, orbitally complete.
2020 Mathematics Subject Classification: 47H10, 54H25.

For being convenient, we use the same terminology and the notations as have been utilized in [3]. In [3], Ri utilized the following lemma to prove Theorem 2, which is the main result of [3].

Lemma 1 ([3], Lemma 2.2)**.** *Assume that the following conditions hold:*

(1) (X,d) *is a complete metric space;*
(2) $\varphi : [0,\infty) \to [0,\infty)$ *is a function with* $\varphi(0) = 0$, $\varphi(t) < t$ *for all* $t > 0$, $\limsup_{s\to t^+} \varphi(s) < t$ *for all* $t > 0$*;*
(3) $f : X \to X$ *is a map such that for all* $x, y \in X$,

$$d(f(x), f(y)) \leq \varphi(d(x,y)). \tag{1}$$

Then for each $x \in X$*, the sequence* $\{f^n(x)\}$ *is a Cauchy sequence.*

Theorem 2 ([3], Theorem 2.1)**.** *Assume that all conditions in Lemma 1 hold. Then* f *has a unique fixed point.*

For the applications to the fractal, Ri [3] obtained the fixed point theorem of some generalized contraction in the fractal space. In 2018, Bisht [1] gave a counterexample without giving proper justification to Lemma 1 as follows:

Example 3 ([1], Example 1.2)**.** Let $X = \left\{ \sum\limits_{k=1}^{n} \frac{1}{k} : n = 1, 2, 3, \ldots \right\}$ and d be the usual metric on X. Define $\varphi : [0, \infty) \to [0, \infty)$ and $f : X \to X$ by $\varphi(t) = \frac{t}{1+t}$ for all $t \in [0, \infty)$ and

$$f\left(\sum_{k=1}^{n} \frac{1}{k}\right) = \sum_{k=1}^{n+1} \frac{1}{k}$$

for all $n = 1, 2, 3, \ldots$ Then we have the following assertions:

(1) f and φ satisfy all the conditions of Lemma 1;
(2) the sequence $\{f^n(x)\}$ is not a Cauchy sequence with $x = 1$.

Unfortunately, we find that for $x = 1$ and $y = 1 + \frac{1}{2} + \frac{1}{3}$,

$$d(f(x), f(y)) = d\left(1 + \frac{1}{2}, 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}\right) = \frac{1}{3} + \frac{1}{4} = \frac{7}{12},$$

$$\varphi\left(d(x, y)\right) = \varphi\left(d\left(1, 1 + \frac{1}{2} + \frac{1}{3}\right)\right) = \varphi\left(\frac{1}{2} + \frac{1}{3}\right) = \varphi\left(\frac{5}{6}\right) = \frac{\frac{5}{6}}{1 + \frac{5}{6}} = \frac{5}{11}.$$

Then $d(f(x), f(y)) = \frac{7}{12} > \frac{5}{11} = \varphi(d(x, y))$. This proves that the condition (1) of Lemma 1 does not hold. In this case, it is important to note that Condition (1) holds for $y = fx$ only. Then Example 3 is not correct. However, Theorem 2 still holds.

Bisht [1] also improved the result of Ri [3] by employing a proper setting as follows:

Theorem 4 ([1], Theorem 2.1)**.** *Let (X, d) be a metric space and $f : X \to X$ be a map. Assume that the following conditions hold:*

(1) *(X, d) is an f-orbitally complete metric space;*
(2) *there exist $x_0 \in X$ and a function $\varphi_{x_0} : (0, \infty) \to (0, \infty)$ with $\varphi_{x_0}(t) < t$ and $\limsup\limits_{s \to t^+} \varphi_{x_0}(s) < t$ for all $t > 0$;*
(3) *for all $x, y \in \overline{O(x_0, f)}$ with $x \neq y$,*

$$\begin{aligned} & d(f(x), f(y)) \\ \leq\ & \varphi_{x_0}\big(\max\{d(x, y), ad(x, f(x)) + (1 - a)d(y, f(y)), \\ & (1 - a)d(x, f(x)) + ad(y, f(y))\}\big), \end{aligned} \tag{2}$$

where $\overline{O(x_0, f)}$ is the closure of

$$O(x_0, f) := \{x_0, fx_0, f^2x_0, f^3x_0, \ldots\}$$

and $0 < a < 1$.

Then we have the following assertions:

(1) *the sequence $\{f^n(x_0)\}$ is a Cauchy sequence in X and $\lim\limits_{n \to \infty} f^n(x_0) = z \in X$;*
(2) *if f is orbitally continuous at z, then z is a fixed point of f;*
(3) *z is the unique fixed point of f in $\overline{O(x_0, f)}$.*

Theorem 5 ([1], Theorem 2.3)**.** *Theorem 4 is still true if we replace Inequality (2) by the following condition:*

$$d(f(x), f(y)) \quad \leq \quad \varphi_{x_0}\big(\max\{d(x, y), d(x, f(x)), d(y, f(y))\}\big) \tag{3}$$

for all $x, y \in \overline{O(x_0, f)}$ *with* $x \neq y$.

We have some comments on Theorem 4 and Theorem 5 as follows.

(1) Theorem 2 assumes the condition for the complete metric space X while Theorem 4 and Theorem 5 assume the condition for the complete metric space $\overline{O(x_0, f)}$. The calculations are the same. This idea first appeared in [2].

(2) The function φ is from $[0, \infty)$ to $[0, \infty)$ in Theorem 2 and the function φ_{x_0} is from $(0, \infty)$ to $(0, \infty)$ in Theorem 4 and Theorem 5. Then Inequality (2) and Inequality (3) are for $x \neq y$ to satisfy that φ_{x_0} is not defined at 0.

(3) The assumption of orbital continuity at z of f in Theorem 4 is redundant. Indeed, from Inequality (2) and $\varphi_{x_0}(t) < t$ for all $t > 0$, we have

$$\begin{aligned} d(f(x), f(y)) \quad \leq \quad & \max\{d(x, y), ad(x, f(x)) + (1-a)d(y, f(y)), \\ & (1-a)d(x, f(x)) + ad(y, f(y))\} \end{aligned}$$

for all $x, y \in \overline{O(x_0, f)}$. Note that $z \in \overline{O(x_0, f)}$. So we have

$$\begin{aligned} d(f^{n+1}(x_0), f(z)) \quad \leq \quad & \max\{d(f^n(x_0), z), \\ & ad(f^n(x_0), f^{n+1}(x_0)) + (1-a)d(z, f(z)), \\ & (1-a)d(f^n(x_0), f^{n+1}(x_0)) + ad(z, f(z))\}. \end{aligned} \tag{4}$$

Letting $n \to \infty$ in (4) and using $\lim_{n\to\infty} f^n(x_0) = z$, we have

$$\begin{aligned} d(z, f(z)) \quad & \leq \quad \max\{(1-a)d(z, f(z)), ad(z, d(z))\} \\ & = \quad \max\{(1-a), a\}d(z, f(z)). \end{aligned} \tag{5}$$

Note that $0 < a < 1$. From (5), we get $d(z, f(z)) = 0$, that is, z is a fixed point of f.

(4) The assumption of orbital continuity at z of f in Theorem 5 is also redundant. Indeed, if there exists n_0 such that $f^n(x_0) = z$ for all $n \geq n_0$, then z is a fixed point of f. Otherwise, there exists a subsequence $\{f^{k_n}(x_0)\}$ of $\{f^n(x_0)\}$ such that $f^{k_n}(x_0) \neq z$ for all k_n. Moreover, the subsequence can be chosen such that the sequence $\{d(f^{k_n}(x_0), z)\}$ is decreasing to 0. Then, from (3) and $z \in \overline{O(x_0, f)}$, we have

$$d(f^{k_n+1}(x_0), f(z)) \leq \varphi_{x_0}\big(\max\{d(f^{k_n}(x_0), z), d(f^{k_n}(x_0), f^{k_n+1}(x_0)), d(z, f(z))\}\big). \tag{6}$$

Note that the sequence

$$\big\{\max\{d(f^{k_n}(x_0), z), d(f^{k_n}(x_0), f^{k_n+1}(x_0)), d(z, f(z))\}\big\}$$

is decreasing to $d(z, f(z))$. Suppose to the contrary that $d(z, f(z)) > 0$. Then letting $n \to \infty$ in (6) and using $\limsup_{s\to t^+} \varphi_{x_0}(s) < t$ for all $t > 0$, we have

$$\begin{aligned} d(z, f(z)) & \leq \limsup_{n\to\infty} \varphi_{x_0}\big(\max\{d(f^{k_n}(x_0), z), d(f^{k_n}(x_0), f^{k_n+1}(x_0)), d(z, f(z))\}\big) \\ & < d(z, f(z)). \end{aligned}$$

This is a contradiction. Therefore, $d(z, f(z)) = 0$, that is, z is a fixed point of f.

Acknowledgement. This project is funded by National Research Council of Thailand (NRCT) N41A640092.

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Received: May 14, 2022; Accepted: July 20, 2022.