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FURTHER RESULTS ON θ-METRIC SPACES

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Abstract. In this paper, we first revise some results and proofs on θ-metric spaces. Next, we construct an explicit metric to metrize a given θ-metric that gives an affirmative answer to an open question on the metrization of θ-metric spaces. After that, we use the obtained result to calculate such metric of known θ-metrics, and reprove a fixed point theorem in θ-metric spaces.
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1. Introduction and preliminaries

There have been many generalized metric spaces to be used in the fixed point theory [13], [15]. One of the approaches to study such spaces is to metrize the given generalized metrics [5], [12], [6], [7], [9], [12], [18]. In [19], Som *et al.* studied the metrization of certain generalized metric spaces including b-metric spaces, $\mathcal{F}$-metric spaces, and θ-metric spaces. The author improved upon the metrization results of An *et al.* [2] for b-metric spaces, provided two shorter proofs of the metrization of $\mathcal{F}$-metric spaces [18], and answered partially to Question 1.1 below regarding the openness of $\mathcal{F}$-open balls in $\mathcal{F}$-metric spaces.

Question 1.1 ([3], Open problem 2.6)**.** *Is every open ball an $\mathcal{F}$-open set in $\mathcal{F}$-metric spaces?*

Actually, Question 1.1 was answered negatively in [10, Examples 5-6]. Moreover, there is a closed ball in an $\mathcal{F}$-metric space that is not $\mathcal{F}$-closed in [10, Example 5].

The authors also posed the following question.

Question 1.2 ([19], Open question on page 271)**.** (1) *Can an explicit metric be constructed with respect to which the given b-metric space with coefficient κ is metrizable?*

(2) *Can an explicit metric be constructed with respect to which the given θ-metric space is metrizable?*

Also, Question 1.2.(1) was answered affirmatively so that there has been an explicit metric to metrize the b-metric spaces, for example see [1, Theorem I] and [16, Proposition on page 4308]. One shows that the class of b-metrics and that of θ-metrics are distinct. There also exists a b-metric that is not an $\mathcal{F}$-metric [11, Proposition 2.1], and there exists an $\mathcal{F}$-metric that is not a b-metric [11, Remark 2.2].

In [14, Definition 15], the authors defined the open ball $B(x, r)$ in the θ-metric space with $r \in \mathrm{Im}\theta$. The assumption $r \in \mathrm{Im}\theta$ has been used to prove [14, Lemma 16] that each open ball is an open set in the θ-metric space. However, with this assumption, the open ball $B(x, \frac{1}{n})$ does not exist if $\frac{1}{n} \notin \mathrm{Im}\theta$. So [14, Lemma 18] which states that the family $\{B(x, \frac{1}{n}) : n \in \mathbb{N}\}$ is a countably local base at x needs to be revised.

Note that every B-action θ is only assumed to be continuous in each variable, see Definition 1.3.(2) below. However, in the proof of [14, Theorem 26], the authors used the continuity of θ at $(0, 0)$ to prove the inequality (21) therein. Also, in the proof of [19, Theorem 3.8], the authors confirmed that the B-action θ is continuous at the point $(0, 0)$ by using *the unproved claim* $\lim_{n\to\infty} \theta(s_n, t_n) = 0$ *as* $\lim_{n\to\infty}(s_n, t_n) = (0, 0)$. It implies that we have to prove again the continuity of the B-action θ at $(0, 0)$.

In this paper, we first prove a revision for [14, Lemma 16] and prove that every B-action θ is continuous at $(0, 0)$. Next, motivated by Frink's technique [8], we construct an explicit metric from a given θ-metric that gives an affirmative answer to Question 1.2.(2) above. After that, we use the obtained result to calculate such metrics for known θ-metrics, and reprove a fixed point theorem in θ-metric spaces.

Now, we recall the notions and properties which will be useful later.

Definition 1.3 ([14], Definition 4)**.** Let $\theta : [0, \infty) \times [0, \infty) \to [0, \infty)$ be a function such that for all $s, t, u, v \in [0, \infty)$,

(1) θ is continuous with respect to each variable.
(2) $\theta(0, 0) = 0$.
(3) $\theta(s, t) = \theta(t, s)$.
(4) $\theta(u, v) < \theta(s, t)$ if either $u \le s$ and $v < t$ or $u < s$ and $v \le t$.
(5) For each $m \in \mathrm{Im}\theta$ and each $t \in [0, m]$, there exists $s \in [0, m]$ such that $\theta(s, t) = m$, where $\mathrm{Im}\theta = \{\theta(s, t) : s, t \ge 0\}$.
(6) $\theta(s, 0) \le s$ for all $s > 0$.

Then θ is called a *B-action*.

Definition 1.4 ([14], Definition 11)**.** Let X be a non-empty set and the function $d : X \times X \to [0, \infty)$ and the B-action θ satisfying the following for all $x, y, z \in X$,

(1) $d(x, y) = 0$ if and only if $x = y$.
(2) $d(x, y) = d(y, x)$.
(3) $d(x, z) \le \theta(d(x, y), d(y, z))$.

Then d is called a *θ-metric* on X with respect to the B-action θ and (X, d, θ) is called a *θ-metric space*.

For other notions and properties of θ-metric spaces, the reader may refer to [4], [14], [17] and the references therein.

2. Main results

First, we prove a revision for [14, Lemma 18] as follows.

Proposition 2.1. *Suppose that (X, d, θ) is a θ-metric space. Then for every $x \in X$, the family $\{B(x, \theta(0, \frac{1}{n})) : n \in \mathbb{N}\}$ is a local base at x, and the topological space (X, d, θ) is first countable.*

Proof. Let $r \in \mathrm{Im}\theta$ and $r > 0$. Then there exists n such that $\frac{1}{n} \leq r$. It follows from Definition 1.3.(5) that there exists $s_n \in [0, r]$ such that $\theta(s_n, \frac{1}{n}) = r$. By Definition 1.3.(4) we have

$$\theta\left(0, \frac{1}{n}\right) < \theta\left(s_n, \frac{1}{n}\right) = r.$$

This proves that $B(x, \theta(0, \frac{1}{n})) \subset B(x, r)$. So the family $\{B(x, \theta(0, \frac{1}{n})) : n \in \mathbb{N}\}$ is a local base at x. Since this family is countable, the topological space (X, d, θ) is first countable. $\square$

Next, we prove that every B-action is continuous at $(0, 0)$.

Proposition 2.2. *Suppose that θ is a B-action. Then θ is continuous at $(0, 0)$.*

Proof. Let $\lim\limits_{n\to\infty} (s_n, t_n) = (0, 0)$ in $[0, \infty) \times [0, \infty)$. If there exists $n_0 \in \mathbb{N}$ such that $s_n = 0$ for all $n \geq n_0$, then, for all $n \geq n_0$,

$$0 \leq \theta(s_n, t_n) = \theta(0, t_n) \leq t_n.$$

It implies that $\lim\limits_{n\to\infty} \theta(s_n, t_n) = 0 = \theta(0, 0)$.

Similarly, if there exists $n_0 \in \mathbb{N}$ such that $t_n = 0$ for all $n \geq n_0$, then we also have

$$\lim_{n\to\infty} \theta(s_n, t_n) = 0 = \theta(0, 0).$$

Now, we can suppose that both of sets $\{n \in \mathbb{N} : s_n > 0\}$ and $\{n \in \mathbb{N} : t_n > 0\}$ are infinite. Let $\varepsilon > 0$. Then $\theta(0, \varepsilon) > \theta(0, 0) > 0$. Since $\lim\limits_{n\to\infty} t_n = 0$, there exists n_0 such that $0 < t_{n_0} < \theta(0, \varepsilon) \leq \varepsilon$. Let $r_{n_0} \in [0, \theta(0, \varepsilon))$ be such that $\theta(r_{n_0}, t_{n_0}) = \theta(0, \varepsilon)$. If $r_{n_0} = 0$, then

$$\theta(r_{n_0}, t_{n_0}) = \theta(0, t_{n_0}) \leq t_{n_0} < \theta(0, \varepsilon).$$

It is a contradiction. So we have $r_{n_0} > 0$. Since $\lim\limits_{n\to\infty} (s_n, t_n) = (0, 0)$, there exists $n_1 > n_0$ such that $s_n < r_{n_0}$ and $t_n < t_{n_0}$ for all $n \geq n_1$. Then

$$\theta(r_n, t_n) \leq \theta(r_{n_0}, t_{n_0}) = \theta(0, \varepsilon) \leq \varepsilon$$

for all $n \geq n_1$. This proves that $\lim\limits_{n\to\infty} \theta(r_n, t_n) = 0 = \theta(0, 0)$.

The above arguments show $\lim\limits_{n\to\infty} \theta(s_n, t_n) = 0 = \theta(0, 0)$, that is, θ is continuous at $(0, 0)$. $\square$

Remark 2.3. (1) Proposition 2.1 and Proposition 2.2 ensure that [14, Theorems 20, 21, 27] still hold but $B(x, \frac{1}{n})$ in their proofs has to be replaced by $B(x, \theta(0, \frac{1}{n}))$.

(2) It follows from Definition 1.3.(1) that the B-action is continuous with respect to each variable. However, known examples of B-actions in the literature are continuous in both variables. It still remains open that there exists a non-continuous B-action or not.

Now, we construct an explicit metric from a given θ-metric with respect to which the θ-metric space is metrizable. This result gives an affirmative answer to Question 1.2.(2).

Theorem 2.4. *Suppose that (X, d, θ) is a θ-metric space. Then*

(1) *For each $\varepsilon \geq 0$, there exists $s_\varepsilon \in [0, \theta(0, \varepsilon)]$ such that*

$$\theta(s_\varepsilon, \theta(0, \varepsilon)) = \theta(0, \varepsilon). \tag{2.1}$$

(2) *If for all $\varepsilon \geq 0$,*

$$\psi(\varepsilon) = \begin{cases} \frac{\varepsilon}{3} & \text{if } s_\varepsilon = 0 \\ \min\{\frac{\varepsilon}{3}, s_\varepsilon\} & \text{if } s_\varepsilon > 0 \end{cases}$$

and for all $x, y \in X$,

$$D(x, y) = D(y, x) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } d(x, y) \geq 1 \\ \frac{1}{2^n} & \text{if } \psi^{n-1}(1) > d(x, y) \geq \psi^n(1), n \in \mathbb{N} \end{cases} \tag{2.2}$$

then the formula

$$\delta(x, y) = \inf\Big\{\sum_{i=0}^{n} D(x_i, x_{i+1}) : x_0 = x, x_1, \ldots, x_n, x_{n+1} = y \in X, n \in \mathbb{N}\Big\} \tag{2.3}$$

defines a metric on X, and for all $x, y \in X$,

$$\frac{D(x, y)}{4} \leq \delta(x, y) \leq D(x, y). \tag{2.4}$$

(3) *If D is a metric, then $\delta = D$.*

(4) *The θ-metric space (X, d, θ) is metrizable by metric δ.*

(5) *The θ-metric space (X, d, θ) is complete if and only if the metric space (X, δ) is complete.*

Proof. (1). For each $\varepsilon \geq 0$, we have $\theta(0, \varepsilon) \in \mathrm{Im}\theta$. By Definition 1.3.(5), there exists $s_\varepsilon \in [0, \theta(0, \varepsilon)]$ such that

$$\theta(s_\varepsilon, \theta(0, \varepsilon)) = \theta(0, \varepsilon).$$

(2). For each $\varepsilon > 0$, we have $0 < \psi(\varepsilon) \leq \frac{\varepsilon}{3} < \frac{\varepsilon}{2}$. Then for all n,

$$0 < \psi^n(1) = \psi(\psi^{n-1}(1)) < \frac{\psi^{n-1}(1)}{2} < \cdots < \frac{\psi(1)}{2^{n-1}} < \frac{1}{2^n}.$$

So we get

$$\lim_{n\to\infty} \psi^n(1) = 0. \tag{2.5}$$

We also have

$$\psi^{n+1}(1) = \psi(\psi^n(1)) < \frac{\psi^n(1)}{2} < \psi^n(1).$$

Then the sequence $\{\psi^n(1)\}$ is strictly decreasing. It implies that the formula (2.2) is well-defined and so the formula (2.3) is.

Let $x, y, z \in X$ and $\varepsilon > 0$. We prove that if $D(x,z) < \varepsilon$ and $D(z,y) < \varepsilon$, then $D(x,y) < 2\varepsilon$. On the contrary, suppose that there exist $\varepsilon_0 > 0$ and $x_0, y_0, z_0 \in X$ with $D(x_0, z_0) < \varepsilon_0$, $D(z_0, y_0) < \varepsilon_0$ and $D(x_0, y_0) \geq 2\varepsilon_0$. Since $D(x_0, y_0) \leq 1$, we have $\varepsilon_0 \leq \frac{1}{2}$. Then there exists m_0 such that $\frac{1}{2^{m_0+1}} < \varepsilon_0 \leq \frac{1}{2^{m_0}}$. Since $D(x_0, z_0) < \varepsilon_0$ and $D(y_0, z_0) < \varepsilon_0$, we have $D(x_0, z_0) < \frac{1}{2^{m_0}}$ and $D(y_0, z_0) < \frac{1}{2^{m_0}}$. By using (2.2) we get $d(x_0, z_0) < \psi^{m_0}(1)$ and $d(y_0, z_0) < \psi^{m_0}(1)$. So, by the definition of ψ, we have

$$d(x_0, z_0) < \psi^{m_0}(1) = \psi(\psi^{m_0-1}(1)) \leq s_{\psi^{m_0-1}(1)}$$

$$d(y_0, z_0) < \psi^{m_0}(1) = \psi(\psi^{m_0-1}(1)) \leq s_{\psi^{m_0-1}(1)}.$$

By using Definition 1.3.(4) and (2.1) we get

$$d(x_0, y_0) \leq \theta(d(x_0, z_0), d(y_0, z_0)) < \theta(s_{\psi^{m_0-1}(1)}, s_{\psi^{m_0-1}(1)})$$

$$\leq \theta(s_{\psi^{m_0-1}(1)}, \theta(0, \psi^{m_0-1}(1))) = \theta(0, \psi^{m_0-1}(1)) \leq \psi^{m_0-1}(1).$$

So $d(x_0, y_0) < \psi^{m_0-1}(1)$. By using (2.2) again we get $D(x_0, y_0) \leq \frac{1}{2^{m_0}} < 2\varepsilon_0$. It is a contradiction.

So, if $D(x,z) < \varepsilon$ and $D(z,y) < \varepsilon$, then $D(x,y) < 2\varepsilon$.

In particular, for $x \neq y$ and $z \in X$, we have $D(x,z) < \max\{D(x,z), D(y,z)\}$ and $D(y,z) < \max\{D(x,z), D(y,z)\}$. So we get

$$D(x,y) < 2\max\{D(x,z), D(y,z)\}. \tag{2.6}$$

We shall prove that

$$D(x,y) \leq 2D(x,x_1) + 4\sum_{i=1}^{n-1} D(x_i, x_{i+1}) + 2D(x_n, y) \tag{2.7}$$

for all $x = x_0, x_1, \ldots, x_n, x_{n+1} = y \in X$ and $n = 0, 1, 2, \ldots$, where $\sum_{i=1}^{n-1} D(x_i, x_{i+1}) = 0$ for the cases $n = 0$ and $n = 1$.

If $x = y$, then (2.7) holds. Let $x \neq y$. Suppose to the contrary that (2.7) is false. Then there exist n_0 and $x^0 = x_0^0, x_1^0, \ldots, x_n^0, x_{n_0+1}^0 = y^0 \in X$ such that

$$D(x^0, y^0) > 2D(x^0, x_1^0) + 4\sum_{i=1}^{n_0-1} D(x_i^0, x_{i+1}^0) + 2D(x_{n_0}^0, y^0). \tag{2.8}$$

It follows from (2.6) that $n_0 \geq 2$. We suppose that n_0 is possible smallest, that is, for all $n < n_0$ we have

$$D(x,y) \leq 2D(x,x_1) + 4\sum_{i=1}^{n-1} D(x_i, x_{i+1}) + 2D(x_n, y) \tag{2.9}$$

For any $i = 1, \ldots, n_0$, if $D(x^0, y^0) > 2D(x^0, x_i^0)$ and $D(x^0, y^0) > 2D(x_i^0, y^0)$, then from (2.6) we get

$$D(x^0, y^0) > 2\max\{D(x^0, x_i^0), D(x_i^0, y^0)\} \geq D(x^0, y^0).$$

It is a contradiction. So either

$$D(x^0, y^0) \leq 2D(x^0, x_i^0) \tag{2.10}$$

or

$$D(x^0, y^0) \leq 2D(x_i^0, y^0). \tag{2.11}$$

If $i = 1$ then from (2.8) we see that (2.10) does not hold. So (2.11) holds. It also follows from(2.8) that (2.11) does not hold for $i = n_0$. Put

$$i_0 = \max\{i = 1, \ldots, n_0 - 1 : D(x^0, y^0) \leq 2D(x_i^0, y^0)\}.$$

Then $i_0 \leq n_0 - 1$ and we have

$$D(x^0, y^0) \leq 2D(x_{i_0}^0, y^0) \tag{2.12}$$

$$D(x^0, y^0) > 2D(x_{i_0+1}^0, y^0). \tag{2.13}$$

It follows from (2.13) and (2.6) that

$$D(x^0, y^0) \leq 2D(x^0, x_{i_0+1}^0). \tag{2.14}$$

Note that $i_0 < n_0$ and $n_0 - 1 - i_0 < n_0$. By using (2.9) for $n = i_0$ and $n = n_0 - 1 - i_0$, we get

$$D(x^0, x_{i_0+1}^0) \leq 2D(x^0, x_1^0) + 4\sum_{i=1}^{i_0-1} D(x_i^0, x_{i+1}^0) + 2D(x_{i_0}^0, x_{i_0+1}^0) \tag{2.15}$$

and

$$D(x_{i_0}^0, y^0) \leq 2D(x_{i_0}^0, x_{i_0+1}^0) + 4\sum_{i=i_0+1}^{n_0-1} D(x_i^0, x_{i+1}^0) + 2D(x_{n_0}, y^0). \tag{2.16}$$

It follows from (2.12), (2.14), (2.15) and (2.16) that

$$\begin{aligned}
D(x^0, y^0) &\leq 2\min\{D(x^0, x_{i_0+1}^0), D(x_{i_0}^0, y^0)\} \\
&\leq D(x^0, x_{i_0+1}^0) + D(x_{i_0}^0, y^0) \\
&\leq 2D(x^0, x_1^0) + 4\sum_{i=1}^{i_0-1} D(x_i^0, x_{i+1}^0) + 2D(x_{i_0}^0, x_{i_0+1}^0) \\
&\quad +2D(x_{i_0}^0, x_{i_0+1}^0) + 4\sum_{i=i_0+1}^{n_0-1} D(x_i^0, x_{i+1}^0) + 2D(x_{n_0}, y^0) \\
&= 2D(x^0, x_1^0) + 4\sum_{i=1}^{n_0-1} D(x_i^0, x_{i+1}^0) + 2D(x_{n_0}^0, y^0).
\end{aligned}$$

It is a contradiction to (2.8). So (2.7) also holds for the case $x \neq y$.

Now (2.7) holds for all $x = x_0, x_1, \dots, x_n, x_{n+1} = y \in X$ and $n = 0, 1, 2, \dots$, where $\sum_{i=1}^{n-1} D(x_i, x_{i+1}) = 0$ for the cases $n = 0$ and $n = 1$. Then

$$\begin{aligned} D(x,y) &\leq 2D(x,x_1) + 4\sum_{i=1}^{n-1} D(x_i,x_{i+1}) + 2D(x_n,y) \\ &\leq 4\sum_{i=0}^{n} D(x_i,x_{i+1}) \end{aligned} \tag{2.17}$$

for all $x = x_0, x_1, \dots, x_n, x_{n+1} = y \in X$. This proves that

$$D(x,y) \leq 4\delta(x,y). \tag{2.18}$$

For all $x, y, z \in X$, we have $\delta(x,y) \geq 0$, $\delta(x,y) = \delta(y,x)$ and

$$\delta(x,y) \leq D(x,y). \tag{2.19}$$

From (2.18) and (2.19) we get

$$\frac{D(x,y)}{4} \leq \delta(x,y) \leq D(x,y). \tag{2.20}$$

So (2.4) holds. Moreover, (2.20) also shows that $\delta(x,y) = 0$ if and only if $D(x,y) = 0$, that is, $x = y$.

Now, we prove that for all $x, y, z \in X$,

$$\delta(x,y) \leq \delta(x,z) + \delta(z,y). \tag{2.21}$$

Indeed, for all $\varepsilon > 0$, it follows from (2.3) that there exist $x = x_0, x_1, \dots, x_k, x_{k+1} = z$ and $z = x_{k+1}, x_{k+2}, \dots, x_n, x_{n+1} = y$ such that

$$\sum_{i=0}^{k} D(x_i, x_{i+1}) < \delta(x,z) + \frac{\varepsilon}{2}$$

and

$$\sum_{j=0}^{n-k-1} D(x_{k+1+j}, x_{k+1+j+1}) < \delta(z,y) + \frac{\varepsilon}{2}.$$

Then we get

$$\begin{aligned} \delta(x,y) &\leq \sum_{i=0}^{n} D(x_i,x_{i+1}) \\ &= \sum_{i=0}^{k} D(x_i,x_{i+1}) + \sum_{j=0}^{n-k-1} D(x_{k+1+j}, x_{k+1+j+1}) \\ &< \delta(x,z) + \frac{\varepsilon}{2} + \delta(z,y) + \frac{\varepsilon}{2} \\ &= \delta(x,z) + \delta(z,y) + \varepsilon. \end{aligned} \tag{2.22}$$

Taking the limit as $\varepsilon \to 0^+$ in (2.22) we get

$$\delta(x,y) \leq \delta(x,z) + \delta(z,y).$$

So δ defines a metric on X.

(3). If D is a metric, then

$$D(x,y) \leq \sum_{i=0}^{n} D(x_i, x_{i+1})$$

for all $x = x_0, x_1, \ldots, x_n, x_{n+1} = y \in X$. It implies that $D(x,y) \leq \delta(x,y)$. By combining with (2.19) we get $\delta = D$.

(4). One shows that every θ-metric space is metrizalbe [14, Theorem 26] or [19, Theorem 3.8]. So, to prove the θ-metric space (X, d, θ) is metrizable by metric δ, we need only to prove that:

$\lim\limits_{n\to\infty} x_n = x$ in (X, d, θ) if and only if $\lim\limits_{n\to\infty} x_n = x$ in (X, δ).

By using (2.4), we have

$\lim\limits_{n\to\infty} x_n = x$ in (X, δ) if and only if $\lim\limits_{n\to\infty} x_n = x$ in (X, D).

So we need only to prove that

$\lim\limits_{n\to\infty} x_n = x$ in (X, d, θ) if and only if $\lim\limits_{n\to\infty} x_n = x$ in (X, D),

that is,

$\lim\limits_{n\to\infty} d(x_n, x) = 0$ if and only if $\lim\limits_{n\to\infty} D(x_n, x) = 0$.

Indeed, let $\lim\limits_{n\to\infty} d(x_n, x) = 0$. For each $\varepsilon > 0$, there exists n_1 such that $\frac{1}{2^{n_1}} < \varepsilon$. Since $\lim\limits_{n\to\infty} d(x_n, x) = 0$, there exists n_2 such that $d(x_n, x) < \psi^{n_1}(1)$ for all $n \geq n_2$. By using (2.2) we get $D(x_n, x) < \frac{1}{2^{n_1}} < \varepsilon$ for all $n \geq n_2$. This proves that $\lim\limits_{n\to\infty} D(x_n, x) = 0$.

Now, let $\lim\limits_{n\to\infty} D(x_n, x) = 0$. For each $\varepsilon > 0$, by using (2.5) there exists n_3 such that $\psi^{n_3}(1) < \varepsilon$. Since $\lim\limits_{n\to\infty} D(x_n, x) = 0$, there exists n_4 such that $D(x_n, x) < \frac{1}{2^{n_3}}$ for all $n \geq n_4$. By using (2.2) we get $d(x_n, x) < \psi^{n_3}(1) < \varepsilon$ for all $n \geq n_4$. This proves that $\lim\limits_{n\to\infty} d(x_n, x) = 0$. □

Now we apply Theorem 2.4 to show the explicit metrics of known θ-metrics in the literature.

Example 2.5. For some $k \in (0, 1]$, consider the B-action $\theta(s,t) = k(s+t)$ in [14, Example 5. (θ_1)]. Suppose that (X, d, θ) is a θ-metric space with such θ.

If $k = 1$, then the θ-metric space is exactly a metric space.

Let $k \in (0, 1)$. We have

$$0 \leq d(x,y) \leq \theta(d(x,y), d(y,y)) = \theta(d(x,y), 0) = k.d(x,y).$$

So $0 \leq d(x,y) \leq k.d(x,y)$. It implies that $d(x,y) = 0$, that is, $x = y$. So the given θ-metric space is trivial in the sense that it has only one point.

Example 2.6. For some $k \in (0, 1]$, consider the B-action $\theta(s,t) = k(s+t+st)$ in [14, Example 5. (θ_2)]. Suppose that (X, d, θ) is a θ-metric space with such θ.

If $k \in (0, 1)$, then similar to Example 2.5 we see that the given θ-metric space is also trivial.

Let $k = 1$. Then $\theta(s,t) = s + t + st$. For each $\varepsilon \geq 0$, we have $\theta(0,\varepsilon) = \varepsilon$. It follows from (2.1) that

$$\theta(s_\varepsilon, \varepsilon) = \theta(s_\varepsilon, \theta(0,\varepsilon)) = \theta(0,\varepsilon).$$

Then, by using Definition 1.3.(4), we get $s_\varepsilon = 0$. It implies that $\psi(\varepsilon) = \frac{\varepsilon}{3}$, and

$$D(x,y) = D(y,x) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } d(x,y) \geq 1 \\ \frac{1}{2^n} & \text{if } \frac{1}{3^{n-1}} > d(x,y) \geq \frac{1}{3^n}, n \in \mathbb{N}. \end{cases}$$

So the metric δ defined by (2.3) is explicit.

Example 2.7. Consider the function $\theta(s,t) = \frac{st}{1+st}$ in [14, Example 5. (θ_3)]. We have $\theta(t,0) = 0$ for all $t \geq 0$. So the given function does not satisfy Definition 1.3.(4). Then the function is not a B-function.

Example 2.8. Consider the B-functions $\theta(s,t) = \sqrt{s^2 + t^2}$, $\theta(s,t) = s + t + \sqrt{st}$, and $\theta(s,t) = (s+t)(1+st)$ in [14, Example 5. (θ_4), (θ_6), (θ_7)]. Suppose that (X,d,θ) is a θ-metric space with such θ.

For each $\varepsilon \geq 0$, we have $\theta(0,\varepsilon) = \varepsilon$. It follows from (2.1) that

$$\theta(s_\varepsilon, \varepsilon) = \theta(s_\varepsilon, \theta(0,\varepsilon)) = \theta(0,\varepsilon).$$

Then, by using Definition 1.3.(4), we get $s_\varepsilon = 0$. It implies that $\psi(\varepsilon) = \frac{\varepsilon}{3}$, and

$$D(x,y) = D(y,x) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } d(x,y) \geq 1 \\ \frac{1}{2^n} & \text{if } \frac{1}{3^{n-1}} > d(x,y) \geq \frac{1}{3^n}, n \in \mathbb{N}. \end{cases}$$

So the metric δ defined by (2.3) is explicit.

Example 2.9. Consider the θ-metric space (X,d,θ) in [14, Example 13], where $X = \{a,b,c\}$ and $\theta(s,t) = s + t + st$ for all $t, s \geq 0$, and

$$d(x,y) = d(y,x) = \begin{cases} 0 & \text{if } x = y \\ 2 & \text{if } (x,y) = (a,b) \\ 6 & \text{if } (x,y) = (a,c) \\ 10 & \text{if } (x,y) = (b,c). \end{cases}$$

It follows from [14, Remark 14] that d is a θ-metric which is not a metric.

Since $d(x,y) \geq 1$ for all $x \neq y$, we have

$$D(x,y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y. \end{cases}$$

So D is a metric. It follows from Theorem 2.4.(3) that $\delta = D$.

Example 2.10. Consider the θ-metric space (X,d,θ) in [4, Example 3.11], where $X = [0,1]$ and $\theta(s,t) = s + t + st$. For each $\varepsilon \geq 0$, we have $\theta(0,\varepsilon) = \varepsilon$. It follows from (2.1) that

$$\theta(s_\varepsilon, \varepsilon) = \theta(s_\varepsilon, \theta(0,\varepsilon)) = \theta(0,\varepsilon).$$

Then, by using Definition 1.3.(4), we get $s_\varepsilon = 0$. It implies that $\psi(\varepsilon) = \frac{\varepsilon}{3}$, and

$$D(x,y) = D(y,x) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } d(x,y) \geq 1 \\ \frac{1}{2^n} & \text{if } \frac{1}{3^{n-1}} > d(x,y) \geq \frac{1}{3^n}, n \in \mathbb{N}. \end{cases}$$

So the metric δ defined by (2.3) is explicit.

Finally, by using Theorem 2.4, we reprove the following fixed point result in [14].

Theorem 2.11 ([14], Theorem 28)**.** *Let (X, d, θ) be a complete θ-metric space and $f : X \to X$ be a map such that*

$$d(fx, fy) \leq \alpha d(x, y) \tag{2.23}$$

for all $x, y \in X$ and for some $\alpha \in [0, 1)$. Then f has a unique fixed point.

Proof. Let $x_0 \in X$ and $x_{n+1} = fx_n$ for all $n \in \mathbb{N}$. We have

$$0 \leq d(x_{n+1}, x_n) \leq \alpha d(x_n, x_{n-1}) \leq \cdots \leq \alpha^n d(x_1, x_0).$$

Since $\alpha \in [0, 1)$, we get $\lim\limits_{n\to\infty} d(x_{n+1}, x_n) = 0$. Then there exists n_0 such that $d(x_{n+1}, x_n) < 1$ for all $n \geq n_0$. By using the notations in Theorem 2.4, we have for each $n \geq n_0$, there exists k_n such that $\psi^{k_n - 1}(1) > d(x_{n+1}, x_n) \geq \psi^{k_n}(1)$. By using (2.2) we get $D(x_{n+1}, x_n) = \frac{1}{2^{k_n}}$. It follows from (2.4) that

$$\delta(x_{n+1}, x_n) \leq D(x_{n+1}, x_n) = \frac{1}{2^{k_n}}.$$

Then we get for all $m \geq n$,

$$\delta(x_n, x_m) \leq d(x_n, x_{n+1}) + \cdots + d(x_{m-1}, x_m) \leq \frac{1}{2^{k_n}} + \cdots + \frac{1}{2^{k_{m-1}}} \leq \sum_{i=k_n}^{\infty} \frac{1}{2^i}.$$

This implies that $\lim\limits_{n\to\infty} \delta(x_n, x_m) = 0$. So $\{x_n\}$ is a Cauchy sequence in the metric space (X, δ). Since the θ-metric space (X, d, θ) is complete, the metric space (X, δ) is complete by Theorem 2.4.(5). Then there exists the limit $\lim\limits_{n\to\infty} x_n = x^*$ in (X, δ). It follows from Theorem 2.4.(4) that $\lim\limits_{n\to\infty} x_n = x^*$ in the θ-metric space (X, d, θ). Note that for all $n \in \mathbb{N}$,

$$d(x_{n+1}, fx^*) = d(fx_n, fx^*) \leq \alpha d(x_n, x^*).$$

Then $\lim\limits_{n\to\infty} d(x_{n+1}, fx^*) = 0$, that is $\lim\limits_{n\to\infty} x_{n+1} = fx^*$ in (X, d, θ). By using [14, Theorem 19], the limit point of a convergent sequence in (X, d, θ) is unique. So $x^* = fx^*$. This proves that x^* is a fixed point of f.

Now, let x^*, y^* be two fixed points of f. Then

$$0 \leq d(x^*, y^*) = d(fx^*, fy^*) \leq \alpha d(x^*, y^*).$$

Note that $\alpha \in [0, 1)$. So $d(x^*, y^*) = 0$, that is, $x^* = y^*$. Then the fixed point of f is unique. □

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