Fixed Point Theory, 25(2024), No. 1, 3-14
DOI: 10.24193/fpt-ro.2024.1.01
http://www.math.ubbcluj.ro/~nodeacj/sfptcj.html

GENERALIZED MULTIVALUED $F-$ CONTRACTION ON ORTHOGONAL METRIC SPACE

ÖZLEM ACAR* AND EŞREF ERDOĞAN**

*Department of Mathematics, Faculty of Science, Selçuk University, 42003, Konya, Turkey
E-mail: acarozlem@ymail.com

**Department of Mathematics, Faculty of Science, Selçuk University, 42003, Konya, Turkey
E-mail: esreferdogan@yahoo.com

Abstract. In this paper, we consider the notion of generalized multivalued $F-$contraction mappings and prove fixed point theorems for this type mappings. Also, we construct non-trivial example to validate the potential of our result. Finally, as application, we apply our corollary to show the existence of a unique solution of the first-order ordinary differential equation.
Key Words and Phrases: Fixed point, F-contraction, orthogonal metric space.
2020 Mathematics Subject Classification: 54H25, 47H10.

1. Introduction and preliminaries

Using the concept of the Hausdorff metric, Nadler [11] introduced the notion of multivalued contraction mapping and gave a multivalued version of the well known Banach contraction principle such as:

Let (M, ρ) be a metric space. Denote by $P(M)$ the family of all nonempty subsets of M, $CB(M)$ the family of all nonempty, closed and bounded subsets of M and $K(M)$ the family of all nonempty compact subsets of M. It is well known that, $H : CB(M) \times CB(M) \to \mathbb{R}$ is defined by, for every $A, B \in CB(M)$,

$$H(A, B) = \max \left\{ \sup_{\varsigma \in A} D(\varsigma, B), \sup_{\omega \in B} D(\omega, A) \right\}$$

is a metric on $CB(M)$, which is called Hausdorff metric induced by ρ, where

$$D(\varsigma, B) = \inf \left\{ \rho(\varsigma, \omega) : \omega \in B \right\}.$$

Let $T : M \to CB(M)$ be a map, then T is called multivalued contraction if for all $\varsigma, \omega \in M$ there exists $L \in [0, 1)$ such that

$$H(T\varsigma, T\omega) \leq L\rho(\varsigma, \omega).$$

Then Nadler [11] proved that every multivalued contraction mappings on complete metric space has a fixed point.

Inspired by his result, various fixed point results concerning multivalued contractions appeared in the last decades [see, [4, 5, 7–10, 14, 15]] Also, combining the ideas of

Wardowski [16] and Nadler, multivalued F-contractions by was introduced in [1] and a fixed point result for these type mappings on complete metric space was given as:

Definition 1.1 ([1])**.** Let (M, ρ) be a metric space and $T : M \to CB(M)$ be a mapping. Then T is said to be a multivalued F-contraction if $F \in \mathcal{F}$ and there exists $\tau > 0$ such that

$$\varsigma, \omega \in M,\ H(T\varsigma, T\omega) > 0 \Rightarrow \tau + F(H(T\varsigma, T\omega)) \leq F(M(\varsigma, \omega))$$

where

$$M(\varsigma, \omega) = \max\left\{\rho(\varsigma, \omega), D(\varsigma, T\varsigma), D(\omega, T\omega), \frac{1}{2}\left[D(\varsigma, T\omega) + D(\omega, T\varsigma)\right]\right\}.$$

Theorem 1.1 ([1])**.** *Let (M, ρ) be a complete metric space and $T : M \to K(M)$ be a multivalued F-contraction, then T has a fixed point in M.*

Recently, Gordji et al. [6] introduced the concept of an orthogonal set and present some fixed point theorems in orthogonal metric spaces. Then Sharma et al. [13] introduced the notion of multivalued orthogonal F-contraction mappings in the framework of orthogonal metric space. Also you can see [2, 3, 12].

Now, we give some fundamental definitions and notations of corresponding mappings and space which are used in this paper.

Definition 1.2 ([6])**.** Let M be a non-empty set and $\curlywedge$ be a binary relation defined on M. If binary relation $\curlywedge$ fulfils the following criteria:

$$\exists\varsigma_0(\forall\omega\in M, \omega\curlywedge\varsigma_0) \text{ or } (\forall\omega \in M, \varsigma_0 \curlywedge \omega),$$

then pair, $(M, \curlywedge)$ known as an orthogonal set. The element ς_0 is called an orthogonal element. We denote this O-set or orthogonal set by $(M, \curlywedge)$.

Definition 1.3 ([6])**.** Let $(M, \curlywedge)$ be an orthogonal set (O-set). Any two elements $\varsigma, \omega \in M$ such that $\varsigma \curlywedge \omega$, then $\varsigma, \omega \in M$ are said to be orthogonally related.

Definition 1.4 ([6])**.** A sequence $\{\varsigma_n\}$ is called an orthogonal sequence (briefly O -sequence) if

$$(\forall n \in \mathbb{N}, \varsigma_n \curlywedge \varsigma_{n+1}) \text{ or } (\forall n \in \mathbb{N}, \varsigma_{n+1} \curlywedge \varsigma_n).$$

Similarly, a Cauchy sequence $\{\varsigma_n\}$ is said to be a orthogonally Cauchy sequence if

$$(\forall n \in \mathbb{N}, \varsigma_n \curlywedge \varsigma_{n+1}) \text{ or } (\forall n \in \mathbb{N}, \varsigma_{n+1} \curlywedge \varsigma_n).$$

Definition 1.5 ([6])**.** Let $(M, \curlywedge)$ be an orthogonal set and ρ be a metric on M. Then $(M, \curlywedge, \rho)$ is called an orthogonal metric space (shortly O-metric space).

Definition 1.6 ([6])**.** Let $(M, \curlywedge, \rho)$ be an orthogonal metric space. Then M is said to be a O-complete if every Cauchy O-sequence is converges in M.

Definition 1.7 ([6]). Let $(M, \perp, \rho)$ be an orthogonal metric space. A function $f : M \to M$ is said to be orthogonally continuous ($\perp$ -continuous) at ς if for each O-sequence $\{\varsigma_n\}$ converging to ς implies $f(\varsigma_n) \to f(\varsigma)$ as $n \to \infty$. Also f is $\perp$-continuous on M if f is $\perp$-continuous at every $\varsigma \in M$.

Definition 1.8 ([6]). Let a pair $(M, \perp)$ be an O-set, where $M(\neq \emptyset)$ be a non-empty set and $\perp$ be a binary relation on set M. A mapping $f : M \to M$ is said to be $\perp$-preserving if $f(\varsigma) \perp f(\omega)$ whenever $\varsigma \perp \omega$ and weakly $\perp$-preserving if $f(\varsigma) \perp f(\omega)$ or $f(\omega) \perp f(\varsigma)$ whenever $\varsigma \perp \omega$.

Definition 1.9 ([13]). Let A and B be two nonempty subsets of an orthogonal set $(M, \perp)$. The set A is orthogonal to set B is denoted by $\perp_1$ and defined as follows:

$$A \perp_1 B, \text{ if for every } a \in A \text{ and } b \in B, a \perp b.$$

Lemma 1.1 ([13]). *Let $(M, \perp, \rho)$ be an orthogonal metric space, $x \in M$ and $A \in K(M)$. Then there exists $a \in A$ such that*

$$D(x, A) = d(x, a).$$

Lemma 1.2 ([13]). *Let $(M, \perp, \rho)$ be an orthogonal metric space, and $A, B \in K(M)$, $a \in A$. Then there exists $b \in B$ such that*

$$d(a, b) \leq H(A, B).$$

Definition 1.10 ([16]). Let $\mathcal{F}$ be the set of all functions $F : (0, \infty) \to \mathbb{R}$ satisfying the following conditions:
$(F1)$ F is strictly increasing, i.e., for all $\alpha, \beta \in (0, \infty)$ such that $\alpha < \beta, F(\alpha) < F(\beta)$,
$(F2)$ for each sequence $\{a_n\}$ of positive numbers,

$$\lim_{n\to\infty} a_n = 0 \text{ if and only if } \lim_{n\to\infty} F(a_n) = -\infty,$$

$(F3)$ there exists $k \in (0, 1)$ such that $\lim_{\alpha\to 0^+} \alpha^k F(\alpha) = 0$.
We consider by $\mathcal{F}$ be the set of all functions F satisfying $(F1)$-$(F3)$ and
$(F4)$ $F(\inf A) = \inf F(A)$ for all $A \subset (0, \infty)$ with $\inf A > 0$.

The following examples will certify this assertion:

Example 1.1 ([16]). Let $F_1 : (0, \infty) \to \mathbb{R}$ be given by the formulae $F_1(\alpha) = \ln \alpha$. It is clear that $F_1 \in \mathcal{F}$.

Example 1.2 ([16]). Let $F_2 : (0, \infty) \to \mathbb{R}$ be given by the formulae $F_2(\alpha) = \alpha + \ln \alpha$. It is clear that $F_2 \in \mathcal{F}$.

We can find some different examples for the function F belonging to $\mathcal{F}$ in [16]. In addition, Wardowski concluded that every F-contraction T is a contractive mapping, i.e.,

$$d(Tx, Ty) < d(x, y), \text{ for all } x, y \in X, Tx \neq Ty.$$

Thus, every F-contraction is a continuous mapping.

Also, Wardowski concluded that if $F_1, F_2 \in \mathcal{F}$ with $F_1(\alpha) \leq F_2(\alpha)$ for all $\alpha > 0$ and $G = F_2 - F_1$ is nondecreasing, then every F_1-contraction T is an F_2-contraction.

He noted that for the mappings $F_1(\alpha) = \ln \alpha$ and $F_2(\alpha) = \alpha + \ln \alpha$, $F_1 < F_2$ and a mapping $F_2 - F_1$ is strictly increasing. Hence, it was obtained that every Banach contraction satisfies the contractive condition. On the other side, Example 2.5 in [16]

Motivated by the significance of the problems mentioned above, in this paper, we consider the notion of generalized multivalued $F-$contraction mappings and prove fixed point theorems for this mappings. Finally, we construct non-trivial example to validate the potential of our result.

2. Main Result

We begin with this section by presenting the new concept of generalized multivalued F-contraction on orthogonal metric space, then we give a fixed point theorems for this type mapping.

Definition 2.1. Let $(M, \curlywedge, \rho)$ be a metric space and $T : M \to CB(M)$ be a mapping. Then T is said to be a generalized multivalued orthogonal F-contraction if $F \in \mathcal{F}$ and there exist $\tau > 0, L > 0$ such that $\varsigma, \omega \in M$ with $\varsigma \curlywedge \omega$,

$$H(T\varsigma, T\omega) > 0 \Rightarrow \tau + F(H(T\varsigma, T\omega)) \leq F(M(\varsigma, \omega) + LN(\varsigma, \omega)), \tag{2.1}$$

where

$$\begin{aligned} M(\varsigma, \omega) &= \max \left\{ \rho(\varsigma, \omega), D(\varsigma, T\varsigma), D(\omega, T\omega), \frac{1}{2} \left[D(\varsigma, T\omega) + D(\omega, T\varsigma) \right] \right\} \\ N(\varsigma, \omega) &= \min \left\{ D(\varsigma, T\omega), D(\omega, T\varsigma) \right\}. \end{aligned}$$

Theorem 2.1. *Let $(M, \curlywedge, \rho)$ be an $O-$complete orthogonal metric space and $T : M \to K(M)$ be a mapping. Assume that the following conditions are satisfied:*
(i) There exists $\varsigma_0 \in M$ such that $\{\varsigma_0\} \curlywedge_1 T\varsigma_0$ or $T\varsigma_0 \curlywedge_1 \{\varsigma_0\}$,
(ii) For all $\varsigma, \omega \in M, \varsigma \curlywedge \omega$ implies $T\varsigma \curlywedge_1 T\omega$,
(iii) If $\{\varsigma_n\}$ is an orthogonal sequence in M such that $\varsigma_n \to \varsigma^*$, then $\varsigma_n \curlywedge \varsigma^*$ or $\varsigma^* \curlywedge \varsigma_n$ for all $n \in \mathbb{N}$,
(iv) T is a generalized multivalued orthogonal F-contraction.
Then, T has at least a fixed point in M.

Proof. By assumption (i), we can choose $\varsigma_1 \in T\varsigma_0$ such that $\varsigma_0 \curlywedge \varsigma_1$ or $\varsigma_1 \curlywedge \varsigma_0$ and from (ii), we get $T\varsigma_0 \curlywedge_1 T\varsigma_1$, that is there exists $\varsigma_2 \in T\varsigma_1$ such that $\varsigma_1 \curlywedge \varsigma_2$ or $\varsigma_2 \curlywedge \varsigma_1$. If $\varsigma_1 \in T\varsigma_1$, then ς_1 is a fixed point of T. Let $\varsigma_1 \notin T\varsigma_1$. Then $D(\varsigma_1, T\varsigma_1) > 0$ since $T\varsigma_1$ is compact. On the other hand, from

$$D(\varsigma_1, T\varsigma_1) \leq H(T\varsigma_0, T\varsigma_1)$$

and $(F1)$, we obtain

$$F(D(\varsigma_1, T\varsigma_1)) \leq F(H(T\varsigma_0, T\varsigma_1)).$$

From (2.1), we can write that

$$\begin{aligned}
& F(D(\varsigma_1, T\varsigma_1)) \\
\leq\ & F(H(T\varsigma_0, T\varsigma_1)) \leq F(M(\varsigma_0, \varsigma_1) + LN((\varsigma_0, \varsigma_1)) - \tau \\
=\ & F\left(\begin{array}{l} \max\left\{ \begin{array}{l} \rho(\varsigma_0, \varsigma_1), D(\varsigma_0, T\varsigma_0), D(\varsigma_1, T\varsigma_1), \\ \frac{1}{2}\left[D(\varsigma_0, T\varsigma_1) + D(\varsigma_1, T\varsigma_0)\right] \end{array} \right\} \\ +L\min\left\{D(\varsigma_0, T\varsigma_1), D(\varsigma_1, T\varsigma_0)\right\} \end{array} \right) - \tau \\
\leq\ & F\left(\max\left\{\rho(\varsigma_0, \varsigma_1), \frac{1}{2}D(\varsigma_0, T\varsigma_1)\right\}\right) - \tau \\
\leq\ & F\left(\max\left\{\rho(\varsigma_0, \varsigma_1), \frac{1}{2}\left[\rho(\varsigma_0, \varsigma_1) + D(\varsigma_1, T\varsigma_1)\right]\right\}\right) - \tau \\
\leq\ & F(\max\left\{\rho(\varsigma_0, \varsigma_1), D(\varsigma_1, T\varsigma_1)\right\}) - \tau \\
=\ & F(\rho(\varsigma_0, \varsigma_1)) - \tau. \qquad (2.2)
\end{aligned}$$

Continuing this process, we can construct an orthogonal sequence $\{\varsigma_n\}$ in M such that $\varsigma_{n+1} \in T\varsigma_n$ for all $n \in \mathbb{N} \cup \{0\}$. Thus, we have $\varsigma_{n+1} \curlywedge \varsigma_n$ or $\varsigma_n \curlywedge \varsigma_{n+1}$ for all $n \in \mathbb{N} \cup \{0\}$. If $\varsigma_k \in T\varsigma_k$ for all $k \in \mathbb{N} \cup \{0\}$ then ς_k is a fixed point of T. So, we may assume that $\varsigma_k \notin T\varsigma_k$ for all $k \in \mathbb{N} \cup \{0\}$. Since $T\varsigma_n$ closed, we have $D(\varsigma_n, T\varsigma_n) > 0$ for all $n \in \mathbb{N} \cup \{0\}$. Also

$$D(\varsigma_n, T\varsigma_n) \leq H(T\varsigma_{n-1}, T\varsigma_n).$$

So using $(F1)$, we have

$$F(D(\varsigma_n, T\varsigma_n)) \leq F(H(T\varsigma_{n-1}, T\varsigma_n)).$$

Further from (iv), we get

$$\begin{aligned}
& F(D(\varsigma_n, T\varsigma_n)) \\
\leq\ & F(H(T\varsigma_{n-1}, T\varsigma_n)) \\
\leq\ & F(M(\varsigma_{n-1}, \varsigma_n) + LN(\varsigma_{n-1}, \varsigma_n)) - \tau \\
=\ & F\left(\begin{array}{l} \max\left\{ \begin{array}{l} \rho(\varsigma_{n-1}, \varsigma_n), D(\varsigma_{n-1}, T\varsigma_{n-1}), D(\varsigma_n, T\varsigma_n), \\ \frac{1}{2}\left[D(\varsigma_{n-1}, T\varsigma_n) + D(\varsigma_n, T\varsigma_{n-1})\right] \end{array} \right\} \\ +L\min\left\{D(\varsigma_{n-1}, T\varsigma_n), D(\varsigma_n, T\varsigma_{n-1})\right\} \end{array} \right) - \tau \\
=\ & F\left(\max\left\{ \begin{array}{l} \rho(\varsigma_{n-1}, \varsigma_n), D(\varsigma_{n-1}, T\varsigma_{n-1}), D(\varsigma_n, T\varsigma_n), \\ \frac{1}{2}\left[D(\varsigma_{n-1}, T\varsigma_n) + D(\varsigma_n, T\varsigma_{n-1})\right] \end{array} \right\}\right) - \tau \\
\leq\ & F(\rho(\varsigma_{n-1}, \varsigma_n)) - \tau.
\end{aligned}$$

Hence from the strictly increasing property of F, we get

$$H(T\varsigma_{n-1}, T\varsigma_n) < \rho(\varsigma_{n-1}, \varsigma_n).$$

We know that $x_{n+1} \in Tx_n$,

$$\rho(\varsigma_n, \varsigma_{n+1}) = D(\varsigma_n, T\varsigma_n) \leq H(T\varsigma_{n-1}, T\varsigma_n) < \rho(\varsigma_{n-1}, \varsigma_n).$$

Therefore the sequence $\{\rho(\varsigma_n, \varsigma_{n+1})\}$ is strictly decreasing sequence. Suppose that $a_n = \rho(\varsigma_n, \varsigma_{n+1}) \to t$ for some $t \geq 0$. Furthermore for all $n \geq n_0$, we have

$$\begin{aligned} \tau + F(\rho(\varsigma_n, \varsigma_{n+1})) &\leq \tau + F(H(T\varsigma_n, T\varsigma_{n-1})) \\ &\leq F(\rho(\varsigma_n, \varsigma_{n-1})). \end{aligned} \tag{2.3}$$

Taking $n \to \infty$ in (2.3), we get a contradiction. So $\rho(\varsigma_n, \varsigma_{n+1}) \to 0$. From $(F3)$ there exists $k \in (0,1)$ such that

$$\lim_{n\to\infty} a_n^k F(a_n) = 0.$$

Then the following holds for all $n \in \mathbb{N}$

$$a_n^k F(a_n) - a_n^k F(a_0) \leq -a_n^k n\tau \leq 0. \tag{2.4}$$

Letting $n \to \infty$ in (2.4), we obtain that

$$\lim_{n\to\infty} n a_n^k = 0. \tag{2.5}$$

From (2.5), there exits $n_1 \in \mathbb{N}$ such that $na_n^k \leq 1$ for all $n \geq n_1$. So we have

$$a_n \leq \frac{1}{n^{1/k}} \tag{2.6}$$

for all $n \geq n_1$. In order to show that $\{\varsigma_n\}$ is a $O-$Cauchy sequence consider $m, n \in \mathbb{N}$ such that $m > n \geq n_1$. Using the triangular inequality for the metric and from (2.6), we have

$$\begin{aligned} \rho(\varsigma_n, \varsigma_m) &\leq \rho(\varsigma_n, \varsigma_{n+1}) + \rho(\varsigma_{n+1}, \varsigma_{n+2}) + \cdots + \rho(\varsigma_{m-1}, \varsigma_m) \\ &= a_n + a_{n+1} + \cdots + a_{m-1} \\ &= \sum_{i=n}^{m-1} a_i \\ &\leq \sum_{i=n}^{\infty} a_i \\ &\leq \sum_{i=n}^{\infty} \frac{1}{i^{1/k}}. \end{aligned}$$

By the convergence of the series $\sum\limits_{i=1}^{\infty} \frac{1}{i^{1/k}}$, we get $\rho(\varsigma_n, \varsigma_m) \to 0$ as $n \to \infty$. This yields that $\{\varsigma_n\}$ is a $O-$Cauchy sequence in (M, ρ). Since (M, ρ) is a $O-$complete metric space, the sequence $\{\varsigma_n\}$ converges to some point $z \in X$, that is, $\lim_{n\to\infty} \varsigma_n = z$.

Now we claim that, $z \in Tz$. Asume that $z \notin Tz$. Hence there exists $n_1 \in \mathbb{N}$ such that $z \notin \{\varsigma_n\}_{n \geq n_1}$, $H(T\varsigma_n, Tz) > 0$. Therefore further by our assumption, $\varsigma_n \curlywedge z$ or $z \curlywedge \varsigma_n$, using (iv), we get

$$
\begin{aligned}
& F(D(\varsigma_{n+1}, Tz)) \\
\leq \ & F(H(T\varsigma_n, Tz)) \\
\leq \ & F(M(\varsigma_n, z) + LN(\varsigma_n, z)) - \tau \\
\leq \ & F\left(\begin{array}{l} \max\left\{ \begin{array}{l} \rho(\varsigma_n, z), D(\varsigma_n, T\varsigma_n), D(z, Tz), \\ \frac{1}{2}\left[D(\varsigma_n, Tz) + D(z, T\varsigma_n)\right] \end{array} \right\} \\ +L \min\left\{D(\varsigma_n, Tz), D(z, T\varsigma_n))\right\} \end{array} \right) - \tau \\
\leq \ & F\left(\begin{array}{l} \max\left\{ \begin{array}{l} \rho(\varsigma_n, z), D(\varsigma_n, T\varsigma_n), D(z, Tz), \\ \frac{1}{2}\left[D(\varsigma_n, Tz) + D(z, T\varsigma_n)\right] \end{array} \right\} \\ +L \min\left\{D(\varsigma_n, Tz), D(z, \varsigma_{n+1}))\right\} \end{array} \right) - \tau \\
= \ & F\left(\begin{array}{l} \max\left\{ \begin{array}{l} \rho(\varsigma_n, z), \rho(\varsigma_n, \varsigma_{n+1}), D(z, Tz), \\ \frac{1}{2}\left[D(\varsigma_n, Tz) + D(z, T\varsigma_n)\right] \end{array} \right\} \\ +L \min\left\{D(\varsigma_n, Tz), D(z, \varsigma_{n+1}))\right\} \end{array} \right) - \tau
\end{aligned}
$$

Taking $n \to \infty$, we get $F(D(z, Tz)) \leq F(D(z, Tz)) - \tau$, which is a contradiction, so $z \in Tz$. This completes the proof. □

By adding the condition $(F4)$ on F, we can consider $CB(M)$ instead of $K(M)$.

Theorem 2.2. *Let $(M, \curlywedge, \rho)$ be an $O-$complete orthogonal metric space and $T : M \to CB(M)$ be a mapping. Assume that the following conditions are satisfied:*
(i) There exists $\varsigma_0 \in M$ such that $\{\varsigma_0\} \curlywedge_1 T\varsigma_0$ or $T\varsigma_0 \curlywedge_1 \{\varsigma_0\}$,
(ii) For all $\varsigma, \omega \in M, \varsigma \curlywedge \omega$ implies $T\varsigma \curlywedge_1 T\omega$,
(iii) If $\{\varsigma_n\}$ is an orthogonal sequence in M such that $\varsigma_n \to \varsigma^*$, then $\varsigma_n \curlywedge \varsigma^*$ or $\varsigma^* \curlywedge \varsigma_n$ for all $n \in \mathbb{N}$,
(iv) T is a generalized multivalued orthogonal F-contraction.
Then, T has at least a fixed point in M.

Proof. Let $\varsigma_0 \in M$. Since $T\varsigma$ is nonempty for all $\varsigma \in M$, by assumption (i), we can choose $\varsigma_1 \in T\varsigma_0$ such that $\varsigma_0 \curlywedge \varsigma_1$ or $\varsigma_1 \curlywedge \varsigma_0$. If $\varsigma_1 \in T\varsigma_1$, then ς_1 is a fixed point of T. Let $\varsigma_1 \notin T\varsigma_1$. Then $D(\varsigma_1, T\varsigma_1) > 0$ since $T\varsigma_1$ is closed. On the other hand, from

$$D(\varsigma_1, T\varsigma_1) \leq H(T\varsigma_0, T\varsigma_1)$$

and $(F1)$, we obtain

$$F(D(\varsigma_1, T\varsigma_1)) \leq F(H(T\varsigma_0, T\varsigma_1)).$$

From (2.1), we can write that

$$\begin{aligned}
& F(D(\varsigma_1, T\varsigma_1)) \\
\leq\ & F(H(T\varsigma_0, T\varsigma_1)) \leq F(M(\varsigma_0, \varsigma_1) + LN((\varsigma_0, \varsigma_1)) - \tau \\
=\ & F\left(\begin{array}{l} \max\left\{ \begin{array}{l} \rho(\varsigma_0, \varsigma_1), D(\varsigma_0, T\varsigma_0), D(\varsigma_1, T\varsigma_1), \\ \frac{1}{2}\left[D(\varsigma_0, T\varsigma_1) + D(\varsigma_1, T\varsigma_0)\right] \end{array} \right\} \\ +L\min\left\{D(\varsigma_0, T\varsigma_1), D(\varsigma_1, T\varsigma_0)\right\} \end{array} \right) - \tau \\
\leq\ & F\left(\max\left\{ \rho(\varsigma_0, \varsigma_1), \frac{1}{2} D(\varsigma_0, T\varsigma_1) \right\} \right) - \tau \\
\leq\ & F\left(\max\left\{ \rho(\varsigma_0, \varsigma_1), \frac{1}{2}\left[\rho(\varsigma_0, \varsigma_1) + D(\varsigma_1, T\varsigma_1)\right] \right\} \right) - \tau \\
\leq\ & F(\max\left\{\rho(\varsigma_0, \varsigma_1), D(\varsigma_1, T\varsigma_1)\right\}) - \tau \\
=\ & F(\rho(\varsigma_0, \varsigma_1)) - \tau. \qquad (2.7)
\end{aligned}$$

From $(F4)$ we get

$$F(D(\varsigma_1, T\varsigma_1)) = \inf_{y \in T\varsigma_1} F(\rho(\varsigma_1, y)).$$

So, from (2.7), we have

$$\begin{aligned}
F(D(\varsigma_1, T\varsigma_1)) &= \inf_{y \in T\varsigma_1} F(\rho(\varsigma_1, y)) \\
&\leq F(H(T\varsigma_0, T\varsigma_1)) \\
&\leq F(\rho(\varsigma_0, \varsigma_1)) - \tau \\
&< F(\rho(\varsigma_0, \varsigma_1)) - \frac{\tau}{2}.
\end{aligned}$$

By assumption (ii), we get $T\varsigma_0 \curlywedge_1 T\varsigma_1$. Continuing this process we constract an orthogonal sequence $\{\varsigma_n\}$ in M such that $\varsigma_{n+1} \in T\varsigma_n$ for all $n \in \mathbb{N} \cup \{0\}$. Thus we have $\varsigma_{n+1} \curlywedge \varsigma_n$ or $\varsigma_n \curlywedge \varsigma_{n+1}$ for all $n \in \mathbb{N} \cup \{0\}$. If $\varsigma_k \in T\varsigma_k$ for all $k \in \mathbb{N} \cup \{0\}$ then ς_k is a fixed point of T. So we may assume that $\varsigma_k \notin T\varsigma_k$ for all $k \in \mathbb{N} \cup \{0\}$. Since $T\varsigma_n$ closed, we have $D(\varsigma_n, T\varsigma_n) > 0$ for all $n \in \mathbb{N} \cup \{0\}$. Also

$$D(\varsigma_n, T\varsigma_n) \leq H(T\varsigma_{n-1}, T\varsigma_n).$$

So using $(F1)$, we have

$$F(D(\varsigma_n, T\varsigma_n)) \leq F(H(T\varsigma_{n-1}, T\varsigma_n)).$$

Further from (iv), we get

$$\begin{aligned}
& F(D(\varsigma_n, T\varsigma_n)) \\
\leq\ & F(H(T\varsigma_{n-1}, T\varsigma_n)) \\
\leq\ & F(M(\varsigma_{n-1}, \varsigma_n) + LN(\varsigma_{n-1}, \varsigma_n)) - \tau \\
=\ & F\left(\begin{array}{l} \max\left\{ \begin{array}{l} \rho(\varsigma_{n-1}, \varsigma_n), D(\varsigma_{n-1}, T\varsigma_{n-1}), D(\varsigma_n, T\varsigma_n), \\ \frac{1}{2}\left[D(\varsigma_{n-1}, T\varsigma_n) + D(\varsigma_n, T\varsigma_{n-1})\right] \end{array} \right\} \\ +L\min\left\{D(\varsigma_{n-1}, T\varsigma_n), D(\varsigma_n, T\varsigma_{n-1})\right\} \end{array} \right) - \tau \\
=\ & F\left(\max\left\{ \begin{array}{l} \rho(\varsigma_{n-1}, \varsigma_n), D(\varsigma_{n-1}, T\varsigma_{n-1}), D(\varsigma_n, T\varsigma_n), \\ \frac{1}{2}\left[D(\varsigma_{n-1}, T\varsigma_n) + D(\varsigma_n, T\varsigma_{n-1})\right] \end{array} \right\} \right) - \tau \\
\leq\ & F(\rho(\varsigma_{n-1}, \varsigma_n)) - \tau \\
<\ & F(\rho(\varsigma_{n-1}, \varsigma_n)) - \frac{\tau}{2}.
\end{aligned}$$

Since

$$F(D(\varsigma_n, T\varsigma_n)) = \inf_{y \in T\varsigma_n} F(\rho(\varsigma_n, y)).$$

Therefore using this equality, we get

$$\begin{aligned}
F(D(\varsigma_n, T\varsigma_n)) &= \inf_{y \in T\varsigma_n} F(\rho(\varsigma_n, y)) \\
&\leq F(H(T\varsigma_{n-1}, T\varsigma_n)) \\
&< F(\rho(\varsigma_{n-1}, \varsigma_n)) - \frac{\tau}{2}. \qquad (2.8)
\end{aligned}$$

So, from (2.8) we can get a sequence $\{\varsigma_n\}$ in M such that $\varsigma_{n+1} \in T\varsigma_n$ and

$$F(\rho(\varsigma_n, \varsigma_{n+1})) < F(\rho(\varsigma_{n-1}, \varsigma_n))$$

for all $n \in \mathbb{N}$. The rest of the proof can be completed as in the proof of Theorem 2.1. □

Example 2.1. Let $M = \{\varsigma_n = \frac{n(n+1)}{2} : n \in \mathbb{N}\}$ and $\rho(\varsigma, \omega) = |\varsigma - \omega|$, $\varsigma, \omega \in M$. Define a relation $\curlywedge$ on M by

$$\varsigma \curlywedge \omega \Longleftrightarrow \varsigma\omega \in \{\varsigma, \omega\} \subset M = \{\varsigma_n\}.$$

Then $(M, \curlywedge, \rho)$ is an $O-$complete metric space. Define the mapping $T : M \to K(M)$ by the:

$$T\varsigma = \begin{cases} \{\varsigma_1\} & , \quad \varsigma = \varsigma_1 \\ \{\varsigma_1, \varsigma_2, \cdots, \varsigma_{n-1}\} & , \quad \varsigma = \varsigma_n \end{cases}.$$

Then T is generalized multivalued orthogonal F-contraction with respect to

$$F(\alpha) = \alpha + \ln\alpha \text{ and } \tau = 1.$$

On the other hand, since

$$\lim_{n\to\infty}\frac{H(T\varsigma_n,T\varsigma_1)}{M(\varsigma_n,\varsigma_1)}=\lim_{n\to\infty}\frac{\varsigma_{n-1}-1}{\varsigma_n-1}=1,$$

then T is not generalized multivalued contraction.

Corollary 2.1. *Let $(M,\curlywedge,\rho)$ be an $O-$complete orthogonal metric space and $T:M\to CB(M)$ be a mapping. Assume that the following conditions are satisfied:*
(i) There exists $\varsigma_0\in M$ such that $\{\varsigma_0\}\curlywedge_1 T\varsigma_0$ or $T\varsigma_0\curlywedge_1\{\varsigma_0\}$,
(ii) For all $\varsigma,\omega\in M,\varsigma\curlywedge\omega$ implies $T\varsigma\curlywedge_1 T\omega$,
(iii) If $\{\varsigma_n\}$ is an orthogonal sequence in M such that $\varsigma_n\to\varsigma^*$, then $\varsigma_n\curlywedge\varsigma^*$ or $\varsigma^*\curlywedge\varsigma_n$ for all $n\in\mathbb{N}$,
(iv) $F\in\mathcal{F}$ and there exists $\tau>0$ such that $\varsigma,\omega\in M$ with $\varsigma\curlywedge\omega$,

$$H(T\varsigma,T\omega)>0\Rightarrow\tau+F(H(T\varsigma,T\omega))\leq F(M(\varsigma,\omega)),$$

where

$$M(\varsigma,\omega)=\max\left\{\rho(\varsigma,\omega),D(\varsigma,T\varsigma),D(\omega,T\omega),\frac{1}{2}\left[D(\varsigma,T\omega)+D(\omega,T\varsigma)\right]\right\}.$$

Then, T has at least a fixed point in M.

Corollary 2.2. *Let $(M,\curlywedge,\rho)$ be an $O-$complete orthogonal metric space and $T:M\to CB(M)$ be a mapping. Assume that the following conditions are satisfied:*
(i) There exists $\varsigma_0\in M$ such that $\{\varsigma_0\}\curlywedge_1 T\varsigma_0$ or $T\varsigma_0\curlywedge_1\{\varsigma_0\}$,
(ii) For all $\varsigma,\omega\in M,\varsigma\curlywedge\omega$ implies $T\varsigma\curlywedge_1 T\omega$,
(iii) If $\{\varsigma_n\}$ is an orthogonal sequence in M such that $\varsigma_n\to\varsigma^*$, then $\varsigma_n\curlywedge\varsigma^*$ or $\varsigma^*\curlywedge\varsigma_n$ for all $n\in\mathbb{N}$,
(iv) $F\in\mathcal{F}$ and there exists $\tau>0$ such that $\varsigma,\omega\in M$ with $\varsigma\curlywedge\omega$,

$$H(T\varsigma,T\omega)>0\Rightarrow\tau+F(H(T\varsigma,T\omega))\leq F(\rho(\varsigma,\omega)).$$

Then, T has at least a fixed point in M.

Corollary 2.3. *Let $(M,\curlywedge,\rho)$ be an $O-$complete orthogonal metric space and $T:M\to M$ be a mapping. Assume that the following conditions are satisfied:*
(i) There exists $\varsigma_0\in M$ such that $\{\varsigma_0\}\curlywedge_1 T\varsigma_0$ or $T\varsigma_0\curlywedge_1\{\varsigma_0\}$,
(ii) For all $\varsigma,\omega\in M,\varsigma\curlywedge\omega$ implies $T\varsigma\curlywedge_1 T\omega$,
(iii) If $\{\varsigma_n\}$ is an orthogonal sequence in M such that $\varsigma_n\to\varsigma^*$, then $\varsigma_n\curlywedge\varsigma^*$ or $\varsigma^*\curlywedge\varsigma_n$ for all $n\in\mathbb{N}$,
(iv) $F\in\mathcal{F}$ and there exists $\tau>0$ such that $\varsigma,\omega\in M$ with $\varsigma\curlywedge\omega$,

$$\tau+F(\rho(T\varsigma,T\omega))\leq F(\rho(\varsigma,\omega)).$$

Then, T has at least a fixed point in M.

3. Applications

Recall that, for any $1\leq \mathrm{p}<\infty$, the space $L^p(M,F,\mu)$ (or $L^p(M)$) consists of all complex-valued measurable functions κ on the underlying space M satisfying

$$\int_M|\kappa(\varsigma)|^{\mathrm{p}}\,d\mu(\varsigma),$$

where F is the σ-algebra of measurable sets and μ is the measure. When p $= 1$, the space $L^1(M)$ consists of all integrable functions κ on M and we define the L^1-norm of κ by

$$\|\kappa\|_1 = \int\limits_M |\kappa(\varsigma)|\, d\mu(\varsigma).$$

In the section, using Theorem 2.1, we show the existence of a solution of the following differential equation:

$$\begin{cases} u'(t) = f(t, u(t)), & a.e.\ t \in I := [0, T] \\ u(0) = a, & a \geq 1, \end{cases} \tag{3.1}$$

where $f : I \times \mathbb{R} \to \mathbb{R}$ is an integrable function satisfying the following conditions:
(i) $f(s, \mathrm{p}) \geq 0$ for all p ≥ 0 and $s \in I$;
(ii) for each $\varsigma, \omega \in L^1(I)$ with $\varsigma(s)\omega(s) \geq \varsigma(s)$ or $\varsigma(s)\omega(s) \geq \omega(s)$ for all $s \in I$, there exist $\kappa \in L^1(I)$ and $\tau > 0$ such that

$$|f(s, \varsigma(s)) - f(s, \omega(s))| \leq \frac{\kappa(s)}{\left(1 + \tau\sqrt{\kappa(s)}\right)^2} |\varsigma(s) - \omega(s)| \tag{3.2}$$

and

$$|\varsigma(s) - \omega(s)| \leq \kappa(s)\, e^{A(s)}$$

for all $s \in I$,where $A(s) := \int\limits_0^s |\kappa(w)|\, dw$.

Theorem 3.1. *Consider the differential Eq. 3.1. If (i) and (ii) are satisfied, then the differential Eq. 3.1 has a unique positive solution.*

Proof. Let $X = \{u \in C(I, \mathbb{R}) : u(t) > 0 \text{ for all } t \in I\}$. Define the orthogonality relation $\perp$ on M by

$$\varsigma \perp \omega \iff \varsigma(s)\omega(s) \geq \varsigma(s)\, or \varsigma(s)\omega(s) \geq \omega(s) \text{ for all } t \in I.$$

Since $A(t) = \int\limits_0^t |\kappa(s)|\, ds$, we have $A'(t) = |\kappa(t)|$ for almost everywhere $t \in I$.
Define a mapping $\rho(\varsigma, \omega) = \|\varsigma - \omega\|_A = \sup\limits_{t \in I} e^{-A(t)} |\varsigma(s) - \omega(s)|$ for all $\varsigma, \omega \in M$. Thus, (X, d) is a metric space and also a complete metric space (see, [6] for details). Define a mapping $\mathbb{G} : M \to M$ by

$$(\mathbb{G}\varsigma)(t) = a + \int\limits_0^t f(s, \varsigma(s))\, ds.$$

Then, we see that $\mathbb{G}$ is $\perp$-continuous. Now, we shot that $\mathbb{G}$ is $\perp$- preserving. For each $\varsigma, \omega \in M$ with $\varsigma \perp \omega$ and $t \in I$, we have

$$(\mathbb{G}\varsigma)(t) = a + \int\limits_0^t f(s, \varsigma(s))\, ds \geq 1.$$

It follows that $[(\mathbb{G}\varsigma)(t)][(\mathbb{G}\omega)(t)] \geq (\mathbb{G}\omega)(t)$ and so $(\mathbb{G}\varsigma)(t) \perp (\mathbb{G}\omega)(t)$. Then $\mathbb{G}$ is $\perp$-preserving.

Now, we can say that $\mathbb{G}$ satisfies Corollary 2.3 with $F(\alpha) = \frac{-1}{\sqrt{\alpha}}$. Hence the differential equation (3.1) has a unique positive solution. $\square$

References

[1] Ö. Acar, G. Durmaz, G. Mınak, *Generalized multivalued F-contractions on complete metric space*, Bull. Iranian Math. Soc., **40**(2014), no. 6, 1469-1478.

[2] Ö. Acar, E. Erdoğan, *Some fixed point results for almost contraction on orthogonal metric space*, Creat. Math. Inform., **31**(2022), no. 2, 147-153.

[3] Ö. Acar, A.S. Özkapu, *Multivalued Rational Type F-Contraction on Orthogonal Metric Space*, Mathematical Foundations of Computing 6 (3)(2022), 303-312.

[4] Lj. B. Ćirić, *Multi-valued nonlinear contraction mappings*, Nonlinear Anal., **71**(2009), 2716-2723.

[5] P.Z. Daffer, H. Kaneko, *Fixed points of generalized contractive multivalued mappings*, J. Math. Anal. Appl., **192**(1995), 655-666.

[6] M.E. Gordji, M. Rameani, M. De La Sen, Y.J. Cho, *On orthogonal sets and Banach fixed point theorem*, Fixed Point Theory, **18**(2017), 569-578.

[7] E. Karapinar, A. Fulga, R.P. Agarwal, *A survey: F-contractions with related fixed point results*, Journal of Fixed Point Theory and Applications, (2020), 22:69 https://doi.org/10.1007/s11784-020-00803-7.

[8] D. Klim, D. Wardowski, *Fixed point theorems for set-valued contractions in complete metric spaces*, J. Math. Anal. Appl., **334**(2007), 132-139.

[9] S. Kumar, M. Asim, *Fixed point theorems for a pair of ordered F-contraction mappings in ordered metric spaces,* Advances in Nonlinear Variational Inequalities, **25**(2022), no. 1, 17-28.

[10] S. Kumar, L. Sholastica, *On some fixed point theorems for multivalued F-contractions in partial metric spaces,* Demonstratio Mathematica, **54**(2021), 151-161.

[11] S.B. Nadler, *Multi-valued contraction mappings*, Pacific J. Math., **30**(1969), 475-488.

[12] K. Sawangsup, W. Sintunavarat, Y.J. Cho, *Fixed point theorems for orthogonal F-contraction mappings on O-complete metric spaces*, Journal of Fixed Point Theory and Applications, **22**(2020), no. 1, 1-14.

[13] R.K. Sharma, S. Chandok, *Multivalued problems, orthogonal mappings, and fractional integro-differential equation*, Journal of Mathematics, Volume 2020, Article ID 6615478, 8 pages.

[14] L. Sholastica, S. Kumar, G. Kakiko, *Fixed points for F-contraction mappings in partial metric spaces*, Lobachevskii Journal of Mathematics, **40**(2019), no. 2, 183-188.

[15] L. Wangwe, S. Kumar, *A common fixed point theorem for generalized F-Kannan-Suzuki type mapping in TVS valued cone metric space with applications,* Journal of Mathematics, Vol. 2022, Article ID 6504663, 17 pages, https://doi.org/10.1155/2022/6504663.

[16] D. Wardowski, *Fixed points of a new type of contractive mappings in complete metric spaces,* Fixed Point Theory Appl. 2012, 2012:94, 6 pp.

Received: October 21, 2021; Accepted: January 23, 2023.