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BEST PROXIMITY POINTS OF GENERALIZED α-ψ-GERAGHTY PROXIMAL CONTRACTIONS IN GENERALIZED METRIC SPACES

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Abstract. In this manuscript, we introduce the class of generalized α-ψ-Geraghty proximal contractions in the context of generalized metric spaces and set up some best proximity point results for these contractions. Our results extend, improve and generalize several existing results in the literature.

Key Words and Phrases: Fixed point, best proximity point, Geraghty proximal contractions, generalized metric spaces.
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1. Introduction

Fixed point theory focuses on the strategies for solving nonlinear equations of the form $Tx = x$, where the function T is defined on some abstract space X. It is well known that the remarkable Banach contraction principal is one of the most useful and fundamental results in modern mathematical analysis. It guarantees the existence and uniqueness of fixed points for certain self-maps in a complete metric space and provides a constructive method to find those fixed points. Due to its practical implication, several authors studied and extended it in various directions and in variety of settings; see for example [3, 5, 6, 9, 11, 27, 24, 28, 25, 31] and the references therein. All these generalizations are only applicable for self-mappings.

Recently, best proximity point theory attracted the attention of several authors. The purpose of best proximity point theory is to address a problem of finding the distance between two closed sets by using non self-mappings from one set to the other. Best proximity point theory analyzes the existence of an approximate solution that is optimal. Some applications of best proximity points in the investigations of market equilibrium in non competitive markets are presented in [13] and [18]. For some interesting known results on best proximity points, we refer readers to [4, 12, 14, 21, 22, 23, 29, 32, 37] and references therein.

Let X be a nonempty set, and let $d : X \times X \to \mathbb{R}^+$, and A, B be two non-empty subsets of a metric space (X, d) and $T : A \to B$ is a mapping, then $d(x, Tx) \geq d(A, B)$ for all $x \in A$. In general, for non self-mapping $T : A \to B$, the fixed point equation $Tx = x$ may not have a solution. In this case, it is focused on the possibility of finding an element $x \in A$ that is an approximate solution such that the error $d(x, Tx)$ is minimum, possibly $d(x, Tx) = d(A, B)$. In case $d(x, Tx) = d(A, B)$, we call that x is a best proximity point of T in A.

A best proximity point becomes a fixed point if the underlying mapping is a self-mapping. Therefore, it can be concluded that best proximity point theorems generalize fixed point theorems in a natural way. In recent years, the existence and convergence of best proximity points is an interesting aspect of optimization theory which attracted the attention of many authors, see for examples [1, 2, 7, 8]. Recently, Asadi et. al. [5] presented the fixed point theorems for α-ψ-Geraghty type contractions in generalized metric spaces and L.B. Kumssa [29] introduced best proximity point of modified Suzuki-Edelstein -Geraghty type proximal contractions in metric spaces. The works of Asadi et al.[5] and L.B. Kumssa [29] have inspired and motivated our work a great deal.

2. Preliminaries

Let A, B be non-empty sets, the following notations and definitions are crucial for the rest of our manuscript:-

1) $d(A, B) = \inf\{d(x, y) : x \in A, y \in B\}$;
2) $A_0 = \{x \in A : d(x, b) = d(A, B), \text{ for some } \ b \in B\}$;
3) $B_0 = \{y \in B : d(a, y) = d(A, B), \text{ for some } \ a \in A\}$.

Let $\mathbb{R}^+ = [0, \infty)$ and $\mathbb{N}$ be the set of positive integers. We denote by $\mathcal{F}$ the class of all functions $\beta : [0, \infty) \to [0, 1)$ satisfying the following condition:

$$\beta(t_n) \to 1 \implies t_n \to 0.$$

Definition 2.1. [10] Let X be a nonempty set, and let $d : X \times X \to \mathbb{R}$ satisfy the following conditions for all $x, y \in X$ and all distinct $u, v \in X$ each of which is different from x and y,

GM1. $d(x, y) = 0 \quad$ if and only if $\quad x = y$,
GM2. $d(x, y) = d(y, x)$,
GM3. $d(x, y) \leq d(x, u) + d(u, v) + d(v, y)$.

The map d above is called a *generalized metric* and abbreviated as GM, and the pair (X, d) is called a *generalized metric space* and abbreviated as GMS. Let d be a generalized metric on X and $\epsilon > 0$, we call $B_d(x, \epsilon) = \{y \in X | d(x, y) < \epsilon\}$ an ϵ-ball centered at x.

In the above definition, the condition (GM3) is called the ***quadrilateral inequality***. Notice also that if d satisfies only (GM1) and (GM2), then it is called semimetric (see, e.g., [38]).

Convergent and Cauchy sequences in GMS, completeness, as well as open balls $B_d(p, r)$ can be introduced in a a standard way. However, we refer readers to an example presented by Sarma et al. in [[35], Example 1.1] (see also [23, 36]), which shows several possible properties of generalized metric, different than in the standard metric case.

Example. [35] Let $A = \{0, 3\}$, $B = \left\{\dfrac{n}{n+1} : n \in \mathbb{N}\right\}$ and $X = A \cup B$. Define $d : X \times X \to [0, +\infty)$ as follows:

$$d(x,y) = \begin{cases} 0, \text{ if } \quad x = y \\ 2, \text{ if } \quad x \neq y \text{ and } \{x, y\} \subset A \text{or } \{x, y\} \subset B \\ y, \text{ if } x \in A, y \in B \\ x, \text{ if } x \in B, y \in A. \end{cases}$$

It is easy to show that (X, d) is a generalized metric space, but not a standard metric space, because it lacks the triangular property. For

$$d(0,3) = 2 \nleq d\left(0, \frac{3}{4}\right) + d\left(\frac{3}{4}, 3\right) = \frac{3}{4} + \frac{3}{4} = \frac{3}{2}.$$

Remark 2.1. We note that

(1) Every metric space is a generalized metric space, but the converse is not true in general, see for example in [10, 15, 30].
(2) In [10] , it was taken for granted that a generalized metric space is a Hausdorff topological space and as in a metric space, the topology of a generalized metric space can be generated by the collection of all ϵ-balls $B_d(x, \epsilon)$ for $x \in X$ and $\epsilon > 0$. But Das and Lahiri [11] showed that these assumptions are not true in an arbitrary generalized metric space (see [11], Example 1 and Example 2). Nevertheless, it is to be observed that the GMS (X, d) becomes a topological space when a subset U of X is said to be open if to each $a \in U$, there exists a positive number ϵ_a such that $B_d(a, \epsilon_a) \subseteq U$. For a nice discussion on the topological structure of GMS, we refer readers to [36].

Definition 2.2. [5] The concepts of convergence, Cauchy sequence, completeness and continuity on a GMS (X, d) are defined as follows.

1) A sequence $\{x_n\}$ in a GMS (X, d) is GMS convergent to a limit x if and only if $d(x_n, x) \to 0$ as $n \to \infty$.

2) A sequence $\{x_n\}$ in a GMS (X,d) is GMS Cauchy if and only if for every $\epsilon > 0$ there exists a positive integer $N(\epsilon)$ such that $d(x_n, x_m) < \epsilon$ for all $n > m > N(\epsilon)$.
3) A GMS (X,d) is said to be complete if every GMS Cauchy sequence in X is GMS convergent.
4) A mapping $T : X \to X$ is continuous if for each sequence $\{x_n\}$ in X such that $d(x_n, x) \to 0$ as $n \to \infty$, we have $d(Tx_n, Tx) \to 0$ as $n \to \infty$.

Proposition. [25] *Suppose that $\{x_n\}$ is a Cauchy sequence in a GMS (X,d) with*

$$\lim_{n\to\infty} d(x_n, u) = 0$$

where $u \in X$. Then

$$\lim_{n\to\infty} d(x_n, v) = d(u, v)$$

for all $v \in X$.

Definition 2.3. [39] Let A and B be two non-empty subsets of (X,d) and $A_0 \neq \emptyset$. We say that the pair (A,B) has weak P-property if

$$\left.\begin{array}{l} d(x_1, y_1) = d(A,B) \\ d(x_2, y_2) = d(A,B) \end{array}\right\} \quad \Rightarrow \quad d(x_1, x_2) \leq d(y_1, y_2),$$

for all $x_1, x_2 \in A$ and $y_1, y_2 \in B$.

Definition 2.4. [17] Let A and B be two non-empty subsets of (X,d) and $\alpha, \eta : A \times A \to [0,\infty)$ be functions. We say that a non self-mapping $T : A \to B$ is α-proximal admissible with respect to η if, for all $x, y, u, v, z, w \in A$,

$$\left.\begin{array}{l} \alpha(x,y) \geq \eta(x,y) \\ d(u, Tx) = d(A,B) \\ d(v, Ty) = d(A,B) \end{array}\right\} \quad \Rightarrow \quad \alpha(u,v) \geq \eta(u,v). \tag{2.1}$$

Definition 2.5. [17] Let A and B be two non-empty subsets of (X,d) and $\alpha : A \times A \to [0,\infty)$ be a function. We say that a non self-mapping $T : A \to B$ is α-proximal admissible if, for all $x, y, u, v \in A$,

$$\left.\begin{array}{l} \alpha(x,y) \geq 1 \\ d(u, Tx) = d(A,B) \\ d(v, Ty) = d(A,B) \end{array}\right\} \quad \Rightarrow \quad \alpha(u,v) \geq 1. \tag{2.2}$$

Definition 2.6. [26] Let A and B be two non-empty subsets of a (X,d) and $\alpha : A \times A \to [0,\infty)$ be a function. We say that a non self-mapping $T : A \to B$ is triangular α-proximal admissible if, for all $x, y, z, x_1, x_2, u_1, u_2 \in A$,

(T1).

$$\left.\begin{array}{l} \alpha(x_1, x_2) \geq 1 \\ d(u_1, Tx_1) = d(A,B) \\ d(u_2, Tx_2) = d(A,B) \end{array}\right\} \quad \Rightarrow \quad \alpha(u_1, u_2) \geq 1.$$

(T2).

$$\left.\begin{array}{l} \alpha(x,z) \geq 1 \\ \alpha(z,y) \geq 1 \end{array}\right\} \quad \Rightarrow \quad \alpha(x,y) \geq 1.$$

Definition 2.7. [17] Let A and B be two non-empty subsets of (X, d) and $\alpha, \eta : A \times A \to [0, \infty)$ be functions. We say that a non self-mapping $T : A \to B$ is α-proximal admissible with respect to η if, for all $x, y, u, v, z, w \in A$,

$$\left.\begin{array}{l} \alpha(x,y) \geq \eta(x,y) \\ d(u,Tx) = d(A,B) \\ d(v,Ty) = d(A,B) \end{array}\right\} \quad \Rightarrow \quad \alpha(u,v) \geq \eta(u,v). \tag{2.3}$$

Definition 2.8. [17] Let A and B be two non-empty subsets of (X, d) and $\alpha : A \times A \to [0, \infty)$ be a function. We say that a non self-mapping $T : A \to B$ is α-proximal admissible if, for all $x, y, u, v \in A$,

$$\left.\begin{array}{l} \alpha(x,y) \geq 1 \\ d(u,Tx) = d(A,B) \\ d(v,Ty) = d(A,B) \end{array}\right\} \quad \Rightarrow \quad \alpha(u,v) \geq 1. \tag{2.4}$$

Definition 2.9. [26] Let A and B be two non-empty subsets of a (X, d) and $\alpha : A \times A \to [0, \infty)$ be a function. We say that a non self-mapping $T : A \to B$ is triangular α-proximal admissible if, for all $x, y, z, x_1, x_2, u_1, u_2 \in A$,

(T1).

$$\left.\begin{array}{l} \alpha(x_1,x_2) \geq 1 \\ d(u_1,Tx_1) = d(A,B) \\ d(u_2,Tx_2) = d(A,B) \end{array}\right\} \quad \Rightarrow \quad \alpha(u_1,u_2) \geq 1.$$

(T2).

$$\left.\begin{array}{l} \alpha(x,z) \geq 1 \\ \alpha(z,y) \geq 1 \end{array}\right\} \quad \Rightarrow \quad \alpha(x,y) \geq 1.$$

Lemma 2.10. [34] *Let $T : X \to X$ be a triangular α-admissible map. Assume that there exists $x \in X$ such that $\alpha(Tx, Ty) \geq 1$. Define a sequence $\{x_n\}$ by $x_{n+1} = Tx_n$. Then we have $\alpha(x_n, x_m) \geq 1$ for all $m, n \in \mathbb{N}$ with $n < m$.*

Let Ψ denote the class of auxiliary functions $\psi : [0, \infty) \to [0, \infty)$ which satisfy the following conditions:

(a) ψ is nondecreasing;
(b) ψ is continuous;
(c) $\psi(t) = 0 \Leftrightarrow t = 0$.

Recently, Asadi et al.[5] introduced the following definition.

Definition 2.11. [5] Let (X, d) be a generalized metric space, and let $\alpha : X \times X \to \mathbb{R}$ be a function. A map $T : X \to X$ is called α-ψ-Geraghty contraction mapping if there exists $\beta \in \mathcal{F}$ such that for all $x, y \in X$,

$$\alpha(x,y)\psi(d(Tx,Ty)) \leq \beta(\psi(d(x,y))\psi(d(x,y))),$$

where $\psi \in \Psi$, $\mathbb{R}$ = set of real numbers.

Remark 2.12. If we take $\psi(t) = t$ in Definition 2.11, then T is called α-Geraghty contraction mapping. Again, if we take $\alpha(x, y) = 1$ for all $x, y \in X$ in Definition 2.11, then T is called ψ-Geraghty contraction mapping.

In this manuscript, inspired and motivated by Asadi et al.[5] and L.B. Kumssa [29] we established best proximity point results of generalized α-ψ-Geraghty proximal contractions in the setting of generalized metric spaces. Our results extend and improve many corresponding results obtained in the literature.

3. Main results

We introduce the following definition.

Definition 3.1. Let A, B be two non-empty subsets of a generalized metric space (X, d), and let $\alpha : X \times X \to \mathbb{R}$ be a function. A non-self map $T : A \to B$ is called a *generalized α-ψ-Geraghty proximal contraction* if there exists $\beta \in \mathcal{F}$ such that for all $x, y \in X$,

$$\alpha(x,y)\psi(d(Tx,Ty)) \leq \beta(\psi(M(x,y)))\psi(\max\{d(x,y), m(x,y) - d(A,B)\}), \tag{3.1}$$

where

$$M(x,y) = \max\{d(x,y), d(x,Tx), d(y,Ty)\}$$

and

$$m(x,y) = \max\{d(x,Tx), d(y,Ty)\}.$$

Note that if $A = B = X$ and take $\psi(t) = t$ and $M(x,y) = d(x,y) = m(x,y)$ in Definition 3, then T is called an α-Geraghty contraction mapping. Again, if we take $A = B = X$ and $\alpha(x,y) = 1$, $M(x,y) = d(x,y) = m(x,y)$ for all $x, y \in X$ in Definition 3, then T is called a ψ-Geraghty contraction mapping. One can see that Definition 3 significantly generalizes Definition 2.

We note that the following lemma appears in [19] and we again prove it here.

Lemma 3.2. [19] *Suppose (X, d) is a generalized metric space and $\{x_n\}$ be a sequence in X such that $d(x_n, x_{n+1}) \to 0$ and $d(x_n, x_{n+2}) \to 0$ as $n \to \infty$. If $\{x_n\}$ is not a Cauchy sequence, then there exist an $\epsilon > 0$ and sequences of positive integers $\{m_k\}$ and $\{n_k\}$ with $m_k > n_k > k$ such that $d(x_{m_k}, x_{n_k}) \geq \epsilon$, $d(x_{m_k-1}, x_{n_k}) < \epsilon$ and*

(i) $\lim_{k\to\infty} d(x_{m_k-1}, x_{n_k+1}) = \epsilon$.
(ii) $\lim_{k\to\infty} d(x_{m_k}, x_{n_k}) = \epsilon$.
(iii) $\lim_{k\to\infty} d(x_{m_k-1}, x_{n_k}) = \epsilon$.
(iv) $\lim_{k\to\infty} d(x_{m_k}, x_{n_k+1}) = \epsilon$.

Proof. If $\{x_n\}$ is not a Cauchy sequence, the there exists an $\epsilon > 0$ and sequences of positive integers $\{m_k\}$ and n_k such that $m_k > n_k > k$ satisfying

$$d(x_{m_k}, x_{n_k}) \geq \epsilon. \tag{3.2}$$

We choose m_k, the least positive integer satisfying (3.2) such that

$$d(x_{m_k-1}, x_{n_k}) < \epsilon. \tag{3.3}$$

We now prove

(i). By using the quadrilateral inequality, we have

$$\epsilon \leq d(x_{m_k}, x_{n_k}) \leq d(x_{m_k}, x_{m_k-1}) + d(x_{m_k-1}, x_{n_k+1}) + d(x_{n_k+1}, x_{n_k}).$$

Taking the limit inferior as $k \to \infty$, we have

$$\epsilon \leq \liminf_{k\to\infty} d(x_{m_k}, x_{m_k-1}) + \liminf_{k\to\infty} d(x_{m_k-1}, x_{n_k+1}) + \liminf_{k\to\infty} d(x_{n_k+1}, x_{n_k}).$$

Now using $d(x_n, x_{n+1}) \to 0$ as $n \to \infty$, we get

$$\epsilon \leq \liminf_{k\to\infty} d(x_{m_k-1}, x_{n_k+1}). \tag{3.4}$$

Again, by using the quadrilateral inequality, we get

$$d(x_{m_k-1}, x_{n_k+1}) \leq d(x_{m_k-1}, x_{m_k}) + d(x_{m_k}, x_{n_k}) + d(x_{n_k}, x_{n_k+1}).$$

Taking the limit superior as $k \to \infty$, we get

$$\limsup_{k\to\infty} d(x_{m_k-1}, x_{n_k+1}) \leq \epsilon. \tag{3.5}$$

From (3.4) and (3.5), we obtain

$$\liminf_{k\to\infty} d(x_{m_k-1}, x_{n_k+1}) = \limsup_{k\to\infty} d(x_{m_k-1}, x_{n_k+1}) = \epsilon.$$

This means $\lim\limits_{k\to\infty} d(x_{m_k-1}, x_{n_k+1})$ exists and so $\lim\limits_{k\to\infty} d(x_{m_k-1}, x_{n_k+1}) = \epsilon$. Hence (i) holds.

(ii). We have $d(x_{m_k}, x_{n_k}) \geq \epsilon$, and hence

$$\epsilon \leq \liminf_{k\to\infty} d(x_{m_k}, x_{n_k}). \tag{3.6}$$

Now,

$$\begin{aligned} d(x_{m_k}, x_{n_k}) &\leq d(x_{m_k}, x_{m_k-2}) + d(x_{m_k-2}, x_{m_K-1} + d(x_{m_k-1}, x_{n_k})) \\ &\leq d(x_{m_k}, x_{m_k-2}) + d(x_{m_k-2}, x_{m_K-1} + \epsilon. \end{aligned}$$

This implies,

$$\limsup_{k\to\infty} d(x_{m_k}, x_{n_k}) \leq \epsilon. \tag{3.7}$$

From (3.6) and (3.7), we get

$$\liminf_{k\to\infty} d(x_{m_k}, x_{n_k}) = \limsup_{k\to\infty} d(x_{m_k}, x_{n_k}) = \epsilon.$$

Therefore, we obtain

$$\lim_{k\to\infty} d(x_{m_k}, x_{n_k}) = \epsilon.$$

Hence (ii) holds.

(iii) We have, $d(x_{m_k-1}, x_{n_k}) < \epsilon$. Hence

$$\limsup_{k\to\infty} d(x_{m_k-1}, x_{n_k}) \leq \epsilon. \tag{3.8}$$

Now,

$$\epsilon \leq d(x_{m_k}, x_{n_k}) \leq d(x_{m_k}, x_{m_k+1}) + d(x_{m_k+1}, x_{m_k-1}) + d(x_{m_k-1}, x_{n_k}).$$

Taking the limit inferior as $k \to \infty$ and using the property that $d(x_n, x_{n+1}) \to 0$ and $d(x_n, x_{n+2}) \to 0$ as $n \to \infty$, we get

$$\epsilon \leq \liminf_{k\to\infty} d(x_{m_k-1}, x_{n_k}). \tag{3.9}$$

From (3.8) and (3.9), we deduce that

$$\liminf_{k\to\infty} d(x_{m_k-1}, x_{n_k}) = \limsup_{k\to\infty} d(x_{m_k-1}, x_{n_k}) = \epsilon.$$

This means $\lim_{k\to\infty} d(x_{m_k-1}, x_{n_k+1}) = \epsilon$. Hence (iii) holds.

(iv) We now prove that $\lim_{k\to\infty} d(x_{m_k}, x_{n_k+1}) = \epsilon$. By the assumption,

$$d(x_{m_k}, x_{m_k+1}) \to 0 \text{ and } d(x_{m_k}, x_{m_k+2}) \to 0.$$

Hence, it is impossible that $m_k = n_k + 1$ or $m_k = n_k + 2$ (because in either of these cases it would be impossible to have $d(x_{m_k}, x_{n_k}) \geq \epsilon$). Then, by the quadrilateral inequality we have

$$d(x_{m_k}, x_{n_k+1}) \leq d(x_{m_k}, x_{n_k}) + d(x_{n_k}, x_{n_k-1}) + d(x_{n_k-1}, x_{n_k+1}).$$

Taking the Taking the limit superior as $k \to \infty$, we get

$$\limsup_{k\to\infty} d(x_{m_k}, x_{n_k+1}) \leq \epsilon. \tag{3.10}$$

Again, by the quadrilateral inequality we have

$$d(x_{m_k}, x_{n_k}) \leq d(x_{m_k}, x_{n_k+1}) + d(x_{n_k+1}, x_{n_k-1}) + d(x_{n_k-1}, x_{n_k}).$$

Taking the limit inferior as $k \to \infty$, we get

$$\epsilon \leq \liminf_{k\to\infty} d(x_{m_k}, x_{n_k+1}). \tag{3.11}$$

From (3.10) and (3.11), we obtain $\lim_{k\to\infty} d(x_{m_k}, x_{n_k+1}) = \epsilon$. Hence (iv) holds.

Remark 3.3. Readers can see a different proof of this lemma in [19].

In fact, Lemma 3 holds in a more general setting. We state the following lemma without proof. It slightly generalizes Lemma 3.

Lemma 3.4. *Suppose (X, d) is a generalized metric space and $\{x_n\}$ be a sequence in X such that $d(x_n, x_{n+1}) \to 0$ and $d(x_n, x_{n+2}) \to 0$ as $n \to \infty$. If $\{x_n\}$ is not a Cauchy sequence, then there exist an $\epsilon > 0$ and sequences of positive integers $\{m_k\}$ and $\{n_k\}$ with $m_k > n_k > k$ such that $d(x_{m_k}, x_{n_k}) \geq \epsilon$, $d(x_{m_k-1}, x_{n_k}) < \epsilon$ and for any fixed $s \in \mathbb{N}$,*

(i) $\lim_{k\to\infty} d(x_{m_k-j}, x_{n_k+j}) = \epsilon$, *for* $1 \leq j \leq s$.
(ii) $\lim_{k\to\infty} d(x_{m_k}, x_{n_k}) = \epsilon$.
(iii) $\lim_{k\to\infty} d(x_{m_k-j}, x_{n_k}) = \epsilon$, *for* $1 \leq j \leq s$.
(iv) $\lim_{k\to\infty} d(x_{m_k}, x_{n_k+j}) = \epsilon$, *for* $1 \leq j \leq s$.

Theorem 3.5. *Let A, B two non-empty subsets of a complete generalized metric space (X, d) and $\alpha : X \times X \to \mathbb{R}^+$ be a function, and let $T : A \to B$ be a generalized α-ψ-Geraghty proximal contraction map(Def.3). Suppose that the following conditions are satisfied:*

1). *$T(A_0) \subseteq B_0$ and the pair (A, B) satisfies the weak P-property;*
2). *T is triangular α-admissible with respect to $\eta(x, y) = 1$;*
3). *T is continuous;*
4). *there exists $x_0, x_1 \in A$ such that $d(x_1, Tx_0) = d(A, B)$ and $\alpha(x_0, x_1) \geq 1$.*

Then T has a unique best proximity in A_0.

Proof. By assumption (4), there exist $x_0, x_1 \in A$ such that

$$d(x_1, Tx_0) = d(A, B) \tag{3.12}$$

and

$$\alpha(x_0, x_1) \geq 1.$$

Since $Tx_0 \in B$, by the definition of A_0, from (3.28), we have $x_1 \in A_0$. Since $T(A_0) \subseteq B_0$, we have $Tx_1 \in B_0$. Hence by definition of B_0, there exists $x_2 \in A$ such that

$$d(x_2, Tx_1) = d(A, B). \tag{3.13}$$

Since T is α-proximal admissible with respect to $\eta(x, y) = 1$, we obtain $\alpha(x_1, x_2) \geq 1$. By continuing this process, we have

$$d(x_{n+1}, Tx_n) = d(A, B), \tag{3.14}$$

and

$$\alpha(x_n, x_{n+1}) \geq 1,$$

for all $n \in \mathbb{N}$. Now, by (3.1) we have

$$\begin{aligned} &\alpha(x_n, x_{n+1})\psi(d(Tx_n, Tx_{n+1})) \\ &\qquad \leq \beta\big(\psi(M(x_n, x_{n+1}))\big)\psi(\max\{d(x_n, x_{n+1}), m(x_n, x_{n+1}) - d(A, B)\}), \end{aligned} \tag{3.15}$$

where

$$M(x_n, x_{n+1}) = \max\{d(x_n, x_{n+1}), d(x_n, Tx_n), d(x_{n+1}, Tx_{n+1})\},$$

and

$$\begin{aligned} m(x_n, x_{n+1}) &= \max\{d(x_n, Tx_n), d(x_{n+1}, Tx_{n+1})\} \\ &= \max\{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2})\}. \end{aligned}$$

Suppose $x_{n_0} = x_{n_0+1}$, for some $n_0 \in \mathbb{N}$. Assume that $x_{n_0+1} \neq x_{n_0+2}$, then by (3.15), it follows that

$$\begin{aligned} &\psi(d(x_{n_0+1}, x_{n_0+2})) \\ &= \psi(d(Tx_{n_0}, Tx_{n_0+1})) \\ &\leq \alpha(x_{n_0}, x_{n_0+1})\psi(d(Tx_{n_0}, Tx_{n_0+1})) \\ &\leq \beta\big(\psi(M(x_{n_0}, x_{n_0+1}))\big)\psi(\max\{d(x_{n_0}, x_{n_0+1}), m(x_{n_0}, x_{n_0+1}) - d(A, B)\}) \\ &< \psi\big(\max\{d(x_{n_0}, x_{n_0+1}), m(x_{n_0}, x_{n_0+1}) - d(A, B)\}\big) \\ &= \psi\big(\max\{d(x_{n_0}, x_{n_0+1}), d(x_{n_0+1}, x_{n_0+2}) - d(A, B)\}\big) \\ &\leq \psi\big(\{d(x_{n_0+1}, x_{n_0+2}) + d(A, B)\} - d(A, B)\big) \\ &= \psi\big(d(x_{n_0+1}, x_{n_0+2})\big), \end{aligned}$$

which is a contradiction. Therefore $x_{n_0+1} = x_{n_0+2}$, hence $x_{n_0} = x_{n_0+1} = x_{n_0+2}$, so from (3.14), it follows that

$$d(x_{n_0}, Tx_{n_0}) = d(x_{n_0+1}, Tx_{n_0}) = d(A, B),$$

i.e., x_{n_0} is a best proximity point of T, a desired result. Therefore, we assume that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N} \cup \{0\}$. From (3.1), we obtain

$$\begin{aligned}
&\psi(d(x_n, x_{n+1})) = \psi(d(Tx_{n-1}, Tx_n)) \\
&\leq \alpha(x_{n-1}, x_n)\psi(d(Tx_{n-1}, Tx_n)) \\
&\leq \beta\big(\psi(M(x_{n-1}, x_n))\big)\psi(\max\{d(x_{n-1}, x_n), m(x_{n-1}, x_n) - d(A, B)\}) \\
&< \psi\big(\max\{d(x_{n-1}, x_n), m(x_{n-1}, x_n) - d(A, B)\}\big) \\
&= \psi\big(\max\{d(x_{n-1}, x_n), \max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\} - d(A, B)\}\big) \\
&\leq \psi\big(\max\{d(x_{n-1}, x_n), \max\{d(x_{n-1}, x_n) + d(A, B), d(x_n, x_{n+1}) + d(A, B)\} - d(A, B)\}\big) \\
&= \psi\big(\max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\}\big). \qquad (*)
\end{aligned}$$

If $\max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\} = d(x_n, x_{n+1})$, then, by (*) we get

$$\psi(d(x_n, x_{n+1})) < \psi(d(x_n, x_{n+1})),$$

which is a contradiction. Hence $\max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\} = d(x_{n-1}, x_n)$, so by (*) we have $\psi(d(x_n, x_{n+1})) < \psi(d(x_{n-1}, x_n))$. Since ψ is non-decreasing, it follows that $d(x_n, x_{n+1}) < d(x_{n-1}, x_n)$, for all $n \geq 1$. Hence we deduce that $\{d(x_n, x_{n+1})\}$ is a decreasing sequence of non-negative real numbers. So there exists $r \geq 0$ such that

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = r. \tag{3.16}$$

Suppose that $r > 0$. From (*), we obtain

$$0 < \frac{\psi(d(x_n, x_{n+1}))}{\psi(d(x_{n-1}, x_n))} \leq \beta\big(\psi(M(x_{n-1}, x_n))\big) < 1, \tag{3.17}$$

where

$$M(x_{n-1}, x_n) = \max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\} = d(x_{n-1}, x_n). \tag{3.18}$$

Taking the limit as $n \to \infty$ in (3.17) and using (3.18), we obtain

$$\lim_{n\to\infty} \beta(\psi(d(x_{n-1}, x_n))) = 1.$$

Since $\beta \in \mathcal{F}$, it follows that $\lim_{n\to\infty} \psi(d(x_{n-1}, x_n)) = 0$. By continuity of ψ, we get

$$\psi\big(\lim_{n\to\infty} d(x_{n-1}, x_n)\big) = 0, \tag{3.19}$$

i.e., $\psi(r) = 0$, so that $r = 0$. That is

$$\lim_{n\to\infty} d(x_{n-1}, x_n) = 0. \tag{3.20}$$

Hence, all elements of the sequence $\{x_n\}$are distinct. Now, we show that $\{x_n\}$ is a Cauchy sequence. Suppose $\{x_n\}$ is not a Cauchy sequence. Then there exists an $\epsilon > 0$ for which we can find sequences of positive integers $\{m_k\}$ and $\{n_k\}$ with $m_k > n_k > k$ such that

$$d(x_{m_k}, x_{n_k}) \geq \epsilon \qquad \text{and} \qquad (x_{m_k-1}, x_{n_k}) < \epsilon. \tag{3.21}$$

Since T is triangular α-proximal admissible with respect to $\eta(x, y) = 1$, we can show that $\alpha(x_n, x_m) \geq 1$ for all $n, m \in \mathbb{N}$ with $n < m$. If $n = m + 1$, we have

$$\alpha(x_n, x_m) \geq 1. \tag{3.22}$$

Suppose that $\alpha(x_n, x_m) \geq 1$ for all $n, m \in \mathbb{N}$ with $n < m$. We shall prove that $\alpha(x_{n_k}, x_{m_k}) \geq 1$ with $n_k < m_k$. From (3.14), we have

$$\alpha(x_m, x_{m+1}) \geq 1. \tag{3.23}$$

Since T is triangular α-proximal admissible with respect to $\eta(x, y) = 1$, then from (3.22) and (3.23), $\alpha(x_n, x_{m+1}) \geq 1$ for all $n, m \in \mathbb{N}$ with $n < m$. Hence, for any $m_k, n_k \in \mathbb{N}$ with $n_k < m_k$, we get $\alpha(x_{n_k}, x_{m_k}) \geq 1$. From (3.20) and (3.21) and by the quadrilateral inequality, we can choose a positive integer $k \in \mathbb{N}$, such that

$$\begin{aligned} d(x_{n_k}, x_{m_k}) &\leq d(x_{n_k}, x_{n_k+1}) + d(x_{n_k+1}, x_{m_k+1}) + d(x_{m_k+1}, x_{m_k}) \\ &= d(x_{n_k}, x_{n_k+1}) + d(Tx_{n_k}, Tx_{m_k}) + d(x_{m_k+1}, x_{m_k}). \end{aligned}. \tag{3.24}$$

Equivalently

$$d(x_{n_k}, x_{m_k}) - d(x_{n_k} x_{n_k+1}) - d(x_{m_k+1}, x_{m_k}) \leq d(Tx_{n_k}, Tx_{m_k}). \tag{3.25}$$

By applying ψ, we get that

$$\begin{aligned} &\psi\big(d(x_{n_k}, x_{m_k}) - d(x_{n_k} x_{n_k+1}) - d(x_{m_k+1}, x_{m_k})\big) \leq \psi(d(Tx_{n_k}, Tx_{m_k})) \\ &\leq \alpha(x_{n_k}, x_{m_k})\psi(d(Tx_{n_k}, Tx_{m_k})) \\ &\leq \beta\big(\psi(M(x_{n_k}, x_{m_k}))\big)\psi\big(\max\{d(x_{n_k}, x_{m_k}), m(x_{n_k}, x_{m_k}) - d(A, B)\}\big). \end{aligned} \tag{3.26}$$

Consider

$$\begin{aligned} &\max\{d(x_{n_k}, x_{m_k}), m(x_{n_k}, x_{m_k}) - d(A, B)\} \\ &= \max\{d(x_{n_k}, x_{m_k}), \max\{d(x_{n_k}, Tx_{n_k}), d(x_{m_k}, Tx_{m_k})\} - d(A, B)\} \\ &= \max\{d(x_{n_k}, x_{m_k}), \max\{d(x_{n_k}, x_{n_k+1}), d(x_{m_k}, x_{m_k+1})\} - d(A, B)\} \\ &\leq \max\{d(x_{n_k}, x_{m_k}), \max\{d(x_{n_k}, x_{n_k+1}) + d(A, B), d(x_{m_k}, x_{m_k+1}) + d(A, B)\} - d(A, B)\} \\ &= \max\{d(x_{n_k}, x_{m_k}), d(x_{n_k}, x_{n_k+1}), d(x_{m_k}, x_{m_k+1})\}. \end{aligned}$$

Applying Lemma 3, we get

$$\begin{aligned} &\lim_{k\to\infty} \max\{d(x_{n_k}, x_{m_k}), m(x_{n_k}, x_{m_k}) - d(A, B)\} \\ &\leq \lim_{k\to\infty} \max\{d(x_{n_k}, x_{m_k}), d(x_{n_k}, x_{n_k+1}), d(x_{m_k}, x_{m_k+1})\} = \epsilon. \end{aligned}$$

Taking the limit as $k \to \infty$ in (3.26), it follows that

$$0 < \frac{\psi(\epsilon)}{\psi(\epsilon)} \leq \lim_{k\to\infty} \beta\big(\psi(M(x_{n_k}, x_{m_k}))\big) \leq 1.$$

Therefore

$$\lim_{k\to\infty} \beta\big(\psi(M(x_{n_k}, x_{m_k}))\big) = 1.$$

Since $\beta \in \mathcal{F}$, we obtain

$$\lim_{k\to\infty} \psi(M(x_{n_k}, x_{m_k})) = 0,$$

or

$$\lim_{k\to\infty} \psi(\max\{d(x_{n_k}, x_{m_k}), d(x_{n_k}, x_{n_K+1}), d(x_{m_k}, x_{m_k+1})\}) = 0.$$

By the continuity of ψ, this yields

$$\psi\big(\lim_{k\to\infty} \max\{d(x_{m_k}, x_{n_k}), d(x_{m_k}, x_{m_K+1})\}, d(x_{n_k}, x_{n_k+1})\}\big) = 0,$$

i.e., $\psi(\epsilon) = 0$ and hence $\epsilon = 0$ which is a contradiction. Hence $\{x_n\}$ is a Cauchy sequence. By the completeness of X and the closed property of A, there exists $x* \in A$ such that $\lim_{n\to\infty} x_n = x*$. Since T is continuos, from (3.14), we obtain

$$d(A, B) = \lim_{n\to\infty} d(x_{n+1}, Tx_n) = d(x*, Tx*),$$

hence $x*$ is the best proximity of T in A_0.

Next, we show the uniqueness of the best proximity point. Suppose that $x*$ and $y*$ are the two distinct best proximity points of T. Since

$$d(x*, Tx*) = d(A, B) = d(y*, Ty*),$$

by the weak P-property of the pair (A, B), we get

$$d(x*, y*) \le d(Tx*, Ty*). \tag{3.27}$$

Now, by (3.1) and (3.27), we have

$$\begin{aligned}
\psi(d(x*, y*)) &\le \alpha(x*, y*)\psi(d(Tx*, Ty*)) \\
&\le \beta\big(\psi(M(x*, y*))\big)\psi\big(\max\{d(x*, y*), m(x*, y*)\} - d(A, B)\}\big) \\
&< \psi\big(\max\{d(x*, y*), m(x*, y*)\} - d(A, B)\}\big) \\
&= \psi\big(\max\{d(x*, y*), \max\{d(x*, Tx*), d(y*, Ty*)\} - d(A, B)\}\big) \\
&= \psi(d(x*, y*)),
\end{aligned}$$

which is a contradiction. Hence $x* = y*$. This completes our proof.

If we take $\psi(t) = t$ in Definition 3, we get the following definition.

Definition 3.6. Let A, B be two non-empty subsets of a generalized metric space (X, d), and let $\alpha : X \times X \to \mathbb{R}$ be a function. A non-self map $T : A \to B$ is called a *generalized α-Geraghty proximal contraction* if there exists $\beta \in \mathcal{F}$ such that for all $x, y \in X$,

$$\alpha(x, y)\psi(d(Tx, Ty)) \le \beta(M(x, y))(\max\{d(x, y), m(x, y) - d(A, B)\}, \tag{3.28}$$

where

$$M(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty)\}$$

and

$$m(x, y) = \max\{d(x, Tx), d(y, Ty)\}.$$

We have the following corollary.

Corollary 3.7. *Let A, B two non-empty subsets of a complete generalized metric space (X, d) and $\alpha : X \times X \to \mathbb{R}$ be a function, and let $T : A \to B$ be a generalized α-Geraghty proximal contraction map(Def.3). Suppose that the following conditions are satisfied:*

1). *$T(A_0) \subseteq B_0$ and the pair (A, B) satisfies the weak P-property;*
2). *T is triangular α-admissible with respect to $\eta(x, y) = 1$;*
3). *T is continuous;*
4). *there exists $x_0, x_1 \in A$ such that $d(x_1, Tx_0) = d(A, B)$ and $\alpha(x_0, x_1) \ge 1$.*

Then T has a unique best proximity in A_0.

Proof. By taking $\psi(t) = t$ in Theorem 3, our result follows.

It is also interesting to remove the continuity of the mapping T by replacing a weaker condition in the above theorem.

Definition 3.8. Let (X, d) be a complete generalized metric space, and $\alpha : X \times X \to \mathbb{R}$ be a function, and let $T : X \to X$ be a map. We say that the sequence $\{x_n\}$ is α-regular, the following condition is satisfied:

If $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all n and $x_n \to x \in X$ as $n \to \infty$, then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\alpha(x_{n_k}, x) \geq 1$ for all k.

Theorem 3.9. *Let A, B two non-empty subsets of a complete generalized metric space (X, d) and $\alpha : X \times X \to \mathbb{R}$ be a function, and let $T : A \to B$ be a generalized α-ψ-Geraghty proximal contraction map (Def.3). Suppose that the following conditions are satisfied:*

1). *$T(A_0) \subseteq B_0$ and the pair (A, B) satisfies the weak P-property;*
2). *T is triangular α-admissible with respect to $\eta(x, y) = 1$;*
3). *$\{x_n\}$ is α-regular;*
4). *there exists $x_0, x_1 \in A$ such that $d(x_1, Tx_0) = d(A, B)$ and $\alpha(x_0, x_1) \geq 1$.*

Then T has a unique best proximity in A_0.

Proof. From the proof of Theorem 3, $\{x_n\}$ is a Cauchy sequence such that $x_n \to x* \in A$ as $n \to \infty$. Since $\{x_n\}$ is α-regular, $x* \in A_0$. We shall prove that $d(x*, Tx*) = d(A, B)$. Suppose $d(x*, Tx*) \neq d(A, B)$. From the proof of Theorem 3, we have $d(x_{n+1}, x_{n+2}) \leq d(x_n, x_{n+1})$ for all $n \in \mathbb{N} \cap \{0\}$. Since $x_n \in A$, $\forall n \in \mathbb{N}$ and $x* \in A$, we obtain $\alpha(x_n, x*) \geq 1$.Thus from (3.1), it follows that

$$\begin{aligned}\psi(d(Tx_n,Tx*)) &\leq \alpha(x_n, x*)\psi(d(Tx_n, Tx*))\\ &\leq \beta\big(\psi(M(x_n, x*))\big)\psi(\max\{d(x_n, x*), m(x_n, x*) - d(A, B)\})\\ &< \psi\big(\max\{d(x_n, x*), m(x_n, x*) - d(A, B)\}\big)\\ &= \psi\big(\max\{d(x_n, x*), \max\{d(x_n, Tx_n), d(x*, Tx*)\} - d(A, B)\}\big).\end{aligned} \tag{3.29}$$

Taking the limit as $n \to \infty$ in inequality (3.29), we obtain

$$\psi\big(\lim_{n\to\infty} d(Tx_n, Tx*)\big) < \psi\big(d(x*, Tx*) - d(A, B)\big). \tag{3.30}$$

By the quadrilateral inequality, we get

$$d(x*, Tx*) \leq d(x*, x_{n+1}) + d(x_{n+1}, Tx_n) + d(Tx_n, Tx*),$$

or

$$d(x*, Tx*) - d(x_{n+1}, Tx_n) \leq d(x*, x_{n+1}) + d(Tx_n, Tx*)$$

From the property of ψ, it follows that

$$\psi\big(d(x*, Tx*) - d(x_{n+1}, Tx_n)\big) \leq \psi\big(d(x*, x_{n+1}) + d(Tx_n, Tx*)\big).$$

This implies

$$\psi\big(d(x*, Tx*) - d(A, B)\big) \leq \psi\big(\lim_{n\to\infty} d(Tx_n, Tx*)\big). \tag{3.31}$$

From (3.30) and (3.31), we have

$$\psi\big(d(x*, Tx*) - d(A, B)\big) < \psi\big(d(x*, Tx*) - d(A, B)\big),$$

which is a contradiction. Hence $d(x*, Tx*) = d(A, B)$. Therefore, $x*$ is the best proximity point of T. Uniqueness follows from the proof of Theorem 3.

Example 3.10. Let $X = \{a, b, c, d, e\}$ and $A = \{a, c, d\}$, $B = \{b, e\}$. Let $d : X \times X \to \mathbb{R}^+$ be a function such that $d(a, b) = 3$, $d(a, c) = d(b, c) = d(b, e) = d(c, e) = 1$, $d(a, d) = d(a, e) = d(b, e) = d(c, d) = d(b, d) = d(d, c) = d(d, e) = 2$ and $d(x, x) = 0$, $\forall x \in X$, $d(x, y) = d(y, x)$, $\forall x, y \in X$. $A_0 = \{c\}$, $B_0 = \{b, e\}$. It is easy to check that (X, d) is a generalized metric space but not a standard metric space because it lacks the triangular property. For

$$d(a, b) = 3 \nleq d(a, c) + d(c, b) = 1 + 1 = 2.$$

Let $T : A \to B$. Define

$$Tx = \begin{cases} b, \text{ if } & x \in \{a, b, c\} \\ e, \text{ otherwise} \end{cases}$$

One can see that T is a contraction. Let $\beta(t) = \dfrac{1+t}{1+2t}$ and $\psi(t) = \dfrac{t}{2}$. Let $\alpha : A \times A \to [0, \infty)$ be defined by

$$\alpha(x, y) = \begin{cases} 1, \text{ if } & x, y \in A \\ \dfrac{1}{4}, \text{ otherwise.} \end{cases}$$

Clearly, $T(A_0) \subseteq B_0, d(A, B) = 1$ and the pair (A, B) satisfies the weak P-property;. We prove that

1). T is triangular α-admissible with respect to $\eta(x, y) = 1$;
2). $\{x_n\}$ is α-regular;
3). there exists $x_0, x_1 \in A$ such that $d(x_1, Tx_0) = d(A, B)$ and $\alpha(x_0, x_1) \geq 1$.

Proof. To show

1). (i). Let $x, y, u, v \in A$ such that $\alpha(x, y) \geq 1$. Then, by the definition of α, we have

$$\left.\begin{array}{l} \alpha(x, y) \geq 1 \\ d(u, Tx) = d(A, B) \\ d(v, Ty) = d(A, B) \end{array}\right\} \quad \Rightarrow \quad \alpha(u, v) \geq 1.$$

(ii) Let $x, y, z \in A$ such that $\alpha(x, y) \geq 1$ and $\alpha(y, z) \geq 1$. Again the definition of α, $x, y, z \in A$ and so $\alpha(x, y) = 1$.

Thus (i) and (ii) imply that T is triangular α-admissible with respect to $\eta(x, y) = 1$.

2). Let $\{x_n\}$ be a sequence in A such that $\alpha(x_n, x_{n+1}) \geq 1$ for all n and $x_n \to x$ as $n \to \infty$. Since $A = \{a, c, d\}$ is closed, we get that $x \in A$. Therefore, the definition of α gives $\alpha(x_n, x) = 1$ for each $n \in \mathbb{N}$.

3). Taking $x_0 = d, x_1 = c$, we have $d(x_1, Tx_0) = d(c, Td) = d(c, e) = d(A, B) = 1$ and $\alpha(x_0, x_1) = \alpha(d, c) = 1$.

Now, one can show that T is a generalized α-ψ-Geraghty proximal contraction mapping. Let $x, y \in A$, then $\alpha(x, y) \geq 1$. It is not difficult to check that, for all $x, y \in X$, we have

$$\beta\big(\psi(M(x,y))\big)\psi(\max\{d(x,y), m(x,y) - d(A,B)\}) - \alpha(x,y)\psi(d(Tx,Ty)) > 0.$$

This means that

$$\alpha(x,y)\psi(d(Tx,Ty)) \leq \beta\big(\psi(M(x,y))\big)\psi(\max\{d(x,y), m(x,y) - d(A,B)\}),$$

for all $x, y \in X$. Hence T is a generalized α-ψ-Geraghty proximal contraction mapping. All conditions of Theorem 3 are satisfied. Therrefore, T has a unique best proximity point in A_0. Here c is the unique best proximity point of T in A_0.

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