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MULTIPLICITY OF PERIODIC SOLUTIONS FOR DYNAMIC LIÉNARD EQUATIONS WITH DELAY AND SINGULAR φ-LAPLACIAN OF RELATIVISTIC TYPE

P. AMSTER*, M.P. KUNA* AND D.P. SANTOS**

*Departamento de Matemática, Facultad de Ciencias Exactas y Naturales,
Universidad de Buenos Aires & IMAS-CONICET,
Ciudad Universitaria, Pabellón I,(1428), Buenos Aires, Argentina
E-mail: pamster@dm.uba.ar, mpkuna@dm.uba.ar

**Universidad Nacional del Centro de la Provincia de Buenos Aires,
Del Pinto 399 (7000), Tandil, Buenos Aires, Argentina
E-mail: dpdallos@exa.unicen.edu.ar

Abstract. We study the existence and multiplicity of periodic solutions for singular φ-Laplacian Liénard-like equations with delay on time scales. We prove the existence of multiple solutions using topological methods based on the Leray-Schauder degree. Special cases are the T-periodic problem for the forced pendulum equation and the sunflower equation with relativistic effects.
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1. Introduction

In this work, we study the existence and multiplicity of T-periodic solutions $x : \mathbb{T} \to \mathbb{R}$ to the following equation with delay on time scales

$$(\varphi(x^{\Delta}(t)))^{\Delta} + h(x(t))x^{\Delta}(t) + g(x(t-r)) = p(t) \quad t \in \mathbb{T}, \tag{1.1}$$

where $\mathbb{T}$ is an arbitrary T-periodic nonempty closed subset of $\mathbb{R}$ (*time scale*), $\varphi : (-a, a) \to \mathbb{R}$ is an increasing homeomorphism with $0 < a < +\infty$ such that $\varphi(0) = 0$, and $h, g : \mathbb{R} \to \mathbb{R}$ are continuous functions. Moreover, we assume that $T > 0$ is a real number and that $p(t + T) = p(t)$ is continuous in $\mathbb{T}$ with

$$\overline{p} := \frac{1}{T}\int_0^T p(t)\Delta t = 0.$$

When $\mathbb{T} \neq \mathbb{R}$, we shall assume that the delay r satisfies $r = qT$ for some $q \in \mathbb{Q}_{\geq 0}$ and that $\mathbb{T} - r \subset \mathbb{T}$.

The time scales theory was introduced in 1988, in the PhD thesis of Stefan Hilger [11], as an attempt to unify discrete and continuous calculus. The time scale $\mathbb{R}$ corresponds to the continuous case and, hence, yields results for ordinary differential equations. If the time scale is $\mathbb{Z}$, then the results apply to difference equations. However, the generality of the set $\mathbb{T}$ produces many different situations in which the time scales formalism is useful in several applications. For example, in the study of hybrid discrete-continuous dynamical systems, see [5].

The methods usually employed to explore the existence of periodic solutions for dynamic equations in time scales are: fixed point theory [12, 14], Mawhin's continuation theorem [7, 13], lower and upper solutions [19, 21] and variational methods [10], [20], [24], among many other works. Some of the above cited references correspond to the semilinear case, that is, $\varphi(x) = x$ and some others to the *p-Laplacian operator*, namely $\varphi_p(x) := |x|^{p-2}x$. However, the literature concerning singular φ-Laplacian operators in time scales is more scarce. A special case of (1.1) with $\mathbb{T} = \mathbb{R}$ is the forced pendulum equation with relativistic effects, namely,

$$\left(\frac{x'}{\sqrt{1-\frac{x'^2}{c^2}}}\right)' + kx'(t) + b\sin x(t) = p(t), \quad t \in \mathbb{R}, \tag{1.2}$$

where $c > 0$ is the speed of light in the vacuum, $k > 0$ is a possible viscous friction coefficient and p is a continuous and T-periodic forcing term with mean value zero. This equation has received much attention by several authors, see e.g. [6, 17, 22]. In particular in [22], employing the Schauder fixed point theorem, Torres proved the existence of at least one T-periodic solution, provided that $2cT \leq 1$. This result was later improved in [23] and finally in [3], where the sharper condition $cT < \sqrt{3}\pi$ was obtained. In the recent paper [8], a new improvement was obtained in terms of k and $\|p\|_{L^1}$ and allows to obtain the uniform condition $cT \leq 2\pi$.

When a positive delay is added to (1.2), a relativistic instance of the sunflower equation is obtained, namely

$$\left(\frac{x'}{\sqrt{1-\frac{x'^2}{c^2}}}\right)' + \frac{k}{r}x'(t) + b\sin x(t-r) = p(t), \quad t \in \mathbb{R}. \tag{1.3}$$

In this work, we generalize several aspects of the results in [3] and [22, 23]. On the one hand, our problem consist of dynamical Liénard-like equations on time scales; on the other hand, the functions g and h are general and the equation may also include a delay. This implies that the use of the Poincaré operator does not reduce the problem to a finite-dimensional one, and requires to employ accurate topological methods such as the Leray-Schauder degree. Moreover, our main theorem is in fact a multiplicity result, which intuitively can be motivated as follows. If we observe for example problems (1.2) and (1.3), it is clear that the periodicity of the sine function implies that if x is a T-periodic solution, then $x+2k\pi$ is also a T-periodic solution for all $k \in \mathbb{Z}$. Such solutions are usually called in the literature *geometrically equivalent*. However, if the term kx' is replaced by $h(x)x'$ for some continuous function h close to a constant, then the problem still admits infinitely many solutions, which may be geometrically distinct if h is not a 2π-periodic function. With this idea in mind, it shall be shown that if the nonlinear term has a more general oscillatory behaviour, then multiple solutions exist.

More specifically, our main result reads as follows:

Theorem 1.1. *Assume that there exists a strictly increasing sequence $\{\alpha_j\}_{j=0}^n$ such that*

$$(-1)^j \int_0^T h(x(t))x^\Delta(t) + g(x(t))\Delta t < 0 \text{ if } x(0) = \alpha_j, \ \left\|x^\Delta\right\|_\infty < a.$$

for every j and each C^1 and T-periodic function $x(t)$. Then, for any continuous T-periodic function $p(t)$ with mean value zero, problem (1.1) has at least n different T-periodic solutions.

In particular, if g is oscillatory over $\mathbb{R}$ and h is locally monotone or locally close to a constant, then (1.1) has infinitely many different T-periodic solutions, provided that the oscillations are sufficiently slow. More precisely, the following corollaries are obtained:

Corollary 1.2. *Assume that there exists a strictly increasing sequence $\{\alpha_j\}_{j=0}^n$ such that*

$$(-1)^j g > 0 \text{ and } (-1)^j h \text{ is nonincreasing over } (\alpha_j - \tfrac{aT}{2}, \alpha_j + \tfrac{aT}{2}).$$

Then, for any continuous T-periodic function $p(t)$ with mean value zero, problem (2.1) has at least n different T-periodic solutions.

Corollary 1.3. *Assume there exists a strictly increasing sequence $\{\alpha_j\}_{j=0}^n$ and constants γ_j such that*

$$a|h(x) - \gamma_j| < (-1)^j g(x) \text{ for all } x \in (\alpha_j - \tfrac{aT}{2}, \alpha_j + \tfrac{aT}{2}).$$

Then, for any continuous T-periodic function $p(t)$ with mean value zero, problem (2.1) has at least n different T-periodic solutions.

The proof of the preceding results shall be based on the search for fixed points of an appropriate compact operator defined on the Banach space of all continuous T-periodic functions on $\mathbb{T}$. The singular nature of φ will be of help in the obtention of the required a priori bounds, thus making possible a Leray-Schauder degree approach. We highlight that, in contrast with the continuous case, the treatment of Liénard-like equations on time scales is more delicate because the average of the term $h(x(t))x^\Delta(t)$ with T-periodic x is not necessarily equal to 0. This is due to the fact that the standard chain rule does not hold and, consequently, extra conditions are required in order to avoid this difficulty.

The paper is organized as follows. In Section 2, we set the notation, terminology, and several preliminary results which will be used throughout this paper. In Section 3, we adapt Mawhin's continuation theorem to the context of times scales in order to prove the existence of at least one T-periodic solution of (1.1). In Section 4, we prove our main theorem with the help of the arguments introduced in the preceding section. Some examples illustrating the results are presented in Section 5.

2. Notation and preliminaries

For fixed $T > 0$, we shall assume that $\mathbb{T}$ is T-periodic, that is, $\mathbb{T}+T = \mathbb{T}$. Moreover, since the equation includes a delay $r \geq 0$, we shall also assume that $\mathbb{T} - r \subset \mathbb{T}$. When $r > 0$, it is observed that if $\mathbb{T} \neq \mathbb{R}$, then r is necessarily commensurable with T, that is, $r = qT$ for some positive $q \in \mathbb{Q}$. Indeed, this is due to the fact that, otherwise, the set $\{e^{-2\pi \frac{r}{T} ni}\}_{n\in\mathbb{N}}$ is dense in $S^1 \subset \mathbb{C}$ and the conclusion follows from the fact that $\mathbb{T}$ is closed. Note, also, that if r is congruent to $\hat{r}$ modulo T, then $x(t-r) = x(t-\hat{r})$ for any T-periodic function x and, thus, we may assume without loss of generality that $r < T$. For convenience, we shall also assume that $0 \in \mathbb{T}$.

Let us denote by $C_T = C_T(\mathbb{T}, \mathbb{R})$ the Banach space of all continuous T-periodic functions on $\mathbb{T}$ endowed with the uniform norm $\|x\|_\infty = \sup_{\mathbb{T}} |x(t)| = \sup_{[0,T]_{\mathbb{T}}} |x(t)|$ and the closed subspace

$$\tilde{C_T} = \Big\{x \in C_T : \int_0^T x(s)\Delta s = 0\Big\}.$$

For an element $x \in C_T$ its maximum and minimum values shall be denoted respectively by x_M and x_m.

Moreover, denote by $C_T^1 = C_T^1(\mathbb{T}, \mathbb{R})$ the Banach space of all continuous T-periodic functions on $\mathbb{T}$ that are Δ-differentiable with continuous Δ-derivatives, endowed with the usual norm

$$\|x\|_1 = \sup_{[0,T]_{\mathbb{T}}} |x(t)| + \sup_{[0,T]_{\mathbb{T}}} \left|x^\Delta(t)\right|.$$

We introduce the following operators and functions:

- The *Nemytskii operator* $N_f : C_T^1 \to C_T$, given by
$$N_f(z)(t) = f(t, x(t), x^\Delta(t), x(t-r)),$$
where $f : \mathbb{T} \times \mathbb{R}^3 \to \mathbb{R}$ is a continuous function;
- The *integration operator* $H : \tilde{C_T} \to C_T^1$,
$$H(z)(t) = \int_0^t z(s)\Delta s,$$
- The continuous linear projectors:
$$Q : C_T \to C_T, \quad Q(x)(t) = \frac{1}{T}\int_0^T x(s)\Delta s,$$
$$P : C_T \to C_T, \quad P(x)(t) = x(0)$$
where, for convenience, the isomorphism between $\mathbb{R}$ and the subspace of constant functions of C_T is omitted.

The above equation (1.1) can be written as follows:

$$(\varphi(x^\Delta(t)))^\Delta = f(t, x(t), x^\Delta(t), x(t-r)), \quad t \in \mathbb{T}, \tag{2.1}$$

A function $x \in C_T^1$ is said to be a solution of (2.1) if $\varphi(x^\Delta)$ is of class C^1 and verifies $(\varphi(x^\Delta(t)))^\Delta = f(t, x(t), x^\Delta(t), x(t-r))$ for all $t \in \mathbb{T}$.

The following lemma is an adaptation of a result of [4] to time scales.

Lemma 2.1. *For each $x \in C_T$, there exists a unique $Q_\varphi = Q_\varphi(x) \in [x_m, x_M]$ such that*

$$\int_0^T \varphi^{-1}(x(t) - Q_\varphi(x))\Delta t = 0.$$

Moreover, the function $Q_\varphi : C_T \to \mathbb{R}$ is continuous and sends bounded sets into bounded sets.

Proof. Let $x \in C_T$ and define the continuous application $G_x : [x_m, x_M] \to \mathbb{R}$ by

$$G_x(s) = \int_0^T \varphi^{-1}(x(t) - s)\Delta t.$$

We claim that the equation

$$G_x(s) = 0 \tag{2.2}$$

has a unique solution $Q_\varphi(x)$. Indeed, Let $r, s \in [x_m, x_M]$ be such that

$$\int_0^T \varphi^{-1}(x(t) - r)\Delta t = 0 = \int_0^T \varphi^{-1}(x(t) - s)\Delta t,$$

then using the fact that φ^{-1} is strictly increasing we deduce that $r = s$. Moreover, It is seen that

$$\int_0^T \varphi^{-1}(x(t) - x_M)\Delta t \le 0 \le \int_0^T \varphi^{-1}(x(t) - x_m)\Delta t,$$

whence

$$G_x(x_m)G_x(x_M) \le 0.$$

Thus, there exists $s \in [x_m, x_M]$ such that $G_x(s) = 0$, that is, equation (2.2) has a unique solution. It follows that function $Q_\varphi : C_T \to \mathbb{R}$ given by $Q_\varphi(x) = s$ is well defined and, furthermore, because $s \in [x_m, x_M]$ we deduce that

$$|Q_\varphi(x)| \le \|x\|_\infty.$$

Therefore, the function Q_φ sends bounded sets into bounded sets.

Finally, let us verify that Q_φ is continuous on C_T. Let $(x_n)_n \subset C_T$ be a sequence such that $x_n \to x$ in C_T. Since the function Q_φ sends bounded sets into bounded sets, the sequence $(Q_\varphi(x_n))_n$ is bounded in $\mathbb{R}$ and, consequently, without loss of generality we may assume that it converges to some $\tilde{a}$.

Because

$$\int_0^T \varphi^{-1}(x_n(t) - Q_\varphi(x_n))\Delta t = 0$$

for all n, by the dominated convergence theorem on time scales [5], we deduce that

$$\int_0^T \varphi^{-1}(x(t) - \widetilde{a})\Delta t = 0,$$

so $Q_\varphi(h) = \widetilde{a}$. Thus, we conclude that the function Q_φ is continuous.

Now, we define a fixed point operator, which is similar to the one employed in [4] (see also [1] for an elementary introduction). In order to transform problem (2.1) into a fixed point problem we use the operators H, Q, N_f, P and Lemma 2.1. The proof of this result is similar to the continuous case and shall not repeated here.

Lemma 2.2. *$x \in C_T^1$ is a solution of (2.1) if and only if x is a fixed point of the operator M_f defined on C_T^1 by*

$$M_f(x) =$$
$$P(x) + Q(N_f(x)) + H\left(\varphi^{-1}\left[H(N_f(x) - Q(N_f(x))) - Q_\varphi(H(N_f(x) - Q(N_f(x))))\right]\right).$$

As the function f is continuous, using the Arzelà-Ascoli theorem it is not difficult to see that M_f is completely continuous.

Using Lemma 2.2, the existence of a T-periodic solution for (2.1) is reduced to the obtention of fixed points of the operator M_f. To this end, we will use topological degree theory.

Consider the following family of problems defined for $\lambda \in [0, 1]$:

$$(\varphi(x^\Delta(t)))^\Delta = \lambda N_f(x)(t) + (1 - \lambda)Q(N_f(x)), \tag{2.3}$$

where the operator N_f is defined by

$$N_f(x)(t) = f(t, x(t), x^\Delta(t), x(t - r)) := -h(x(t))x^\Delta(t) - g(x(t - r)) + p(t), \quad t \in \mathbb{T}.$$

For each $\lambda \in [0,1]$, consider the nonlinear operator $M(\lambda,\cdot)$, where M is defined on $[0,1]\times C^1_T$ by

$$M(\lambda,x) = P(x) + Q(N_f(x)) + H(\varphi^{-1}\left[\lambda H(N_f(x) - Q(N_f(x))) - Q_\varphi(\lambda H(N_f(x) - Q(N_f(x))))\right]). \tag{2.4}$$

Observe that $M(1,x) = M_f$; moreover, similarly as above, it is easy to see that M is completely continuous and that, for $\lambda > 0$, the existence of solution to equation (2.3) is equivalent to the problem

$$x = M(\lambda, x).$$

We claim that the previous assertion is true also for $\lambda = 0$. Indeed, because $Q_\varphi(c) = c$ for any constant c, it is clear that $M(0,x) = P(x) + Q(N_f(x))$. If $x = M(0,x)$ then x is constant and $x = P(x)$, that is, $Q(N_f(x)) = 0$ and (2.3) with $\lambda = 0$ is trivially satisfied. Conversely, if $(\varphi(x^\Delta(t)))^\Delta \equiv Q(N_f(x))$ then we obtain, upon integration, $\int_0^T Q(N_f(x))\Delta t = 0$ which, in turn, implies that $Q(N_f(x)) = 0$. Thus x^Δ is constant and, by periodicity, $x^\Delta \equiv 0$, that is, x is constant and, consequently, $x = P(x) = P(x) + Q(N_f(x)) = M(0,x)$.

Remark 2.3. It is worthy to notice that, for any $\lambda \in [0,1]$, if x is a fixed point of M then $Q(N_f(x)) = 0$.

3. Continuation theorem

In this section, we establish the continuation theorem that shall be employed for the proof of our main result. Let us denote by deg_B and deg_{LS} the Brouwer and Leray-Schauder degrees respectively. The following result is obtained as in the continuous case; we include a proof for the sake of completeness.

Theorem 3.1. *Assume that Ω is an open bounded set in C^1_T such that the following conditions hold:*

(1) *For each $\lambda \in (0,1)$ the problem*

$$(\varphi(x^\Delta(t)))^\Delta = \lambda N_f(x) \tag{3.1}$$

has no solution on $\partial\Omega$.

(2) *The equation*

$$g(y) = 0,$$

has no solution on $\partial\Omega \cap \mathbb{R}$, where we consider the natural identification of $\mathbb{R}$ with the subspace of constant functions of C^1_T.

(3) *The Brouwer degree of g satisfies:*

$$deg_B(g, \Omega\cap\mathbb{R}, 0) \neq 0.$$

Then problem (1.1) has at least one T-periodic solution.

Proof. Let $\lambda \in (0,1]$. If x is a solution of (3.1), then $Q(N_f(x)) = 0$, hence x is a solution of problem (2.3). On the other hand, for $\lambda \in (0,1]$, let x be a solution of (2.3) and since

$$Q\left(\lambda N_f(x) + (1-\lambda)Q(N_f(x))\right) = Q(N_f(x)),$$

it follows that $Q(N_f(x)) = 0$, whence x is a solution of (3.1). It is deduced that, for $\lambda \in (0,1]$, problems (2.3) and (3.1) have the same solutions. We assume that (2.3) has no solutions on $\partial\Omega$ for $\lambda = 1$, since otherwise we are done with the proof. It follows that (2.3) has no solutions for $(\lambda, x) \in (0,1] \times \partial\Omega$. If x is a solution of (2.3) for $\lambda = 0$, then we conclude as before that $Q(N_f(x)) = 0$ and $x(t) \equiv b \in \mathbb{R}$. Thus, using the fact that $\int_0^T p(t)\Delta t = 0$

$$0 = \frac{1}{T}\int_0^T f(t,b,0,b)\Delta t = -g(b),$$

which, together with hypothesis 2, implies that $b \notin \partial\Omega$.

Summarizing, we proved that (2.3) has no solution on $\partial\Omega$ for all $\lambda \in [0,1]$. Thus, for each $\lambda \in [0,1]$, the Leray-Schauder degree $deg_{LS}(I - M(\lambda,\cdot),\Omega,0)$ is well defined and, by the homotopy invariance property,

$$deg_{LS}(I - M(0,\cdot),\Omega,0) = deg_{LS}(I - M(1,\cdot),\Omega,0).$$

On the other hand,

$$deg_{LS}(I - M(0,\cdot),\Omega,0) = deg_{LS}(I - (P + QN_f),\Omega,0).$$

But the range of the mapping

$$z \mapsto P(z) + QN_f(z)$$

is contained in the subspace of constant functions of C^1_T, identified with $\mathbb{R}$. Thus, using the reduction property of the Leray-Schauder degree [9, 16]

$$\begin{aligned} deg_{LS}(I - (P + QN_f),\Omega,0) &= deg_B\left(I - (P + QN_f)\,|_{\overline{\Omega\cap\mathbb{R}}}\,,\Omega\cap\mathbb{R},0\right) \\ &= deg_B(g,\Omega\cap\mathbb{R},0) \neq 0. \end{aligned}$$

Then, $deg_{LS}(I - M(1,\cdot),\Omega,0) \neq 0$ and, in consequence, there exists $x \in \Omega$ such that $M_f(x) = M(1,x) = x$, which is a solution of (2.1) and therefore a solution of (1.1).

With the help of Theorem 3.1 we shall be able to prove the existence of fixed points of M_f. With this aim, for $\lambda \in (0,1]$ we consider the equation

$$\left(\varphi\left(x^{\Delta}(t)\right)\right)^{\Delta} + \lambda h(x(t))x^{\Delta}(t) + \lambda g(x(t-r)) = \lambda p(t) \quad t \in \mathbb{T}, \tag{3.2}$$

which is the explicit expression of problem (3.1).

The next example shows that the $\int_0^T h(x(t))x^{\Delta}(t)\Delta t$ is not necessarily equal to zero. This is due to the fact that the standard chain rule does not hold for time scales.

Example 3.2. Let $\mathbb{T}$ be 3-periodic with $[0,3]_{\mathbb{T}} = [0,1] \cup \{2,3\}$, let $h(x) = x$, and let $x : \mathbb{T} \to \mathbb{R}$ be the 3-periodic function defined on $[0,3)_{\mathbb{T}}$ by

$$x(t) = \begin{cases} t & if \quad 0 \le t \le 1 \\ \\ 2 & if \qquad t = 2. \end{cases}$$

It follows by direct computation that $\int_0^3 x(t)x^{\Delta}(t)\Delta t = -\frac{5}{2}$.

Lemma 3.3. *Assume that h is nondecreasing (resp. nonincreasing) over the range of $x \in C^1_T$. Then*

$$\int_0^T h(x(t))x^{\Delta}(t)\Delta t \leq 0 \quad (\textit{resp.} \geq 0).$$

Proof. Consider the primitive of h, namely

$$\mathcal{H}(u) = \int_0^u h(s)ds$$

and observe that $\mathcal{H} \circ x$ is Δ-differentiable. Moreover, if t_0 is right dense, then $(\mathcal{H} \circ x)^{\Delta}(t_0) = h(x(t_0))x^{\Delta}(t_0)$. On the other hand, if t_0 is right-scattered, then

$$(\mathcal{H} \circ x)^{\Delta}(t_0) = h(\xi)x^{\Delta}(t_0)$$

for some ξ between $x(t_0)$ and $x(\sigma(t_0))$. If h is nondecreasing over the range of x, it readily follows that

$$(\mathcal{H} \circ x)^{\Delta}(t_0) \geq h(x(t_0))x^{\Delta}(t_0)$$

and, consequently,

$$\int_0^T h(x(t))x^{\Delta}(t)\Delta t \leq \int_0^T (\mathcal{H} \circ x)^{\Delta}(t)\Delta t = 0.$$

The opposite inequality is obtained if we assume, instead, that h is nonincreasing.

4. Multiplicity of periodic solutions

In this section we establish the existence of at least n different solutions of problem (1.1). The statements in the introduction are repeated here, for the sake of clarity.

Theorem 4.1. *Assume that there exists a strictly increasing sequence $\{\alpha_j\}_{j=0}^n$ such that for all j and $x \in C^1_T$,*

$$(-1)^j \int_0^T [h(x(t))x^{\Delta}(t) + g(x(t))]\Delta t < 0 \ \textit{if} \ \ x(0) = \alpha_j, \ \left\| x^{\Delta} \right\|_{\infty} < a. \tag{4.1}$$

Then, for any continuous T-periodic function $p(t)$ with mean value zero, problem (1.1) has at least n different T-periodic solutions.

Proof. Assume that $x \in C^1_T(\mathbb{T}, \mathbb{R})$ is a solution of (3.2) with $\lambda \in (0, 1]$, then $|x^{\Delta}(t)| < a$ and

$$\int_0^T [h(x(t))x^{\Delta}(t) + g(x(t - r))]\Delta t = 0.$$

From the periodicity of x we deduce from (4.1) that $x(0) \neq \alpha_j$, for any $j = 0, \ldots, n$. Moreover, (4.1) for $x \equiv \alpha_j$ also implies that $(-1)^j g(\alpha_j) < 0$. Therefore, problem (2.3) has no solution in $\partial\Omega_j$ for all $j = 0, \ldots, n-1$, where

$$\Omega_j := \left\{ x \in C^1_T(\mathbb{T}, \mathbb{R}) / x(0) \in (\alpha_j, \alpha_{j+1}), \ \left\| x^{\Delta} \right\|_{\infty} < a \right\}.$$

From the homotopy invariance of the Leray-Schauder degree, we obtain

$$\begin{aligned} deg_{LS}(I - M(1,\cdot),\Omega_j,0) &= deg_{LS}(I - M(0,\cdot),\Omega_j,0) = \\ &= deg_{LS}(I-(P+QN_f),\Omega_j,0) \\ &= deg_B\left(I-(P+QN_f)\Big|_{\overline{\Omega_j\cap\mathbb{R}}},\Omega_j\cap\mathbb{R},0\right) \\ &= deg_B(g,\Omega_j\cap\mathbb{R},0) \\ &= deg_B(g,(\alpha_j,\alpha_{j+1}),0) = (-1)^j. \end{aligned}$$

We conclude that the operator $M(1,\cdot) = M_f$ has a fixed point $x_j \in \Omega_j$. Finally, observe that $x_j(0) \in (\alpha_j,\alpha_{j+1})$ hence all the solutions are different.

Remark 4.2. It is clear that the sign in condition (4.1) may be reversed, that is:

$$(-1)^j\int_0^T [h(x(t))x^\Delta(t)+g(x(t))]\Delta t > 0 \text{ if } x(0)=\alpha_j,\ \left\|x^\Delta\right\|_\infty < a.$$

The next corollary shows that condition (4.1) can be obtained from appropriate explicit assumptions on g and h.

Corollary 4.3. *Assume that there exists a strictly increasing sequence $\{\alpha_j\}_{j=0}^n$ such that*

$$(-1)^j g > 0 \textit{ and } (-1)^j h \textit{ is nonincreasing over } (\alpha_j - \tfrac{aT}{2}, \alpha_j + \tfrac{aT}{2}).$$

Then, for any continuous T-periodic function $p(t)$ with mean value zero, problem (2.1) has at least n different T-periodic solutions.

Proof. From the previous proof and Lemma 3.3 it suffices to verify that if $x \in C^1_T$ satisfies $x(0)=\alpha_j$ and $\|x^\Delta\|_\infty < a$, then $x(t) \in (\alpha_j - \frac{aT}{2}, \alpha_j + \frac{aT}{2})$ for all t. To this end, observe that if $|x(t)-\alpha_j| \geq \frac{aT}{2}$ for some $t \in (0,T)_{\mathbb{T}}$, then

$$\frac{aT}{2} \leq |x(t)-\alpha_j| \leq \int_0^t |x^\Delta(s)|\Delta s < at,$$

whence $t > \frac{T}{2}$. Due to the periodicity, we also deduce that $T-t > \frac{T}{2}$, a contradiction.

Remark 4.4. In particular, the conditions in the previous theorem imply that

$$\alpha_{j+1}-\alpha_j \geq aT \text{ for } j=0,1,\ldots,n-1.$$

The alternative condition that h is locally close to a constant is obtained in the following corollary:

Corollary 4.5. *Assume there exists a strictly increasing sequence $\{\alpha_j\}_{j=0}^n$ and constants γ_j such that*

$$a|h(x)-\gamma_j| < (-1)^j g(x) \textit{ for all } x \in (\alpha_j - \tfrac{aT}{2}, \alpha_j + \tfrac{aT}{2}).$$

Then, for any continuous T-periodic function $p(t)$ with mean value zero, problem (2.1) has at least n different T-periodic solutions.

Proof. Suppose for example that x is a solution of (3.2) with j even and $x(0)=\alpha_j$. Because $\|x^\Delta\|_\infty < a$ and $x(t) \in (\alpha_j - \frac{aT}{2}, \alpha_j + \frac{aT}{2})$, then

$$(h(x(t))-\gamma_j)x^\Delta(t) + g(x(t)) \geq g(x(t)) - a|h(x(t))-\gamma_j| > 0.$$

Since $\int_0^T \gamma_j x^\Delta(t)\Delta t = 0$, we deduce that

$$\int_0^T [h(x(t))x^\Delta(t) + g(x(t-r))]\Delta t = \int_0^T [h(x(t))x^\Delta(t) + g(x(t))]\Delta t > 0.$$

An analogous reasoning for j odd shows that condition (4.1) is fulfilled.

Remark 4.6. In particular, suppose that g has slow oscillations, that is, there exists a sequence of zeros $x_j \nearrow +\infty$ such that $(-1)^j g(x) > 0$ for $x \in (x_j, x_{j+1})$, with $x_{j+1} - x_j > aT$, then the problem has infinitely many solutions, provided that $(-1)^j h$ is nonincreasing or $a|h(x)| < |g(x)|$ in $(\alpha_j - \frac{aT}{2}, \alpha_j + \frac{aT}{2})$ for all j, where $\alpha_j = \frac{x_j + x_{j+1}}{2}$.

5. Examples

In order to illustrate the above results, we consider some examples.

Example 5.1. Let us consider the equation

$$\left(\frac{x^\Delta(t)}{\sqrt{1 - x^\Delta(t)^2}}\right)^\Delta + e^{-x^2(t)}x^\Delta(t) + \arctan(x(t)) = \sin(4\pi t) \quad t \in \mathbb{T} \tag{5.1}$$

where $\mathbb{T}$ is a $1/2$-periodic time scale with

$$[0, 1/2]_\mathbb{T} = [0, 1/8] \cup \{3/16\} \cup \{1/4\} \cup [5/16, 3/8] \cup [7/16, 1/2]\,.$$

By Corollary 4.3 or Corollary 4.5 with $\alpha_0 \ll 0 \ll \alpha_1$ we deduce that (5.1) has at least one $1/2$-periodic solution.

Example 5.2. Let $h : \mathbb{R} \to \mathbb{R}$ be continuous. Let us study the existence of a 2π-periodic solution to the following problem

$$\left(\frac{x^\Delta(t)}{\sqrt{1 - \frac{x^\Delta(t)^2}{c^2}}}\right)^\Delta + h(x(t))x^\Delta(t) + x^3(t-r) = \cos(t), \quad t \in \mathbb{R}, \tag{5.2}$$

where $c > 0$ and $r \geq 0$. Using Corollaries 4.3 and 4.5, it follows that problem (5.2) has at least one 2π-periodic solution if one of the following assumptions is verified:

(1) There exists $R > 0$ such that
$$h(y) \leq h(x) \qquad \text{for } y \geq x \geq R \text{ or } y \leq x \leq -R.$$
(2) $\limsup_{x\to\pm\infty} \left|\frac{h(x)}{x^3}\right| < 1$.

Example 5.3. Let us consider the relativistic pendulum equation on time scales

$$\left(\frac{x^\Delta(t)}{\sqrt{1 - \frac{x^\Delta(t)^2}{c^2}}}\right)^\Delta + h(x(t))x^\Delta(t) + \sin(x(t)) = p(t) \quad t \in \mathbb{T}, \tag{5.3}$$

where $h, p : \mathbb{R} \to \mathbb{R}$ are continuous functions and p is T-periodic with mean value zero. If $cT \leq \pi$, then problem (5.3) has infinitely many T-periodic solutions under one of the following assumptions:

(1) $(-1)^j h$ is nonincreasing in $(\alpha_j - \frac{cT}{2}, \alpha_j + \frac{cT}{2})$, where $\alpha_j = (2j+1)\frac{\pi}{2}$ for $j \in \mathbb{Z}$.

(2) $c\,|h(x)-\gamma_j| < |\sin(x)|$ for $x \in (\alpha_j - \frac{cT}{2}, \alpha_j + \frac{cT}{2})$ for $j \in \mathbb{Z}$ and some constants γ_j.

Clearly, both conditions are satisfied when h is constant although, in this case, the solutions are not necessarily different in geometric sense (see [22]). It is worth observing that the restriction $cT \le \pi$, which comes from Remark 4.4, improves the one in the original work by Torres, but it is slightly worse than the one obtained in [23] which, as mentioned in the introduction, reads $cT < 2\sqrt{3} = 3.46\ldots$ However, the method in [23] involves a change of variables that cannot be extended to a general time scale. The sharper bound given in [3] is easily obtained in the continuous case, due to the Sobolev inequality

$$\|x - \overline{x}\|_\infty^2 \le \frac{T}{12}\|x'\|_{L^2}^2,$$

which holds for T-periodic functions. Indeed, it suffices to observe that, if we replace P by Q in the definition of the operator M in (2.4) then our main theorem is also valid, changing $x(0)$ by $\overline{x}$ in condition (4.1) and the definition of Ω. Thus, any possible solution of (2.3) satisfying for example $\overline{x} = \frac{\pi}{2}$ verifies $|x(t) - \frac{\pi}{2}| \le \frac{cT}{2\sqrt{3}}$ for all t. If $cT \le \sqrt{3}\pi$, then $x(t) \in [0,\pi]$ for all t and

$$0 = \int_0^T \sin(x(t))\,dt > 0,$$

a contradiction. For a general time scale, the argument is essentially the same and yields the condition $c\sqrt{Ts(\mathbb{T})} \le \frac{\pi}{2}$, where $s(\mathbb{T})$ is the constant of the corresponding Sobolev inequality. A refinement of this method was recently introduced in [2] and, to our knowledge, constitutes the best bound that has been obtained until now for the continuous case. We recall that, in the continuous case, the obtention of the value $s(\mathbb{R}) = \frac{T}{12}$ relies on the Fourier series expansion for periodic functions (see e.g. [15]), which should be adapted accordingly to the general context. For example, a rapid computation shows, for arbitrary $\mathbb{T}$, that $s(\mathbb{T}) \le \frac{T}{4}$ which, applied to this case, retrieves the condition $cT \le \pi$. The same conclusions are obtained for the sunflower-like equation

$$\left(\frac{x^\Delta(t)}{\sqrt{1-\frac{x^\Delta(t)^2}{c^2}}}\right)^\Delta + h(x(t))x^\Delta(t) + \sin(x(t-r)) = p(t) \quad t \in \mathbb{T}.$$

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