Fixed Point Theory, 24(2023), No. 1, 413-418
DOI: 10.24193/fpt-ro.2023.1.23
http://www.math.ubbcluj.ro/~nodeacj/sfptcj.html

GENERALIZATIONS OF BOLZANO INTERMEDIATE VALUE THEOREM FOR BALLS AND CONVEX DOMAINS

XIAO-SONG YANG

School of Mathematics and Statistics, Huazhong University of Science and Technology,
Wuhan, 430074, China
E-mail: yangxs@hust.edu.cn
and
Hubei Key Laboratory of Engineering Modeling and Scientific Computing,
Huazhong University of Science and Technology, Wuhan 430074, China

Abstract. This note presents some new results on existence of zeros in continuous mappings for convex domains that can be regarded as generalizations of the Bolzano intermediate value theorem concerning the cube and the Borsuk-Ulam theorem for the unit ball. Based on the classic Borsuk-Ulam theorem, the main result is proved that under some antipodal-type inequality conditions any continuous map defined on a convex domain has zero point in the domain. As an application, we present a new proof of the well known Poincaré–Miranda theorem, also showing that the Borsuk-Ulam theorem implies the Poincaré–Miranda theorem.
Key Words and Phrases: Bolzano intermediate value theorem, the Borsuk-Ulam theorem, the Poincaré-Miranda theorem, continuous mappings, convex domains.
2020 Mathematics Subject Classification: 55M20, 54M25.

1. Introduction

A well known text-book theorem in mathematical analysis is the Bolzano intermediate value theorem stating that if a continuous function $f(x)$ defined on a closed interval $[a, b]$ satisfies $hf(a) \leq 0$ and $hf(b) \geq 0$ for $h \in \{-1, 1\}$ then $f(x)$ has at least one zero on this interval. This theorem was later extended to continuous mappings from n-dimensional cube into $\mathbf{R}^n$ by H. Poincaré and C. Miranda [3, 4, 5] which is well known as the Poincaré-Miranda theorem. Let us recall this theorem for reader's convenience. Let

$$I^n = [a_1, b_1] \times [a_2, b_2] \times \ldots \times [a_n, b_n] \subset R^n.$$

Consider a continuous mapping

$$f : I^n \to R^n,\ f(x) = (f_1(x), \ldots, f_n(x)),\ x = (x_1, x_2, \ldots, x_n).$$

The Poincaré-Miranda theorem can be stated as follows

The Poincaré-Miranda theorem. [4] *Suppose that a continuous mapping $f : I^n \to R^n$ satisfies*

$$h_i f_i(x_1, \ldots, x_{i-1}, a_i, x_{i+1}, \ldots, x_n) \geq 0$$

and

$$h_i f_i(x_1, \ldots, x_{i-1}, b_i, x_{i+1}, \ldots, x_n) \leq 0$$

for $i = 1, 2, \ldots, n$, *where* $h_i \in \{-1, 1\}$.

Then there exists a point $\bar{x} \in I^n$ *such that* $f(\bar{x}) = 0$.

The Poincaré-Miranda theorem as an n-dimensional generalization of the Bolzano intermediate value theorem is usually more suitable than the Brouwer fixed point theorem in many applications to the existence of periodic solutions of ordinary differential equations. Thus several proofs [3, 4, 5, 6, 7] from the elementary analytical to the topological were given to this wonderful theorem. It is also of significance to get a generalization of the Bolzano intermediate value theorem that can have more applications. Recently, this theorem has been extended to infinite dimensional Banach spaces [1], and some new versions of the Bolzano intermediate value theorem has been generalized to the case where the domain is a simplex [8, 9].

In this note, we will present some new generalizations of the Bolzano intermediate value theorem under the so called antipodal-type inequality condition. Our main result can be stated as follows.

The main theorem. *Let* $U \subset R^n$ *be a bounded open convex domain, and let* $f : \bar{U} \to R^n$ *be a continuous map. Suppose that there is point* $p \in U$ *such that for every straight line* $L(p)$ *passing through* p, *the following holds*

$$\langle f(x_L), f(y_L) \rangle < \|f(x_L)\| \|f(y_l)\|$$

where $\{x_L, y_L\} = L(p) \cap \partial\bar{U}$. *Then there is a point* $\bar{x} \in U$ *such that* $f(\bar{x}) = 0$.

The rest of this paper is organized as follows: Section 2 first considers the continuous map defined on the unit ball, and presents a result which is a general version of Borsuk-Ulam theorem. Section 3 provides the proof of the main result on continuous maps defined on convex domains, which seem more natural and succinct compared with the one dimensional Bolzano intermediate value theorem. As an application a new proof of the Poincaré-Miranda theorem is given in Section 4.

2. Bolzano Intermediate Value Theorem under antipodal inequality condition

The Bolzano intermediate value theorem can also be restated as follows. If a continuous function $f(x)$ defined on a closed interval $[-a, a]$ satisfies $f(-a)f(a) \leq 0$, then $f(x)$ has at least one zero on this interval. In the same spirit to this formulation, a more elegant generalization can be formulated in the context of the ball centered at the origin.

Let $D^n(r) \subset R^n$ be the closed ball centered at the origin with radius r. We have the following fact.

Proposition 2.1. *Consider a continuous map* $f : D^n(r) \to R^n$. *Assume that for each* $x \in \partial D^n(r)$, *the following inequality condition holds*

$$\langle f(-x), f(x) \rangle \leq 0 \text{ for every } x \in \partial D^n(r). \tag{2.1}$$

Then there exists at least one point $\bar{x} \in D^n$ such that $f(\bar{x}) = 0$.

A special case of the above statement is an equivalent version of the famous Borsuk-Ulam theorem:

The Borsuk-Ulam theorem. *Consider a continuous map $f : D^n(r) \to R^n$. If*

$$f(x) = -f(-x), \quad \forall x \in \partial D^n(r).$$

Then there exists a point $\bar{x} \in D^n(r)$ such that $f(\bar{x}) = 0$.

However, Proposition 1 is just a special case of the following result.

Theorem 2.1. *Consider a continuous map $f : D^n(r) \to R^n$. Assume that $\|f(x)\| \neq 0$, for each $x \in \partial D^n(r)$, and the following inequality condition holds*

$$\langle f(-x), f(x)\rangle < \|f(-x)\|\|f(x)\| \text{ for every } x \in \partial D^n(r).$$

Then the degree of the map f restricted to the boundary is an odd number, therefore there exists at least one point $\bar{x} \in D^n(r)$ such that $f(\bar{x}) = 0$.

For convenience the above inequality is called the boundary antipodal inequality.

Proof. Without loss of generality, suppose that f has no zero point on $\partial D^n(r) = S^{n-1}(r)$. Consider $\tilde{f} = f|_{\partial D^n(r)} : S^{n-1}(r) \to R^n$ and define

$$\theta = \frac{\tilde{f}}{\|\tilde{f}\|} : S^{n-1}(r) \to s^{n-1}$$

which is well defined on the sphere. Because of the above boundary antipodal inequality, the inequality $\theta(x) \cdot \theta(-x) < 1$ holds for all $x \in \partial D^n(r) = S^{n-1}(r)$.

It is easy to see that the map θ is not nullhomotopic. In fact, we can construct a homotopy $H : [0,1] \times S^{n-1}(r) \to S^{n-1}$ as follows

$$H(t,x) = \frac{\theta(x) - t\theta(-x)}{\|\theta(x) - t\theta(-x)\|}, \quad (t,x) \in [0,1] \times S^{n-1}(r).$$

Note that

$$\langle\theta(x), \theta(x)\rangle - t\langle\theta(-x), \theta(x)\rangle = 1 - t\langle\theta(x), \theta(x)\rangle > 0 \text{ for } \forall x \in S^{n-1}(r),$$

this homotopy makes sense. Therefore $\theta(x)$ is homotopic to the antipodal map

$$h(x) = \frac{\theta(x) - \theta(-x)}{\|\theta(x) - \theta(-x)\|}$$

which is not nullhomotopic by the well known Borsuk antipodal theorem ([2], page 93) consequently θ is not nullhomotopic. Therefore there exists at least one point $\bar{x} \in \mathrm{int}D^n(r)$ such that $f(\bar{x}) = 0$. □

Remark 2.1. In fact, it is easy to prove that the degree of an antipodal map is an odd number by virtue of the degree theory.

As an application of Theorem 2.1, we can give a new proof of the well-known Brower fixed point theorem.

Brouwer fixed point theorem. *Consider a continuous map $f : D^n \to R^n$. Assume that $f(D^n) \subseteq D^n$, then f has a fixed point.*

Proof. Without loss of generality suppose that f has no fixed point on the boundary ∂D^n. Then construct a continuous map

$$g(x) = f(x) - x.$$

It is easy to see that

$$\langle g(x), g(-x) \rangle < \|g(x)\| \|g(-x)\|, \ \forall x \in \partial D^n$$

and the equality holds only if

$$g(x) = f(x) - x = g(-x) = f(-x) + x, \ \forall x \in \partial D^n.$$

This yields

$$f(x) - f(-x) = 2x.$$

It is follows that

$$\langle x, f(x) \rangle - \langle x, f(-x) \rangle = 2, \ \forall x \in \partial D^n.$$

Since f has no fixed point on the boundary ∂D^n, $\langle x, f(x) \rangle < 1$, and this is in contradiction to the above equation. Therefore the strict inequality holds,

$$\langle g(x), g(-x) \rangle < \|g(x)\| \|g(-x)\|, \ \forall x \in \partial D^n.$$

By Theorem 2.1, g has a zero point in D^n, hence f has a fixed point in D^n. □

3. Bolzano Intermediate Value Theorem for convex domains

Theorem 3.1. *Let $K \subset \mathbf{R}^n$ be a bounded convex closed domain that is symmetric with respect to the origin, and let $f : K \to \mathbf{R}^n$ be a continuous map. Assume that $\|f(x)\| \neq 0$, for each $x \in \partial K$, and the following inequality condition holds*

$$\langle f(-x), f(x) \rangle < \|f(-x)\| \|f(x)\|, \ \textit{for every } x \in \partial K.$$

Then there is a point $p \in int\ K$ such that $f(p) = 0$.

Proof. Without loss of generality, suppose that F has no zero point on ∂K. Let $S^{n-1}(r) \subset \text{int } K$ be the sphere with radius $r > 0$. Consider the map g that sends the point $x \in S^{n-1}(r)$ to the point where $S^{n-1}(r)$ intersects with the half line $\{tx : t > 0\}$. Clearly $g : S^{n-1}(r) \to \partial K$ is a homeomorphism. Now the composition $f \circ g : S^{n-1}(r) \to \mathbf{R}^n$ satisfies the antipodal inequality, and it is easy to see that its degree number is odd in view of Remark 2.1 and Theorem 2.1, and this implies that the map $f : \partial K \to \mathbf{R}^n$ has an odd degree number. Therefore f has at least a zero in int K. □

It is easy to see that the above theorem can be extended to star-shape domains.

Proposition 3.1. *Let $K \subset \mathbf{R}^n$ be a bounded star-shape closed domain with respect to the origin, and let $f : K \to \mathbf{R}^n$ be a continuous map. Assume that $\|f(x)\| \neq 0$, for each $x \in \partial K$, and the following inequality condition holds*

$$\langle f(-x), f(x) \rangle < \|f(-x)\| \|f(x)\|, \ \textit{for every } x \in \partial K.$$

Then there is a point $p \in int\ K$ such that $f(p) = 0$.

More generally, we have the following statement.

Theorem 3.2. *Let $U \subset \mathbf{R}^n$ be a bounded open convex domain, and let $f : \bar{U} \to \mathbf{R}^n$ be a continuous map. Suppose that there is point $p \in U$ such that for every straight line $L(p)$ passing through p, the following holds*

$$\langle f(x_L), f(y_L)\rangle < \|f(x_L)\|\|f(y_L)\|,$$

where $\{x_L, y_L\} = L(p) \cap \partial\bar{U}$. Then there is a point $\bar{x} \in U$ such that $f(\bar{x}) = 0$.

Proof. Without loss of generality, suppose that f has no zero point on ∂U. Let $S^{n-1}(p,r) \subset U \subset \mathbf{R}^n$ be the sphere centered at p with radius $r > 0$. Consider the map g that sends the point $x \in S^{n-1}(p,r)$ to the point where $x \in S^{n-1}(p,r)$ intersect with the half line $\{t(x-p)+p : t > 0\}$. Clearly $g : S^{n-1}(p,r) \to \partial U$ is a homeomorphism because of convexity of ∂U. Now the composition $f \circ g : S^{n-1}(p,r) \to \mathbf{R}^n$ satisfies the antipodal inequality, and it is easy to see that its degree number is odd in view of Remark 2.1 and Theorem 2.1, and this implies that the map $f : \partial\bar{U} \to \mathbf{R}^n$ has also an odd degree number. Therefore, f has at least a zero in U, the proof is completed. □

The above theorem can be restated as follows.

Theorem 3.3. *Let $U \subset \mathbf{R}^n$ be a bounded convex open domains, and let $f : \bar{U} \to \mathbf{R}^n$ be a continuous map. Suppose that there is point $p \in U$ such that for every straight line $L(p)$ passing through p, the following holds:*

There is no $\lambda > 0$ satisfying $f(x_L) = \lambda f(y_L)$,

where $\{x_L, y_L\} = L(p) \cap \partial\bar{U}$.

Then there is a point $\bar{x} \in U$ such that $f(\bar{x}) = 0$.

4. A new proof of The Poincaré-Miranda theorem

As an application of the above arguments, we can give a new short proof of the well known Poincaré-Miranda theorem.

First, suppose that the strict boundary inequality holds

$$h_i f_i(x_1, \cdots, x_{i-1}, a_i, x_{i+1}, \cdots, x_n) > 0,$$

and

$$h_i f_i(x_1, \cdots, x_{i-1}, b_i, x_{i+1}, \cdots, x_n) < 0,$$

for $i = 1, 2, \cdots, n$, where $h_i \in \{-1, 1\}$.

The cube is $I^n = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n] \subset \mathbf{R}^n$. Denote

$$I_i^+ = \{x = (x_1, \cdots, x_i, \cdots, x_n) \in I^n, x_i = b_i\},$$
$$I_i^- = \{x = (x_1, \cdots, x_i, \cdots, x_n) \in I^n, x_i = a_i\}.$$

Obviously I^n is a convex domain. Take the center point of I^n,

$$p = \left(\frac{a_1+b_1}{2}, \frac{a_2+b_2}{2}, \cdots, \frac{a_n+b_n}{2}\right) \in I^n$$

and consider the line $L = \{t(x-p)+p : t \in \mathbf{R}\}, x \in \partial I^n$.

Without loss of generality, suppose that $x \in I_i^+$, then L intersects with $x \in I_i^-$ at $-x+2p$, since the boundary ∂I^n is symmetric with respect to p. It is easy to see that no $\lambda > 0$ satisfies $f(x) = \lambda f(-x+2p)$, since $f_i(x)f_i(-x+2p) < 0$. Then Theorem 3.3 applies to ensure the existence of zero in the interior of I^n. Now assume that the weak inequality (2.1) holds, and suppose that f has no zero on I^n. Then there is a $\delta > 0$ such that $f(x) \geq \delta, \forall x \in I^n$. Thus it is easy to perturb f by a small continuous map η with $2\|\eta(x)\| < \delta, \forall x \in I^n$ such that $\bar{f}(x) = f(x) + \eta(x)$ has no zero on the cube and the perturbed map satisfies

$$h_i\bar{f}_i(x_1, \cdots, x_{i-1}, a_i, x_{i+1}, \cdots, x_n) > 0 \text{ and } h_i\bar{f}_i(x_1, \cdots, x_{i-1}, b_i, x_{i+1}, \cdots, x_n) < 0.$$

Now in view of Theorem 3.3 we get a contradiction.

Remark. Since Theorem 3.3 is derived from the Borsuk-Ulam theorem, this shows that the Borsuk-Ulam theorem implies the Poincaré-Miranda theorem.

References

[1] D. Ariza-Ruiz, J. Garcia-Falset, S. Reich, *The Bolzano-Poincaré-Miranda theorem in infinite dimensional Banach spaces*, Journal of Fixed Point Theory and Applications, **21**(2019), no. 2, 1-12.

[2] A. Granas, J. Dugundji, *Fixed Point Theory*, Springer New York, 2003.

[3] W. Kulpa, *The Poincaré-Miranda theorem*, The American Mathematical Monthly, **104**(1997), 545-550.

[4] J. Mawhin, *Variations on Poincaré-Miranda's theorem*, Advanced Nonlinear Studies, **13**(2013), 209-217.

[5] C. Miranda, *Un'osservazione su un teorema di Brouwer*, Boll Union Mat. Ital., **3**(1940-41), no. 2, 5-7.

[6] C. Morales, *A Bolzano's theorem in the new millenium*, Nonlinear Analysis, **51**(2002), 679-691.

[7] M. Turzański, *The Bolzano-Poincaré-Miranda theorem-discrete version*, Topology & Its Applications, **159**(2012), 3130-3135.

[8] M.N. Vrahatis, *Generalization of Bolzano theorem for simplices*, Topology & Its Applications, **202**(2016), 40-46.

[9] M.N. Vrahatis, *Intermediate value theorem for simplices for simplicial approximation of fixed points and zeros*, Topology and its Applications, **275**(2020), 107036.

Received: May 17, 2022; Accepted: June 20, 2022.