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VISCOSITY APPROXIMATION METHOD FOR FIXED POINT OF PSEUDOCONTRACTION MAPPING IN HADAMARD SPACES

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Abstract. In this paper, we study a viscosity approximation method for finding a nearest fixed point of a continuous pseudo-contraction mapping in a Hadamard space. We applied our result to solve some convex minimization problems. We also present an example to validate our new findings. This work substantially improves and generalizes some well-known results in the literature.
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1. Introduction

In this paper, our concern is to find a common nearest fixed point of continuous pseudo-contraction mappings in a Hadamard space. The motivation of this problem is the strong relationship between pseudo-contractions and accretive operators which has a connection with equations of evolution (see [3, 5]). Physically, significant problems can be modelled as initial value problem of the form:

$$u'(t) + Bu(t) = 0,\ u(0) = u_0 \tag{1.1}$$

where B is an accretive operator on a given Banach space. Examples where such evolution equations occur are heat, wave or Schrödinger's equations (see [37]). If $u(t)$ is independent of t, then (1.1) becomes $Bu = 0$, the solution of this problem corresponds to the equilibrium point of (1.1). Since generally B is nonlinear, there is no closed form solution of this equation. The standard technique is to introduce an operator $T := I \setminus B$ where I is the identity map on the Banach space. Such a T is known as pseudo-contraction mapping. Clearly, any zero of B is a fixed point of T. One of the first fundamental results in the theory of monotone operators by Browder [5], states that the solution of initial value problem (1.1) exists if B is locally Lipschits and accretive on a Banach space. An operator B on a subset C of a Hilbert space H into itself is accretive if $\langle Bx - By, x - y\rangle \geq 0$ for any $x, y \in C$. A mapping T on C into itself is called a pseudo-contraction if

$$\langle Tx - Ty, x - y\rangle \leq ||x - y||^2, \text{ for all } x, y \in C.$$

The mapping T is called *Lipschitian* if there exists $L \geq 0$ such that

$$||Tx - Ty|| \leq L||x - y||,$$

for all $x, y \in C$. If $L = 1$, then T is called *nonexpansive*, and if $L \in [0, 1)$, then T is called a *contraction*. We observe that B is accretive if and only if $T := I \setminus B$ is pseudo-contraction; thus a zero of monotone operator B which is defined as

$$N(B) := \{x \in D(B) : Bx = 0\},$$

is a fixed point of T, that is $Tx = x$. This describes a strong relationship between an accretive operator and a pseudo-contraction mapping. Also a zero of a monotone operator is a solution of a variational inequality associated with the monotone mapping and denoted $VI(C; B)$. If the monotone operator is subdifferential of a convex function, then a zero of the monotone operator is also a solution of a minimization problem for a convex function [17, 22]. Thus, considerable research efforts have been devoted for approximating zero of monotone operators, fixed points of pseudo-contraction mappings (see, for example [17],[8] and the references contained in them). A general iterative formula for approximation of fixed points of pseudo-contraction mappings and nonexpansive mappings was introduced by Mann [23]. The sequence generated by Mann iterations only converges weakly to the fixed point of the given mapping. Genel and Lindenstrass's [12] showed by means of a counterexample that the sequence generated by Mann iteration does not necessarily converge strongly to a fixed point of a nonexapansive mapping. In the framework of Hilbert space, Halpern [14] was the first one to introduce a modified Mann iteration formula which converges strongly to a fixed point of a nonexpansive mapping. Later in 2000, Moudafi [24] introduced the Viscosity approximation in a Hilbert space to generalize the ideas of Halpern work. Zegeye [36] continued that work to extend the result of Moudafi to the class of Lipschitz pseudo-contraction mappings in Banach spaces. Since then viscosity approximations have been extensively studied in the context of convex optimization, linear programming, monotone inclusions and elliptic differential equations (see [26, 30, 33]). One of the obstacles in carrying out results from Banach space to complete CAT(0) space setting lies in the substantial use of linear structure of the Banach spaces. In 2008, Berg and Nikolaev [2] introduced the notion of an inner product-like notion (quasilinearization) in complete CAT(0) spaces to resolve these difficulties.

Question: Can we give an analogue of the above results in a nonlinear domain, namely, complete CAT(0) space?

Motivated by the work of Kakavandi and Amini [16] and Khatibzadeh and Ranjbar [18], we first give affirmative answer to the above question. Secondly, inspired by the work of Moudafi [24], Zegeye [35], Ugwunnadi and Ali [31] and Kumam and Chaipunya [20], we study a viscosity approximation algorithm for finding a nearest fixed point of a continuous pseudo-contraction mapping in a Hadamard space.

2. Preliminaries

Let (X, d) be a metric space and $x, y \in X$ with $d(x, y) = l$. A geodesic path from x to y is a function ν from a closed interval $[0, l]$ to X such that $\nu(0) = x$ and $\nu(l) = y$. The image of ν is called a *geodesic segment* from the point x to y. A metric space X is a (uniquely) *geodesic space*, if any two points of X are joined by (only one) geodesic segment.

CN-Inequality: Let $x, x_0, x_1, x_2 \in X$. If $d(x_0, x_1) = d(x_0, x_2) = \frac{1}{2}d(x_1, x_2)$, then

$$d^2(x, x_0) \leq \frac{1}{2}d^2(x, x_1) + \frac{1}{2}d^2(x, x_2) - \frac{1}{4}d^2(x_1, x_2).$$

If a geodesic space X satisfies CN-inequality, then X is known as a CAT(0) space. So, every CAT(0) space is a uniquely geodesic space. A complete CAT(0) space is known as a Hadamard space. For more definitions and basic properties of geodesic spaces, see ([1, 4, 6, 13, 15] and references therein). Let $x, y \in X$ and $\lambda \in [0, 1]$, we write $\lambda x \oplus (1-\lambda)y$ for the unique point z on the geodesic segment joining x to y such that

$$d(z, x) = (1 - \lambda)d(x, y) \quad \text{and} \quad d(z, y) = \lambda d(x, y). \tag{2.1}$$

We also denote by $[x, y]$ the geodesic segment joining x to y, that is,

$$[x, y] = \{\lambda x \oplus (1 - \lambda)y : \lambda \in [0, 1]\}.$$

A subset C of a CAT(0) space is convex if $[x, y] \subseteq C$ for all $x, y \in C$.

Berg and Nikolaev [2] introduced the concept of *quasilinearization* in a metric space X as follows: Let us denote a pair $(a, b) \in X \times X$ by $\overrightarrow{ab}$ and call it a vector. A quasilinearization is a map $\langle ., . \rangle : (X \times X) \times (X \times X) \to \mathbb{R}$ defined by

$$\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle = \frac{1}{2}\Big(d^2(a, d) + d^2(b, c) - d^2(a, c) - d^2(b, d)\Big), \quad \forall a, b, c, d \in X. \tag{2.2}$$

It is easy to verify that

$$\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle = \langle \overrightarrow{cd}, \overrightarrow{ab} \rangle, \langle \overrightarrow{ab}, \overrightarrow{cd} \rangle = -\langle \overrightarrow{ba}, \overrightarrow{cd} \rangle$$

and

$$\langle \overrightarrow{ax}, \overrightarrow{cd} \rangle + \langle \overrightarrow{xb}, \overrightarrow{cd} \rangle = \langle \overrightarrow{ab}, \overrightarrow{cd} \rangle$$

for all $a, b, c, d \in X$. We say that X satisfies the Cauchy-Schwarz inequality if

$$\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle \leq d(a, b)d(c, d) \tag{2.3}$$

for all $a, b, c, d \in X$. It is known that a geodesically connected metric space is a CAT(0) space if and only if it satisfies the Cauchy-Schwarz inequality (see [2]). A thorough discussion of these spaces and their important role in various branches of mathematics are given in [4, 6].

The notion of the dual space of a Hadamard space was introduced in 2010 by Kakavandi and Amini [16] as follows: Consider the map $\Theta : \mathbb{R} \times X \times X \to C(X)$ defined by

$$\Theta(t, a, b)(x) = t\langle \overrightarrow{ab}, \overrightarrow{ax} \rangle, \tag{2.4}$$

where $C(X)$ is the space of all continuous real-valued functions on X. Then, it follows from Cauchy-Schwarz that $\Theta(t, a, b)$ is a Lipschitz semi-norm

$$L(\Theta(t, a, b)) = |t|d(a, b) \quad \text{for all } a, b \in X, \tag{2.5}$$

where

$$L(f) = \sup\Big\{\frac{f(x) - f(y)}{d(x, y)} : x, y \in X, x \neq y\Big\},$$

is the Lipschitz semi-norm of the function $f : X \to \mathbb{R}$. Now, consider the pseudometric D defined on $\mathbb{R} \times X \times X$ by

$$D((t, a, b), (s, c, d)) = L(\Theta(t, a, b) - \Theta(s, c, d)). \tag{2.6}$$

$D((t, a, b), (s, c, d)) = 0$ if and only if $t\langle \overrightarrow{ab}, \overrightarrow{xy} \rangle = s\langle \overrightarrow{cd}, \overrightarrow{xy} \rangle$ for all $x, y \in X$ (see [16, Lemma 2.1]). For a Hadamard space (X, d), the pseudometric space $(\mathbb{R} \times X \times X, D)$ can be considered as a subspace of the pseudometric space $(\text{Lip}(X, \mathbb{R}), L)$ of all real-valued Lipschitz functions. Also, the metric D defines an equivalent relation on $\mathbb{R} \times X \times X$, where the equivalence class of (t, a, b) is

$$[t\overrightarrow{ab}] = \{s\overrightarrow{cd} : D((t, a, b), (s, c, d)) = 0 \ \forall x, y \in X\}. \tag{2.7}$$

Then, the dual space of a metric space (X, d), is the metric space (X^*, D), where

$$X^* := \{[t\overrightarrow{ab}] : (t, a, b) \in \mathbb{R} \times X \times X\}.$$

Also, $[\overrightarrow{aa}] = [\overrightarrow{bb}]$ for all $a, b \in X$ and zero of dual space is written as $\mathbf{0} = [\overrightarrow{ww}]$ for any fixed vector $w \in X$. If X is a closed and convex subset of a Hilbert space H with non-empty interior, then $X^* = H$ and $t(b-a) \equiv [t\overrightarrow{ab}]$ for all $t \in \mathbb{R}, a, b \in H$ (see [16]). The following relation is adopted from [16], that is:

$$\langle \alpha x^* + \beta y^*, \overrightarrow{xy} \rangle := \alpha\langle x^*, \overrightarrow{xy} \rangle + \beta\langle y^*, \overrightarrow{xy} \rangle, \ (\alpha, \beta \in \mathbb{R}, x, y \in X, x^*, y^* \in X^*),$$

where $\langle x^*, \overrightarrow{xy} \rangle := t\langle \overrightarrow{ab}, \overrightarrow{xy} \rangle$, $(x^* = [t\overrightarrow{ab}] \in X^*, (x, y) \in X \times X)$. Let X be a Hadamard space and X^* be its dual space. A multivalued mapping $B : X \to 2^{X^*}$ with domain

$$\mathbb{D}(B) := \{x \in X : Bx \neq \emptyset\}$$

is called monotone if for all $x, y \in \mathbb{D}(B),\ x^* \in Bx,\ y^* \in By$,

$$\langle x^* - y^*, \overrightarrow{yx} \rangle \geq 0 \ \ \text{(see [18])}. \tag{2.8}$$

A monotone mapping B is called maximal if the graph $G(B)$ defined by

$$G(B) := \{(x, x^*) \in X \times X^* : \ x^* \in B(x)\},$$

is not properly contained in the graph of any other monotone mapping. The resolvent of a monotone mapping B of order $\lambda > 0$ is the multivalued mapping $J_\lambda^B : X \to 2^X$ defined by (see [18])

$$J_\lambda^B(x) := \{z \in X \mid [\frac{1}{\lambda}\overrightarrow{zx}] \in Bz\}.$$

We say that the mapping B satisfies the range condition if for every $\lambda > 0$, $\mathbb{D}(J_\lambda^B) = X$ (see [18]). The variational inequality problem in X associated with the monotone operator B is to:

$$\text{find } \ x \in X \text{ such that } \ \langle x^*, \overrightarrow{xy} \rangle \geq 0 \ \forall \ x^* \in Bx \text{ and } y \in X.$$

The set of solutions of a variational inequality problem is denoted by $VI(X, B)$.

Definition 2.1 Let (X, d) be a Hadamard space with dual X^* and C a nonempty closed and convex subset of X. The mapping $T : C \to X$ is said to be:

(1) Lipschitian if there exist $L \geq 0$ such that

$$d(Tx, Ty) \leq Ld(x, y) \quad \text{for all} \quad x, y \in C,$$

and T is called nonexpansive (contraction) if $L = 1 (L \in [0, 1))$, respectively.

(2) firmly nonexpansive if

$$d^2(Tx, Ty) \leq \langle \overrightarrow{TxTy}, \overrightarrow{xy} \rangle \quad \text{for all} \quad x, y \in C.$$

(3) pseudo-contraction if

$$\langle \overrightarrow{TxTy}, \overrightarrow{xy} \rangle \leq d^2(x, y), \quad \text{for all } x, y \in C. \tag{2.9}$$

Lemma 2.2 [18] Let (X, d) be a Hadamard space with dual X^* and C a nonempty closed and convex subset of X. The mapping $T : C \to X$ is firmly nonexpansive if and only if

$$\langle \overrightarrow{TxTy}, \overrightarrow{(Tx)x} \rangle + \langle \overrightarrow{TxTy}, \overrightarrow{(Ty)y} \rangle \leq 0, \quad \text{for all } x, y \in C. \tag{2.10}$$

Definition 2.3 Let C be a nonempty closed and convex subset of a CAT(0) space X.

(1) Let $f : X \to \mathbb{R} \cup \{\infty\}$ be a function. The domain of f is the set defined by

$$D(f) := \{x \in X : f(x) < \infty\}.$$

The function f is called proper if $D(f) \neq \emptyset$. That is, there exists at least one point $u \in D(f)$ such that $f(u) \in \mathbb{R}$.

(2) A bifunction $f : C \times C \to \mathbb{R}$ is called monotone if $f(x, y) + f(y, x) \leq 0$ for all $x, y \in C$.

(3) A point $v \in C$ is called an equilibrium point of $f : C \times C \to \mathbb{R}$ if

$$f(v, y) \geq 0, \quad \text{for all} \quad y \in C.$$

The set of equilibrium points of f is denoted by $EP(C, f)$.

Definition 2.4 [20] Let C be a nonempty closed and convex subset of a Hadamard space X and $f : C \times C \to \mathbb{R}$ be a bifunction. For any $z \in C$ define

$$\overline{f}_z(x, y) := f(x, y) - \langle \overrightarrow{xz}, \overrightarrow{xy} \rangle, \quad \text{for all } x, y \in C.$$

Theorem 2.5 [20] Let C be a nonempty closed and convex subset of a Hadamard space X and $f : C \times C \to \mathbb{R}$. Then the resolvent operator $J^f : X \to 2^C$ of f is defined as follows:

$$J^f(x) := \{z \in C : f(z, y) - \langle \overrightarrow{zx}, \overrightarrow{zy} \rangle \geq 0, \ \forall \ y \in C\}$$

where $x \in X$. Assume that f has the following properties:

(i) $f(x, x) = 0$ for all $x \in X$,

(ii) f is monotone,

(iii) For each $x \in C, y \mapsto f(x,y)$ is convex and lower semicontinuous,
(iv) for each $y \in C$, $f(x,y) \geq \limsup_{t\downarrow 0}(tx \oplus (1-t)z)$,for all $x, x \in C$.

If $D(J^f) \neq \emptyset$, then the following results hold:

(1) $D(J^f) = X$ and J^f is single-valued.
(2) If $D(J^f) \supset C$, then J^f is nonexpansive restricted to C.
(3) If $D(J^f) \supset C$, then $F(J^f) = EP(C,f)$.

Remark 2.6 By Lemma 2.2, it easy to see that J^f in Theorem 2.5 is a firmly nonexpansive mapping.

Lemma 2.7 [20] Let X be a Hadamard space with dual X^*. Suppose that $B : X \rightrightarrows X^*$ is a monotones with $\mathrm{dom}(B) = X$. Define

$$f_B(x,y) := \sup_{z\in B(x)} \langle z, \overrightarrow{xy}\rangle,\ \forall x,y \in X.$$

Then, $B^{-1}(0) = EP(X, f_B)$, where $B^{-1}(0) := \{w \in X; 0 \in Bw\}$.
By Theorem 2.5 and Lemma 2.7, we obtain the following.

Lemma 2.8 Let C be a nonempty closed and convex subset of a Hadamard space X and $B : C \to 2^{X^*}$ be a continuous monotone mapping. Then there exists a resolvent operator $S_\lambda : X \to 2^C$ of B with order $\lambda > 0$ and defined as

$$S_\lambda(x) := \{z \in C : \langle x^*, \overrightarrow{zy}\rangle + \frac{1}{\lambda}\langle \overrightarrow{xz}, \overrightarrow{zy}\rangle \geq 0,\ \forall\ y \in C, x^* \in Bz\}$$

where $x \in X$. Assume that S_λ is proper. Then the following hold:

(1) S_λ is single-valued.
(2) If $D(S_\lambda) \supset C$, then S_λ is firmly nonexpansive restricted to C.
(3) If $D(S_\lambda) \supset C$, then $F(JS_\lambda) = VI(C,B)$.

Lemma 2.9 Let X be a CAT(0) space, $w, x, y, z \in X$ and $\tau, \lambda \in [0,1]$. Then

(i) $d(\lambda x \oplus (1-\lambda)y, z) \leq \lambda d(x,z) + (1-\lambda)d(y,z)$, (see [11]).
(ii) $d^2(\lambda x \oplus (1-\lambda)y, z) \leq \lambda d^2(x,z) + (1-\lambda)d^2(y,z) - \lambda(1-\lambda)d^2(x,y)$, (see [11]).
(iii) $d(\lambda w \oplus (1-\lambda)x, \lambda y \oplus (1-\lambda)z) \leq \lambda d(w,y) + (1-\lambda)d(x,z)$, (see [4]).
(iv) $d(\tau x \oplus (1-\tau)y, \lambda x \oplus (1-\lambda)y) \leq |\tau - \lambda| d(x,y)$, (see [7]).

Let $\{x_n\}$ be a bounded sequence in a Hadamard space X. For $x \in X$, we set

$$r(x, \{x_n\}) = \limsup_{n\to\infty} d(x, x_n).$$

The asymptotic radius $r(\{x_n\})$ of $\{x_n\}$ is given by

$$r(\{x_n\}) = \inf\{r(x, \{x_n\}) : x \in X\}$$

and the asymptotic center $A(\{x_n\})$ of $\{x_n\}$ is the set

$$A(\{x_n\}) = \{x \in X : r(x, \{x_n\}) = r(\{x_n\})\}.$$

It is well known that in a Hadamard space, $A(\{x_n\})$ consists of exactly one point (see [10, Proposition 7]).

Lemma 2.10 [19] Every bounded sequence in a Hadamard space always has a $\triangle-$convergent subsequence.

Lemma 2.11 [9] If C is a closed and convex subset of a Hadamard space and $\{x_n\}$ is a bounded sequence in C, then the asymptotic center of $\{x_n\}$ is in C.

Let C be a closed and convex subset of X which contains a bounded sequence $\{x_n\}$. We employ the notation:

$$\{x_n\} \rightharpoonup w \Leftrightarrow \limsup_{n\to\infty} d(x_n, w) = \inf_{x\in C}(\limsup_{n\to\infty} d(x_n, x)).$$

We note that (see [25])

$$\{x_n\} \rightharpoonup w \text{ if and only if } A(\{x_n\}) = \{w\}. \tag{2.11}$$

The following lemmas are very useful for proving our main results:

Lemma 2.12 [25] If C is a closed and convex subset of a Hadamard space X and $\{x_n\}$ is a bounded sequence in C, then $\triangle - \lim_{n\to\infty} x_n = p$ implies that $\{x_n\} \rightharpoonup p$.

Lemma 2.13 [29] Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a metric space of hyperbolic type X and $\{\beta_n\}$ be a sequence in [0,1] with

$$\liminf_{n\to\infty} \beta_n < \limsup_{n\to\infty} \beta_n < 1.$$

Suppose that $x_{n+1} = \beta_n x_n \oplus (1-\beta_n)y_n$ for all $n \geq 0$ and

$$\limsup_{n\to\infty}(d(y_{n+1}, y_n) - d(x_{n+1}, x_n)) \leq 0.$$

Then $\lim_{n\to\infty} d(y_n, x_n) = 0$.

Definition 2.14 Let C be a nonempty closed and convex subset of a Hadamard space X. The metric projection $P_C : X \to C$ is defined by

$$u = P_C(x) \;\Leftrightarrow\; d(u, x) = \inf\{d(y, x) : y \in C\}, \text{ for all } x \in X.$$

Lemma 2.15 [2] Let C be a nonempty closed and convex subset of a Hadamard space X. For any $x \in X$ and $u \in C$, $u = P_C(x)$ if and only if

$$\langle \overrightarrow{yu}, \overrightarrow{ux} \rangle \geq 0. \tag{2.12}$$

Lemma 2.16 [9] Let X be a Hadamard space and $T : X \to X$ be a nonexpansive mapping. Then the conditions that $\{x_n\}$ Δ-converges to x and $d(x_n, Tx_n) \to 0$, imply $x = Tx$.

Lemma 2.17 [32] Let X be a CAT(0) space. For any $u, v, \in X$ and $t \in (0, 1)$, let $u_t = tu \oplus (1-t)v$. Then for all $x, y \in X$,

(i) $\langle \overrightarrow{u_t x}, \overrightarrow{u_t y} \rangle \leq t\langle \overrightarrow{ux}, \overrightarrow{u_t y} \rangle + (1-t)\langle \overrightarrow{vx}, \overrightarrow{u_t y} \rangle$;

(ii) $\langle \overrightarrow{u_t x}, \overrightarrow{uy} \rangle \leq t\langle \overrightarrow{ux}, \overrightarrow{uy} \rangle + (1-t)\langle \overrightarrow{vx}, \overrightarrow{uy} \rangle$
and $\langle \overrightarrow{u_t x}, \overrightarrow{vy} \rangle \leq t\langle \overrightarrow{ux}, \overrightarrow{vy} \rangle + (1-t)\langle \overrightarrow{vx}, \overrightarrow{vy} \rangle$.

Lemma 2.18 [32] Let X be a complete CAT(0) space. Then for all $u, x, y \in X$, the following inequality holds:

$$d^2(x, u) \leq d^2(y, u) + 2\langle \overrightarrow{xy}, \overrightarrow{xu} \rangle.$$

Lemma 2.19 [21] Let X be a complete CAT(0) space. Then for all $u, x, y \in X$ and $\alpha \in [0, 1]$. Let $z_1 = \alpha x \oplus (1-\alpha)u$ and $z_2 = \alpha y \oplus (1-\alpha)u$, then the following inequality holds:

$$\langle \overrightarrow{z_1 z_2}, \overrightarrow{xz_2} \rangle \leq \alpha\langle \overrightarrow{xy}, \overrightarrow{xu} \rangle.$$

The following lemma gives the conditions for the convergence of a nonnegative real sequences.

Lemma 2.20 [34] Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying the following relation:

$$a_{n+1} \leq (1-\delta_n)a_n + \delta_n\sigma_n + \theta_n, n \geq 0,$$

where
(*i*) $\{\delta_n\} \subset [0,1]$, $\sum \delta_n = \infty$;
(*ii*) $\lim sup\ \sigma_n \leq 0$; (*iii*) $\theta_n \geq 0$; $(n \geq 0)$, $\sum \theta_n < \infty$.
Then, $a_n \to 0$ as $n \to \infty$.

3. Main results

Lemma 3.1 Let (X,d) be a Hadamard space with dual X^* and $B : X \to 2^{X^*}$ be a continuous monotone map. If $Bx := [\overrightarrow{(Tx)x}]$, for any $x \in X$ where T is a map from X into itself. Then T is a continuous pseudo-contraction map. Furthermore, if the set of fixed points of T, $F(T) \neq \emptyset$, then $N(B) := \{x \in X : x \in B^{-1}(0)\} = F(T)$.
Proof. Let $B : X \to 2^{X^*}$ be a monotone operator and $T : X \to X$ be a map. Then, for every $x, y \in X$, there exist $x^* \in Bx := [\overrightarrow{(Tx)x}]$ and $y^* \in By := [\overrightarrow{(Ty)y}]$ by hypothesis on B. Now using (2.8), we get

$$\begin{array}{rcl} 0 & \leq & \langle [\overrightarrow{(Tx)x}] - [\overrightarrow{(Ty)y}], \overrightarrow{yx}\rangle \\ & = & \langle \overrightarrow{Txx}, \overrightarrow{yx}\rangle - \langle \overrightarrow{Tyy}, \overrightarrow{yx}\rangle \\ & = & \langle \overrightarrow{TxTy}, \overrightarrow{yx}\rangle + \langle \overrightarrow{Tyx}, \overrightarrow{yx}\rangle + \langle \overrightarrow{yTy}, \overrightarrow{yx}\rangle \\ & = & \langle \overrightarrow{TxTy}, \overrightarrow{yx}\rangle + \langle \overrightarrow{yx}, \overrightarrow{yx}\rangle \\ & = & -\langle \overrightarrow{TxTy}, \overrightarrow{xy}\rangle + \langle \overrightarrow{xy}, \overrightarrow{xy}\rangle \\ & = & -\langle \overrightarrow{TxTy}, \overrightarrow{xy}\rangle + d^2(x,y). \end{array}$$

Therefore,

$$\langle \overrightarrow{TxTy}, \overrightarrow{xy}\rangle \leq d^2(x,y);$$

hence by (2.9), we conclude that T is a pseudo-contraction map. Next, we show that $N(B) := \{x \in X : x \in B^{-1}(0)\} = F(T)$. Let $x \in X$ be a zero of the dual space. Then for fix $w \in X$, $[\overrightarrow{ww}] = \mathbf{0} \in Bx$, but $Bx := [\overrightarrow{(Tx)x}]$, so by (2.7), (2.6) and (2.5) with L as a seminorm, we obtain

$$\begin{array}{rcl} 0 & = & D((1,Tx,x),(1,w,w)) \\ & = & L(\Theta(1,Tx,x) - \Theta(1,w,w)) \\ & \geq & \Big| L(\Theta(1,Tx,x)) - L(\Theta(1,w,w)) \Big| \\ & = & |1 \cdot d(Tx,x) - 1 \cdot d(w,w)| \\ & = & d(Tx,x). \end{array}$$

Therefore $d(Tx,x) \leq 0$, which implies that $Tx = x$.
Lemma 3.2 Let C be a nonempty closed and convex subset of a Hadamard space X and $T : C \to C$ be a continuous pseudo-contraction mapping. Then there exists a

resolvent operator $T_\lambda : X \to 2^C$ of T with order $\lambda > 0$ and defined as

$$J_\lambda(x) := \{z \in C : \langle \overrightarrow{Tzz}, \overrightarrow{yz} \rangle + \frac{1}{\lambda}\langle \overrightarrow{xz}, \overrightarrow{zy} \rangle \geq 0,\ \forall\ y \in C\}, \tag{3.1}$$

where $x \in X$. Assume that J_λ is proper. Then the following hold:

(1) J_λ is single-valued.
(2) If $D(J_\lambda) \supset C$, then J_λ is firmly nonexpansive restricted to C.
(3) If $D(J_\lambda) \supset C$, then $F(J_\lambda) = F(T)$.

Proof. Let $x \in X$ and $Bx := [\overrightarrow{(Tx)x}]$. Then by Lemma 3.1, B is a continuous monotone operator. Hence the result by Lemma 2.8.

Lemma 3.3 Let C be a nonempty closed and convex subset of a Hadamard space X and $T : C \to C$ be a continuous pseudo-contraction mapping. For $x \in X$ and $\lambda > \mu > 0$, let the mapping J_λ be as in Lemma 3.2. If $F(J_\lambda) \neq \emptyset$, then for any $x, y \in X$ and $u \in F(J_\lambda)$, the following hold:

(1) $d(J_\lambda x, J_\mu y) \leq d(x, y) + \frac{|\lambda-\mu|}{\lambda} d(J_\lambda x, y)$.
(2) $d^2(J_\lambda x, x) \leq d^2(x, u) - d^2(J_\lambda x, u)$.

In particular, $d(J_\lambda x, J_\lambda y) \leq d(x, y)$ for any $\lambda > 0$ and $x, y \in X$, that is, J_λ is nonexpansive.

Proof. (i) Let $x, y \in X$, and define $z_1 := J_\lambda x,\ z_2 := J_\mu y$ for any $\lambda > \mu > 0$. Then by (3.1), we get

$$\langle \overrightarrow{Tz_1z_1}, \overrightarrow{uz_1} \rangle + \frac{1}{\lambda}\langle \overrightarrow{xz_1}, \overrightarrow{z_1u} \rangle \geq 0,\ \forall\ u \in C, \tag{3.2}$$

and

$$\langle \overrightarrow{Tz_2z_2}, \overrightarrow{vz_2} \rangle + \frac{1}{\mu}\langle \overrightarrow{yz_2}, \overrightarrow{z_2v} \rangle \geq 0,\ \forall\ v \in C. \tag{3.3}$$

If, in particular, $u = z_2$ and $v = z_1$, then by (3.2) and (3.3), we obtain

$$\langle \overrightarrow{Tz_1z_1}, \overrightarrow{z_2z_1} \rangle + \frac{1}{\lambda}\langle \overrightarrow{xz_1}, \overrightarrow{z_1z_2} \rangle \geq 0 \tag{3.4}$$

and

$$\langle \overrightarrow{Tz_2z_2}, \overrightarrow{z_1z_2} \rangle + \frac{1}{\mu}\langle \overrightarrow{yz_2}, \overrightarrow{z_2z_1} \rangle \geq 0. \tag{3.5}$$

Adding (3.4) and (3.5) and using (2.9), we get

$$\begin{aligned} 0 &\leq \langle\overrightarrow{Tz_1z_1}, \overrightarrow{z_2z_1}\rangle + \langle\overrightarrow{Tz_2z_2}, \overrightarrow{z_1z_2}\rangle \\ &\quad + \frac{1}{\lambda}\langle\overrightarrow{xz_1}, \overrightarrow{z_1z_2} + \frac{1}{\mu}\langle\overrightarrow{yz_2}, \overrightarrow{z_2z_1}\rangle \\ &= \langle\overrightarrow{Tz_1Tz_2}, \overrightarrow{z_2z_1}\rangle + \langle\overrightarrow{Tz_2z_1}, \overrightarrow{z_2z_1}\rangle \\ &\quad \frac{1}{\lambda}[\langle\overrightarrow{xy}, \overrightarrow{z_1z_2}\rangle + \langle\overrightarrow{yz_1}, \overrightarrow{z_1z_2}\rangle] + \frac{1}{\mu}[\langle\overrightarrow{yz_1}, \overrightarrow{z_2z_1}\rangle + \langle\overrightarrow{z_1z_2}, \overrightarrow{z_2z_1}\rangle] \\ &= -\langle\overrightarrow{Tz_1Tz_2}, \overrightarrow{z_1z_2}\rangle + \langle\overrightarrow{z_1Tz_2}, \overrightarrow{z_1z_2}\rangle \\ &\quad \frac{1}{\lambda}[\langle\overrightarrow{xy}, \overrightarrow{z_1z_2}\rangle - \langle\overrightarrow{z_1y}, \overrightarrow{z_1z_2}\rangle] + \frac{1}{\mu}[\langle\overrightarrow{yz_1}, \overrightarrow{z_2z_1}\rangle - \langle\overrightarrow{z_1z_2}, \overrightarrow{z_1z_2}\rangle] \\ &= -\langle\overrightarrow{Tz_1Tz_2}, \overrightarrow{z_1z_2}\rangle + \langle\overrightarrow{z_1z_2}, \overrightarrow{z_1z_2}\rangle + \frac{1}{\lambda}\langle\overrightarrow{xy}, \overrightarrow{z_1z_2}\rangle \\ &\quad \Big(\frac{1}{\mu} - \frac{1}{\lambda}\Big)\langle\overrightarrow{z_1y}, \overrightarrow{z_1z_2}\rangle - \frac{1}{\mu}\langle\overrightarrow{z_1z_2}, \overrightarrow{z_1z_2}\rangle \\ &\leq -d^2(z_1, z_2) + d^2(z_1, z_2) + \frac{1}{\lambda}\langle\overrightarrow{xy}, \overrightarrow{z_1z_2}\rangle \\ &\quad \Big(\frac{1}{\mu} - \frac{1}{\lambda}\Big)\langle\overrightarrow{z_1y}, \overrightarrow{z_1z_2}\rangle - \frac{1}{\mu}d^2(z_1, z_2) \\ &\leq \frac{1}{\lambda}d(x, y)d(z_1, z_2) + \Big(\frac{1}{\mu} - \frac{1}{\lambda}\Big)d(z_1, y)d(z_1, z_2) - \frac{1}{\mu}d^2(z_1, z_2). \end{aligned}$$

Therefore,

$$\frac{1}{\mu}d^2(z_1, z_2) \leq \frac{1}{\lambda}d(x, y)d(z_1, z_2) + \Big(\frac{1}{\mu} - \frac{1}{\lambda}\Big)d(z_1, y)d(z_1, z_2).$$

Hence, in view of $\mu > \lambda$ and $\frac{\mu}{\lambda} < 1$, we obtain

$$d(z_1, z_2) \leq d(x, y) + \frac{|\lambda - \mu|}{\lambda}d(z_1, y).$$

(ii) As is J_λ is firmly nonexpansive, so for any $x \in X$ and $u \in F(J_\lambda)$, by (2.10) and (2.2), we obtain

$$\begin{aligned} 0 &\geq \langle\overrightarrow{J_\lambda xJ_\lambda u}, \overrightarrow{J_\lambda xx}\rangle + \langle\overrightarrow{J_\lambda uJ_\lambda x}, \overrightarrow{J_\lambda uu}\rangle \\ &= \langle\overrightarrow{J_\lambda xu}, \overrightarrow{J_\lambda xx}\rangle + \langle\overrightarrow{uJ_\lambda x}, \overrightarrow{uu}\rangle \\ &= \frac{1}{2}\Big[d^2(J_\lambda x, x) + d^2(u, J_\lambda x) - d^2(J_\lambda x, J_\lambda x) - d^2(u, x)\Big]. \end{aligned}$$

Therefore,

$$d^2(J_\lambda x, x) \leq d^2(x, u) - d^2(J_\lambda x, u).$$

Let $T : C \to C$ be a continuous pseudo-contraction mapping. Then in view of Lemma 3.2, we make the following assumption in the rest of the paper. For any $\lambda_n \in (0, +\infty)$ and $x \in X$,

$$J_{\lambda_n}(x) := \{z \in C : \langle\overrightarrow{Tzz}, \overrightarrow{yz}\rangle + \frac{1}{\lambda_n}\langle\overrightarrow{xz}, \overrightarrow{zy}\rangle \geq 0,\ \forall\ y \in C\}.$$

Now, we prove our main result.

Theorem 3.4 Let (X, d) be a Hadamard space and $C \subset X$ a nonempty closed and convex set. Let $T : C \to C$ be continuous pseudo-contraction mapping such that $F(T) \neq \emptyset$. Let g be a contraction of C into itself with a contraction constant $\gamma \in (0, \frac{1}{2})$. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence generated by:

$$\begin{cases} x_1 \in C; \\ x_{n+1} = \beta_n x_n \oplus (1-\beta_n) y_n, \\ y_n = (1-\alpha_n) J_{\lambda_n} x_n \oplus \alpha_n g(x_n), \end{cases} \tag{3.6}$$

where $\lambda_n \in (0, +\infty)$ with $\liminf_{n\to\infty} \lambda_n > 0$; $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in $[0, 1]$ satisfying the following conditions:

(C1) $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(C2) $\lim_{n\to\infty} |\alpha_{n+1} - \alpha_n| = 0$ and $\lim_{n\to\infty} |\lambda_{n+1} - \lambda_n| = 0$.

(C3) $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$.

Then the sequence $\{x_n\}_{n=1}^{\infty}$ converges strongly to $u := P_{F(T)} g(u)$ where $P_{F(T)}$ is the projection of C onto $F(T)$.

Proof. First, we show that there exists a unique element $u \in F(T)$ such that $u = P_{F(T)} g(u)$. Let $x, y \in C$. Then by nonexpansivity of $P_{F(T)}$ and definition of g, we get

$$d(P_{F(T)} g(x), P_{F(T)} g(y)) \le d(g(x), g(y)) \le \gamma d(x, y).$$

Thus, $P_{F(T)} g$ is a contraction on C. Since C is a closed subset of X and X is complete, there exists a unique element say u in C such that $u = P_{F(T)} g(u)$. Next, we show that $\{x_n\}_{n=1}^{\infty}$ is bounded. Let $p \in F(T)$. By Lemma 3.3 , we get

$$d(J_{\lambda_n} x_n, p) \le d(x_n, p).$$

Now, by (3.6) and Lemma 2.9, we obtain

$$\begin{aligned} d(y_n, p) &= d((1-\alpha_n) J_{\lambda_n} x_n \oplus \alpha_n g(x_n), p) \\ &\le (1-\alpha_n) d(J_{\lambda_n} x_n, p) + \alpha_n d(g(x_n), g(p)) + \alpha_n d(g(p), p) \\ &\le [1 - \alpha_n (1-\gamma)] d(x_n, p) + \alpha_n d(g(p), p). \end{aligned} \tag{3.7}$$

and

$$\begin{aligned} d(x_{n+1}, p) &= d(\beta_n x_n \oplus (1-\beta_n) y_n, p) \\ &\le \beta_n d(x_n, p) + (1-\beta_n) d(y_n, p) \\ &\le [1 - \alpha_n (1-\beta_n)(1-\gamma)] d(x_n, p) \\ &\quad + \alpha_n (1-\beta_n)(1-\gamma) \frac{d(g(p), p)}{1-\gamma} \\ &\le \max\left\{ d(x_n, p), \frac{d(g(p), p)}{1-\gamma} \right\} \\ &\vdots \\ &\le \max\left\{ d(x_1, p), \frac{d(g(p), p)}{1-\gamma} \right\}. \end{aligned}$$

Therefore $\{x_n\}_{n=1}^{\infty}$ is bounded. So $\{y_n\}$, $\{g(x_n)\}$ and $\{J_{\lambda_n}x_n\}$ are bounded. Next we show that $\lim_{n\to\infty} d(x_{n+1},x_n) = 0$. Letting $v_n := J_{\lambda_n}x_n$, then by Lemma 2.9 and Lemma 3.3, we obtain

$$\begin{aligned}
d(y_{n+1},y_n) &= d(\alpha_{n+1}g(x_{n+1}) \oplus (1-\alpha_{n+1})v_{n+1}, \alpha_n g(x_n) \oplus (1-\alpha_n)v_n) \\
&\leq d(\alpha_{n+1}g(x_{n+1}) \oplus (1-\alpha_{n+1})v_{n+1}, \alpha_{n+1}g(x_{n+1}) \oplus (1-\alpha_{n+1})v_n) \\
&\quad + d(\alpha_{n+1}g(x_{n+1}) \oplus (1-\alpha_{n+1})v_n, \alpha_{n+1}g(x_n) \oplus (1-\alpha_{n+1})v_n) \\
&\quad + d(\alpha_{n+1}g(x_n) \oplus (1-\alpha_{n+1})v_n, \alpha_n g(x_n) \oplus (1-\alpha_n)v_n) \\
&\leq (1-\alpha_{n+1})d(v_{n+1},v_n) + \alpha_{n+1}d(g(x_{n+1}),g(x_n)) \\
&\quad + |\alpha_{n+1}-\alpha_n| d(g(x_n),v_n) \\
&\leq [1-\alpha_{n+1}(1-\gamma)]d(x_{n+1},x_n) + |\alpha_{n+1}-\alpha_n| d(g(x_n),v_n) \\
&\quad + (1-\alpha_{n+1})\frac{|\lambda_{n+1}-\lambda_n|}{\lambda_{n+1}} d(v_{n+1},x_n). \qquad (3.8)
\end{aligned}$$

Therefore,

$$\begin{aligned}
&d(y_{n+1},y_n) + [\alpha_{n+1}(1-\gamma)-1]d(x_{n+1},x_n) \\
&\leq (1-\alpha_{n+1})\frac{|\lambda_{n+1}-\lambda_n|}{\lambda_{n+1}} d(v_{n+1},x_n) \\
&\quad + |\alpha_{n+1}-\alpha_n| d(g(x_n),v_n). \qquad (3.9)
\end{aligned}$$

Hence, by (C1), (C2) and (3.9), we obtain

$$\limsup_{n\to\infty}(d(y_{n+1},y_n) - d(x_{n+1},x_n)) \leq 0.$$

So by Lemma 2.13, we obtain

$$\lim_{n\to\infty} d(y_n,x_n) = 0. \qquad (3.10)$$

Thus, by (3.6) we get

$$\lim_{n\to\infty} d(x_{n+1},x_n) = \lim_{n\to\infty} \beta_n d(y_n,x_n) = 0. \qquad (3.11)$$

By (3.6) and (C1), we get

$$\lim_{n\to\infty} d(y_n, J_{\lambda_n}x_n) = \lim_{n\to\infty} \alpha_n d(g(x_n), J_{\lambda_n}x_n) = 0. \qquad (3.12)$$

Thus from (3.11) and (3.12), we obtain

$$\lim_{n\to\infty} d(x_n, J_{\lambda_n}x_n) \leq \lim_{n\to\infty} d(x_n,y_n) + \lim_{n\to\infty} d(y_n, J_{\lambda_n}x_n) = 0. \qquad (3.13)$$

Also with the nonexpansivity of J_{λ_n} for any $n \geq 1$, we get

$$\begin{aligned}
d(y_n, J_{\lambda_n}y_n) &\leq d(y_n,x_n) + d(x_n, J_{\lambda_n}x_n) + d(J_{\lambda_n}x_n, J_{\lambda_n}y_n) \\
&\leq 2d(x_n,y_n) + d(x_n, J_{\lambda_n}x_n),
\end{aligned}$$

thus by (3.10) and (3.13), we have

$$\lim_{n\to\infty} d(y_n, J_{\lambda_n}y_n) = 0. \qquad (3.14)$$

Next, we show that

$$\limsup_{n\to\infty}\langle\overrightarrow{g(u)u},\overrightarrow{x_n u}\rangle\leq 0,$$

where $u=P_{F(T)}g(u)$. Since $\{x_n\}$ is bounded, by Lemma 2.10, there exists, a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ which $\Delta-$converges to u in X. Now by Lemma 2.12, we get $x_{n_k}\rightharpoonup u\in X$ as $k\to\infty$. Then by (2.11) and Lemma 2.11, we obtain $u\in C$. Furthermore, since J_{λ_n} is nonexpansive by Lemma 2.16 we have $u\in F(J_{\lambda_n})$, thus by Lemma 3.2(3), we obtain $u\in F(T)$.

Let $\Psi_m x:=\beta_m x\oplus(1-\beta_m)y$, where $y=(1-\alpha_m)J_{\lambda_m}x\oplus\alpha_m g(x)$, for $x\in X$. Then Ψ_m is a contraction mapping for each $m\geq 1$. Thus by Contraction mapping principle, there exists a unique fixed point w_m of Ψ_m for each $m\geq 1$. That is, $w_m=\beta_m w_m\oplus(1-\beta_m)y_m$, where $y_m=(1-\alpha_m)J_{\lambda_m}w_m\oplus\alpha_m g(w_m)$. It follows from [28] that $\lim\limits_{m\to\infty}w_m=u$. Thus , by Lemma 2.17, we get

$$\begin{aligned}
d^2(w_m,y_n) &= \langle\overrightarrow{w_my_n},\overrightarrow{w_my_n}\rangle\\
&= \langle\overrightarrow{[\beta_m w_m\oplus(1-\beta_m)y_m]y_n},\overrightarrow{w_my_n}\rangle\\
&\leq \beta_m\langle\overrightarrow{w_my_n},\overrightarrow{w_my_n}\rangle+(1-\beta_m)\langle\overrightarrow{y_my_n},\overrightarrow{w_my_n}\rangle\\
&= \beta_m d^2(w_m,y_n)+(1-\beta_m)\langle\overrightarrow{y_my_n},\overrightarrow{w_my_n}\rangle
\end{aligned}$$

that is,

$$d^2(w_m,y_n)\leq\langle\overrightarrow{y_my_n},\overrightarrow{w_my_n}\rangle.$$

Since $\{x_n\}$ and $\{\beta_n\}$ are bounded, then there exists some $M>0$ such that

$$M\geq\sup_{m,n\geq 1}\{d(w_m,x_{n+1})\},$$

also with the fact that J_{λ_m} for each $m\geq 1$ is nonexpansive, by (2.2), Lemma 2.17 and (3.6), we obtain

$$\begin{aligned}
d^2(w_m,y_n) &\leq \langle\overrightarrow{y_my_n},\overrightarrow{w_my_n}\rangle\\
&\leq \alpha_m\langle\overrightarrow{g(w_m)y_n},\overrightarrow{w_my_n}\rangle+(1-\alpha_m)\langle\overrightarrow{J_{\lambda_m}w_my_n},\overrightarrow{w_my_n}\rangle\\
&= \alpha_m\langle\overrightarrow{g(w_m)g(u)},\overrightarrow{w_my_n}\rangle+\alpha_m\langle\overrightarrow{g(u)u},\overrightarrow{w_my_n}\rangle+\alpha_m\langle\overrightarrow{uw_m},\overrightarrow{w_my_n}\rangle\\
&\quad+\alpha_m\langle\overrightarrow{w_my_n},\overrightarrow{w_my_n}\rangle+(1-\alpha_m)\langle\overrightarrow{J_{\lambda_m}w_mJ_{\lambda_m}y_n},\overrightarrow{w_my_n}\rangle\\
&\quad+(1-\alpha_m)\langle\overrightarrow{J_{\lambda_m}y_ny_n},\overrightarrow{w_my_n}\rangle\\
&\leq \alpha_m\gamma d(w_m,u)d(w_m,y_n)+\alpha_m\langle\overrightarrow{g(u)u},\overrightarrow{w_my_n}\rangle\\
&\quad+\alpha_m d(u,w_m)d(w_m,y_n)+\alpha_m d^2(w_m,y_n)\\
&\quad+(1-\alpha_m)d(J_{\lambda_m}w_m,J_{\lambda_m}y_n)d(w_m,y_n)\\
&\quad+(1-\alpha_m)d(J_{\lambda_m}y_n,y_n)d(w_m,y_n)\\
&\leq \alpha_m(1+\gamma)Md(w_m,u)+\alpha_m\langle\overrightarrow{g(u)u},\overrightarrow{w_my_n}\rangle\\
&\quad+\alpha_m d^2(w_m,y_n)+(1-\alpha_m)d^2(w_m,y_n)\\
&\quad+(1-\alpha_m)d(J_{\lambda_m}y_n,y_n)M.
\end{aligned}$$

Therefore

$$\langle\overrightarrow{g(u)u},\overrightarrow{y_nw_m}\rangle \leq (1+\gamma)Md(w_m,u)+\frac{d(J_{\lambda_m}y_n,y_n)}{\alpha_m}M.$$

Now, first taking limit superior as $n\to\infty$ and then as $m\to\infty$, using (3.14) and $\lim_{m\to\infty} w_m = u$, we get

$$\limsup_{m\to\infty}\limsup_{n\to\infty}\langle\overrightarrow{g(u)u},\overrightarrow{y_nw_m}\rangle \leq 0.$$

Furthermore, we note that

$$\begin{aligned}\langle\overrightarrow{g(u)u},\overrightarrow{y_nu}\rangle &= \langle\overrightarrow{g(u)u},\overrightarrow{y_nw_m}\rangle+\langle\overrightarrow{g(u)u},\overrightarrow{w_nu}\rangle\\ &\leq \langle\overrightarrow{g(u)u},\overrightarrow{y_nw_m}\rangle+d(g(u),u)d(w_m,u).\end{aligned}$$

thus

$$\limsup_{n\to\infty}\langle\overrightarrow{g(u)u},\overrightarrow{y_nu}\rangle \leq \limsup_{n\to\infty}\langle\overrightarrow{g(u)u},\overrightarrow{y_nw_m}\rangle+d(g(u),u)d(w_m,u).$$

As $\lim_{m\to\infty} w_m = u$ and since the left hand term in the above inequality is independent of m, so we get

$$\limsup_{n\to\infty}\langle\overrightarrow{g(u)u},\overrightarrow{y_nu}\rangle \leq 0. \tag{3.15}$$

Finally, we show that $x_n \to u := P_{F(T)}g(u)$ as $n\to\infty$. For any $n\geq 1$, let (z_n) in C be defined by $z_n = \alpha_n u \oplus (1-\alpha_n)J_{\lambda_n}x_n$. Then by (3.6), Lemma 2.18, 2.19, (2.1), (2.3) and (3.7), we obtain

$$\begin{aligned}d^2(y_n,u) &\leq d^2(z_n,u)+2\langle\overrightarrow{x_nz_n},\overrightarrow{y_nu}\rangle\\ &= (1-\alpha_n)^2d^2(J_{\lambda_n}x_n,u)+2\langle\overrightarrow{z_ny_n},\overrightarrow{uy_n}\rangle\\ &\leq (1-\alpha_n)^2d^2(x_n,u)+2\alpha_n\langle\overrightarrow{ug(x_n)},\overrightarrow{uy_n}\rangle\\ &= (1-\alpha_n)d^2(x_n,u)+2\alpha_n[\langle\overrightarrow{ug(u)},\overrightarrow{uy_n}\rangle+\langle\overrightarrow{g(u)g(x_n)},\overrightarrow{uy_n}\rangle]\\ &\leq (1-\alpha_n)d^2(x_n,u)+2\alpha_n\langle\overrightarrow{g(u)u},\overrightarrow{y_nu}\rangle+2\alpha_n\gamma d(u,x_n)d(u,y_n)\\ &\leq (1-\alpha_n)d^2(x_n,u)+2\alpha_n\langle\overrightarrow{g(u)u},\overrightarrow{y_nu}\rangle\\ &\quad +2\alpha_n\gamma d(x_n,u)\Big[d(x_n,u)+\alpha_n d(g(u),u)\Big]\\ &= \Big[1-\alpha_n(1-2\gamma)\Big]d^2(x_n,u)\\ &\quad +2\alpha_n\Big[\langle\overrightarrow{g(u)u},\overrightarrow{x_{n+1}u}\rangle+\alpha_n\gamma d(g(u),u)d(x_n,u)\Big].\end{aligned}$$

Furthermore, with (x_{n+1}) in (3.6); by Lemma 2.10(ii) and (3.16), we get

$$\begin{aligned}d^2(x_{n+1},u) &= d^2(\beta_nx_n\oplus(1-\beta_n)y_n,u)\\ &\leq \beta_nd^2(x_n,u)+(1-\beta_n)d^2(y_n,u)\\ &\leq \Big[1-\alpha_n(1-\beta_n)(1-2\gamma)\Big]d^2(x_n,u)\\ &\quad +2\alpha_n(1-\beta_n)\Big[\langle\overrightarrow{g(u)u},\overrightarrow{y_nu}\rangle+\alpha_n\gamma d(g(u),u)d(x_n,u)\Big].\end{aligned}$$

Hence

$$d^2(x_{n+1}, u) \leq (1 - \delta_n)d^2(x_n, u) + \delta_n\theta_n,$$

where

$$\theta_n := \frac{2\alpha_n(1-\beta_n)(1-2\gamma)[\langle\overrightarrow{g(u)u}, \overrightarrow{y_n u}\rangle + \alpha_n\gamma d(g(u), u)d(x_n, u)]}{1-2\gamma}$$

and

$$\delta_n := 2\alpha_n(1-\beta_n)(1-2\gamma).$$

Thus, it follows from condition (C1) and (3.15) that $\lim_{n\to\infty} \delta_n = 0$, $\sum \delta_n = \infty$ and

$$\limsup_{n\to\infty} \theta_n \leq 0.$$

Therefore, by Lemma 2.20 , we get $d(x_n, u) \to 0$ as $n \to \infty$ that is $x_n \to u$ as $n \to \infty$, where $u := P_{F(T)}g(u)$. This complete the the proof.

Let C be a nonempty closed and convex subset of a Hadamard space X and $B : C \to 2^{X^*}$ be a continuous monotone mapping. Then there exists a resolvent operator $S_\lambda : X \to 2^C$ of B with order $\lambda_n > 0$ for each $n \geq 1$ and defined as

$$S_{\lambda_n}(x) := \{z \in C : \langle x^*, \overrightarrow{zy}\rangle + \frac{1}{\lambda_n}\langle\overrightarrow{xz}, \overrightarrow{zy}\rangle \geq 0,\ \forall\ y \in C, x^* \in Bz\}$$

where $x \in X$.

Corollary 3.5 Let (X, d) be a Hadamard space and $C \subset X$ a nonempty closed and convex set. Let $B : C \to 2^{X^*}$ be a continuous monotone mapping such that $N(B) \neq \emptyset$. Let g, λ_n, $\{\alpha_n\}$ and $\{\beta_n\}$ be as in Theorem 3.4. Let $\{x_n\}_{n=1}^\infty$ be a sequence generated by:

$$\begin{cases} x_1 \in C; \\ x_{n+1} = \beta_n x_n \oplus (1-\beta_n)y_n, \\ y_n = (1-\alpha_n)S_{\lambda_n}x_n \oplus \alpha_n g(x_n). \end{cases} \tag{3.16}$$

Then the sequence $\{x_n\}_{n=1}^\infty$ converges strongly to $u \in N(B)$, where $u = P_{N(B)}g(u)$.

Proof. Let $Bx = [Txx]$ for all $x \in X$, then by Lemma 3.1 we get that T is continuous pseudo-contraction and $N(B) = \{x \in C : x \in B^{-1}(0)\} = F(T)$. Thus, we obtain the desire result from Theorem 3.4

4. Application

In this section, using Theorem 3.4, we obtain important and new result that is associated with minimization of lower semicontinuous and convex functions in CAT(0) space.

Definition 4.1[16] Let (X, d) be a Hadamard space with a dual X^* and $h : X \to (-\infty, +\infty]$ a proper function with domain $D(h) = \{x \in X : h(x) < +\infty\}$. The subdifferential of h is the multivalued mapping $\partial h : X \rightrightarrows 2^{X^*}$ defined by

$$\partial h = \{x^* \in X^* : \langle x^*, \overrightarrow{xz}\rangle \leq h(z) - h(x) \quad (z \in X)\},$$

for $x \in D(h)$ and $\partial h = \emptyset$, otherwise.

Theorem 4.2 [16] If $h : X \to (-\infty, +\infty]$ is a proper lower semicontinuous and convex function, where X is a Hadamard space, then

(i) h attains its minimum at $x \in X$ if and only if $\mathbf{0} \in \partial h(x)$, where $\mathbf{0} = [\overrightarrow{ww}]$, for fix $w \in X$.
(ii) $\partial h : X \rightrightarrows 2^{X^*}$ is a monotone mapping.
(iii) for each $y \in X$, there exists a point $x \in X$, such that $[\overrightarrow{xy}] \in \partial h(x)$.

Note that (iii) of Theorem 4.2 shows that $D(\partial h) = X$.

Let $h : X \to (-\infty, +\infty]$ be proper lower semicontinuous and convex function on a Hadamard space X with dual X^* and their subdifferentials $\partial h : X \rightrightarrows 2^{X^*}$ satisfy all the conditions of Theorem 4.2. For $\lambda > 0$ and $x \in X$, let

$$K_{\lambda_n}(x) := \{z \in X : \langle x^*, \overrightarrow{zy}\rangle + \frac{1}{\lambda_n}\langle \overrightarrow{xz}, \overrightarrow{zy}\rangle \geq 0,\ \forall y \in X, x^* \in \partial hz\}.$$

Lemma 4.3 [27] Let X be a Hadamard space with dual X^*. Let $h : X \to (-\infty, +\infty]$ be a proper, lower semicontinuous and convex function. Then

$$J^{\partial h}_{\lambda_n}(x) = \text{argmin}_{z \in X}\{h(z) + \frac{1}{2\lambda_n} d^2(z, x)\},\ \forall\ \lambda_n > 0 \text{ and } x \in X,$$

and $F(J^{\partial h}_{\lambda_n}) = (\partial h)^{-1}(0)$.

Now with the help of Lemma 4.3, we obtain the following result. **Theorem 4.4** Let (X, d) be a Hadamard space with dual space X^*. Let $h : X \to (-\infty, +\infty]$ be a proper lower semicontinuous function such that $F := (\partial h)^{-1}(\mathbf{0}) \cap \arg\min(\mathbf{h}) \neq \emptyset$ Let g be a contraction of X into itself with a contraction constant $\gamma \in (0, 1)$. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence generated by:

$$\begin{cases} x_1 \in X; \\ x_{n+1} = \beta_n x_n \oplus (1 - \beta_n) y_n, \\ y_n = (1 - \alpha_n) K_{\lambda_n} x_n \oplus \alpha_n g(x_n), \end{cases}$$

where $\lambda \in (0, \infty)$; $\{\alpha_n\}$ and $\{\beta_n\}$ be a real sequences in $[0, 1]$ satisfy the following conditions:

(C1) $\lim\limits_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(C2) $\lim\limits_{n\to\infty} |\alpha_{n+1} - \alpha_n| = 0$ and $\lim\limits_{n\to\infty} |\lambda_{n+1} - \lambda_n| = 0$.

(C3) $0 < \liminf\limits_{n\to\infty} \beta_n \leq \limsup\limits_{n\to\infty} \beta_n < 1$.

Then the sequence $\{x_n\}_{n=1}^{\infty}$ converges strongly to $u \in F$, where $u = P_F g(u)$.

Proof. Note that $(\partial h)^{-1}(0) = N(\partial h)$, since ∂h is monotone. Thus, the conclusion follows from Lemma 3.1 and Theorem 3.4.

4.1. **Numerical Example.** We provide a numerical result in support of Theorem 3.4. Consider $X = \mathbb{R}$ with usual metric. Then for $\lambda > 0$ and $x \in \mathbb{R}$, by Lemma 3.2, there exists $z \in \mathbb{R}$ such that for each $y \in \mathbb{R}$, we have

$$
\begin{aligned}
& \langle \overrightarrow{Tzz}, \overrightarrow{yz} \rangle + \frac{1}{\lambda}\langle \overrightarrow{xz}, \overrightarrow{zy} \rangle \geq 0 \\
\Leftrightarrow \quad & \frac{1}{2}\Big(d^2(Tz,z) + d^2(z,y) - d^2(Tz,y) - d^2(z,z)\Big) \\
& + \frac{1}{2\lambda}\Big(d^2(x,y) + d^2(z,z) - d^2(x,z) - d^2(z,y)\Big) \geq 0 \\
\Leftrightarrow \quad & \frac{1}{2}\Big(|Tz - z|^2 + |z-y|^2 - |Tz - y|^2\Big) \\
& + \frac{1}{2\lambda}\Big(|x-y|^2 - |x-z|^2 - |z-y|^2\Big) \geq 0 \\
\Leftrightarrow \quad & \frac{1}{2}\Big(((Tz)^2 - 2(Tz)z + z^2) + (z^2 - 2zy + y^2) - ((Tz)^2 - 2(Tz)y + y^2)\Big) \\
& + \frac{1}{2\lambda}\Big((x^2 - 2xy + y^2) - (x^2 - 2xz + z^2) - (z^2 - 2zy + y^2)\Big) \geq 0 \\
\Leftrightarrow \quad & y\Big((Tz - z) + \frac{z-x}{\lambda}\Big) + z(z-a) + \frac{z(x-z)}{\lambda} \geq 0.
\end{aligned}
$$

Put

$$G(y) := 0y^2 + y\Big((Tz - z) + \frac{z-x}{\lambda}\Big) + z(z-a) + \frac{z(x-z)}{\lambda}.$$

Then G is a quadratic function of y with coefficient $a = 0$, $b = \Big((Tz - z) + \frac{z-x}{\lambda}\Big)$ and $c = z(z-a) + \frac{z(x-z)}{\lambda}$. Its discriminant $\Delta := b^2 - 4ac = b^2$. Note that $G(y) \geq 0$ for all $y \in \mathbb{R}$. If it has at most one solution in $\mathbb{R}$, then, $G(y) = 0$; hence

$$\lambda(Tz - z) + z - x = 0. \tag{4.1}$$

Choose $C = [0,1]$, $Tx = 1 - x^{\frac{2}{3}}$, $\lambda = 1$, $\alpha_n = \frac{1}{2n}$, $\beta_n = \frac{n}{7n+1}$ and $g(x_n) = \frac{1}{4}x_n$. Then, by (4.1), we get $(1 - z^{\frac{2}{3}} - z) + z - x = 0$. Thus $z = (1-x)^{\frac{3}{2}}$. Therefore $J_\lambda(x) = (1-x)^{\frac{3}{2}}$. Hence the algorithm (3.6) can be simplified as:

$$
\begin{cases}
x_{n+1} = \frac{n}{7n+1}x_n + \frac{6n+1}{7n}y_n, \\
y_n = \frac{2n-1}{2n}(1-x_n)^{\frac{3}{2}} + \frac{1}{8n}x_n.
\end{cases}
$$

By taking the initial value $x_1 = 0.5$, the numerical experiment result using MATLAB is given in Figure 1, which shows that this iteration process converges to 0.4297.

Remark 4.5

(i) Theorem 3.4 extends the results of Zegeye [35] and Ugwunnadi and Ali [31] from Hilbert space setting to Hadamard spaces.

(ii) Theorem 3.4 extends and generalizes the results of Moudafi [24] and Zegeye [36] in the following aspects:

(a) a real Hilbert space in [24] is replaced by a nonlinear domain, namely, a Hadamaed space.

(b) a nonexpansive mapping in [24] is replaced by continuous pseudo-contraction mapping.

(c) a real Banach space in [36] is replaced by a Hadamard space.

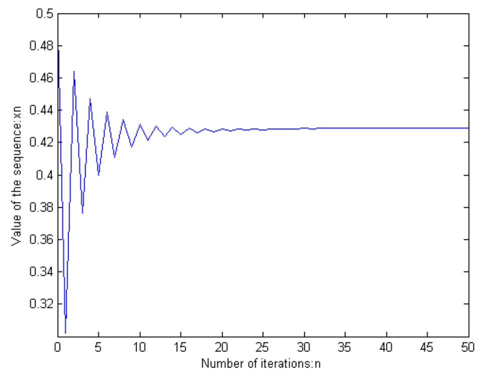

FIGURE 1. $x_1 = 0.5$, the convergence process of the sequence $\{x_n\}$.

(iii) As T is continuous pseudo-contraction if and only if $T := I \setminus B$ is monotone so our Theorem 3.4 also holds for monotone mapping on Hadamard spaces.

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