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BEST PROXIMITY POINTS OF SET-VALUED GENERALIZED CONTRACTIONS

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Abstract. We establish a unique best proximity point theorem for generalized set-valued contractions on metric spaces without involving the Hausdorff distance. This result subsumes and generalizes few important fixed point and best proximity point results for set-valued mappings. In particular, our result enables us to derive the Mizoguchi-Takahashi's fixed point result for closed valued map rather than closed and bounded valued. Moreover, we obtain a best proximity point result for maps satisfying Mizoguchi-Takahashi contractions uniformly locally.
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1. Introduction

Consider a space X with metric d and a map $T : A \to B$ in which $A, B \subseteq (X, d)$. There need not be an element $x^* \in A$ meeting $Tx^* = x^*$ if A does not intersects B. In that situation, we are interested to look for a nearest solution of $Tx = x$. The idea of best proximity point was initiated with an aim for finding a best approximant point $x^* \in A$ which optimizes the problem $\min_{x\in A} d(x, Tx)$. As for each x in A, $d(x, Tx) \geq \inf\{d(a, b) \ : a \in A,\ b \in B\} = dist(A, B)$, an element $x^* \in A$ satisfying $d(x^*, Tx^*) = dist(A, B)$ will optimize $\min_{x\in A} d(x, Tx)$. We call those element x^*, a *best proximity point* [9, 8, 10] for T in general. In fact, Basha and Veeramani [2, 3] have studied for ensuring best proximity points of set-valued maps on a normed linear space. For a set-valued mapping $T : A \to 2^B$, the idea of best proximity point was initiated for finding a point in A which optimizes the problem $\min_{x\in A} D(x, Tx)$ where $D(x, C) = \inf_{z\in C} d(x, z)$ for $C \subseteq X$. Therefore a point $x^* \in A$ meeting the condition $D(x^*, Tx^*) = dist(A, B)$ is familiar as best proximity point [1, 15] of the map T.

Nadler [12] in 1969 first initiated the study of fixed points for multi-valued contractions on (X, d). Let us now define the renowned Hausdorff metric H defined on the set $CB(X) = \{M \subseteq X : M$ is closed, bounded and $M \neq \emptyset\}$. Let $F : X \to CB(X)$ be such that $H(F(x), F(y)) \leq kd(x, y)\ \forall\, x, y \in X$ and for some $k \in [0, 1)$. Nadler [12] deduced that $\exists\, x^* \in X$ satisfying $x^* \in F(x^*)$ if (X, d) is complete. After that, Mizoguchi and Takahashi [11] derived an extension of Nadler's result for generalized set-valued contractions. It is worth to mention that Reich [14] studied the existence of fixed points for such generalized contractions defined from X to $K(X)$ (where $K(X)$ is the set of non-empty compact subsets of X). In [16], examples has been given to illustrate that such contractions is not necessarily Nadler contractions in general.

Theorem 1.1. [11] *Assume $F : X \to CB(X)$ is a map such that*

$$H(F(x), F(y)) \leq \beta(d(x, y))d(x, y) \; \forall\, x, y \in X, \; x \neq y, \tag{1.1}$$

in which $\beta : (0, \infty) \to [0, 1)$ satisfy $\limsup_{s \to t+} \beta(s) < 1$ for any t. If (X, d) is complete, then $\exists\, x^* \in X$ with $x^* \in F(x^*)$.

Feng and Liu [6] proved Nadler's result in a new direction without using the Hausdorff metric. In fact, they consider the map F to be $Cl(X)$ valued where $Cl(X) = \{M \subseteq X : M \text{ is closed and } M \neq \emptyset\}$. The result derived by them is sated below.

Theorem 1.2. [6] *Suppose $F : X \to Cl(X)$ fulfills for $x \in X, y \in F(x)$*

$$D(y, F(y)) \leq kd(x, y) \quad \text{for some } k \in [0, 1). \tag{1.2}$$

Then we have $x^* \in X$ satisfying $x^* \in F(x^*)$ when (X, d) is complete and $x \to D(x, F(x))$ becomes lower semi-continuous.

On the other hand, Akbar and Gabeleh [1] obtained a theorem on best proximity point for maps satisfying set-valued contractive conditions on metric spaces. This result extends the set-valued fixed point result proved by Nadler [12]. The result due to Akbar and Gabeleh [1] is stated below.

Theorem 1.3. [1] *Assume that $A, B \subseteq X$ are non-empty closed sets with the P-property and $A_0 \neq \emptyset$. Suppose $F : A \to CB(B)$ satisfying $F(A_0) \subseteq B_0$ and*

$$H(F(u), F(v)) \leq kd(u, v) \; \forall\, u, v \in A_0, \tag{1.3}$$

for some $0 \leq k < 1$. If (X, d) is complete, then F will have a best proximity point.

Recently, Sahin et al. [15] have used a unique approach to derive the presence of best proximity point for multi-valued contractions (not necessarily self) on metric spaces without using the Hausdorff distance by considering the technique used by Feng and Liu [6] for ensuring fixed points. In fact, the following result is presented by Sahin et al. [15].

Theorem 1.4. [15] *Assume that $A, B \subseteq X$ are non-empty closed sets with the P-property and $A_0 \neq \emptyset$. Suppose that $F : A \to Cl(B)$ satisfies $F(A_0) \subseteq B_0$ and for each $x \in A_0$, $y \in F(x)$,*

$$D(y, F(z)) \leq cd(x, z) \qquad \text{for some } c \in [0, 1),$$

where $z \in A_0$ with $d(z, y) = dist(A, B)$. Then F will have a best proximity point when (X, d) is complete and $f : A \times B \to \mathbb{R}$ formed by $f(x, y) = D(y, F(x))$ becomes lower semi-continuous.

Following Sahin et al. [15], we derive a unique best proximity point result of set-valued generalized contractions without involving the Hausdorff distance. Interestingly, our this theorem unifies the above mentioned results for set-valued maps on fixed point and best proximity points. Indeed, this result enables us to extend Theorem 1.1 in which the mapping is assumed to be closed valued rather than closed and bounded valued. Moreover, we have derived another result for ensuring the presence of best proximity points for those maps who satisfy eqn. (1.1) uniformly locally. This result yields an extension of Theorem 1.3 for mappings satisfying the contractive condition (1.3) locally.

2. Preliminaries

We will utilize the subsequent symbols and definitions through this article.

Let X be a space with metric d. For the sets $P, Q \subseteq (X, d)$, we set

$P_0 := \{p \in P : d(p, q) = dist(P, Q) \text{ where } q \in Q\}$
$Q_0 := \{q \in Q : d(p, q) = dist(P, Q) \text{ where } p \in P\}$.

Definition 2.1. For $P, Q \subseteq X$, the couple set (P, Q) holds the P-property [13] if $p, p' \in P$ and $q, q' \in Q$ with $d(p, q) = dist(P, Q)$ and $d(p', q') = dist(P, Q)$, we have $d(p, p') = d(q, q')$.

Consider a real Hilbert space H and $P, Q \subseteq H$ are nonempty convex and closed sets. Then (P, Q) holds the P-property (see [13]). The stated below example illustrates that the P-property need not hold if $P, Q \subseteq H$ are not convex.

Example 2.2. Let us take the space $(\mathbb{R}^2, ||.||_2)$. Suppose that

$$P = \{(-5, x) : x \in [0, 1]\} \cup \{(5, x) : x \in [0, 1]\}$$

and

$$Q = \{(-1, x) : x \in [0, 1]\} \cup \{(1, x) : x \in [0, 1]\}$$

are two non-empty closed subsets of $\mathbb{R}^2$. Evidently, P_0 coincides with P, Q_0 coincides with Q and $dist(P, Q) = 4$. It is important to mention that (P, Q) does not hold the P-property.

We are now providing a nontrivial example for a couple set (P, Q) of a metric space satisfying the P-property.

Example 2.3. Let $X = \mathbb{R}^3$ with the metric

$$d((x_1, x_2, x_3), (y_1, y_2, y_3)) = |x_1 - y_1| + |x_2 - y_2| + |x_3 - y_3|.$$

Suppose that

$$P = \{(0, 0, 0), (0, 0, 4), (0, 4, 0), (0, 4, 5), (0, 5, 4)\}$$

and

$$Q = \{(1, 0, 0), (1, 0, 4), (1, 4, 0), (1, 4, 5), (1, 5, 4)\}$$

are two non-empty subsets of $\mathbb{R}^3$. It is easy to see that $P_0 = P$, $Q_0 = Q$ and $dist(P, Q) = 1$. Moreover, (P, Q) holds the P-property.

We now bring back the stated below lemma proved by Sahin et al. [15] which will be utilized in the subsequent section.

Lemma 2.4. [15] *Let $A, B \subseteq X$ and $F : A \to 2^B$ be upper semicontinuous. Then, $f : A \times B \to \mathbb{R}$ formed by $f(a,b) = D(b, F(a))$ becomes lower semi-continuous.*

The symbol S indicates the set consists of all $\beta : (0,\infty) \to [0,1)$ fulfilling $\limsup_{s\to t+} \beta(s) < 1,\ \forall\, t \in [0,\infty)$.

3. Main results

We now present the stated below result for ensuring the presence of best proximity points for multi-valued maps on a space (X,d) where the map is assumed to be closed valued rather than closed and bounded valued.

Theorem 3.1. *Assume $A, B \subseteq X$ are nonempty closed sets with the P-property. Let $F : A \to Cl(B)$ satisfy $F(A_0) \subseteq B_0$ where $A_0 \neq \emptyset$. Assume that $\exists\, \beta \in S$ so that for each $x \in A_0$ and $y \in F(x)$*

$$D(y, F(z)) \leq \beta(d(x,z))d(x,z), \tag{3.1}$$

where $z \in A_0$ with $d(z,y) = dist(A,B)$. Then F will have a best proximity point when (X,d) is complete and the map $f : A \times B \to \mathbb{R}$ formed as $f(x,y) = D(y, F(x))$ becomes lower semicontinuous.

Proof. Let us take an arbitrary point $x_0 \in A_0$ and $y_0 \in F(x_0)$. As $y_0 \in F(A_0) \subseteq B_0$, then $\exists\, x_1 \in A_0$ with $d(x_1, y_0) = dist(A,B)$. Therefore for this $x_1 \in A_0$

$$D(y_0, F(x_1)) \leq \beta(d(x_0,x_1))d(x_0,x_1).$$

Considering that $[\beta(d(x_0,x_1))]^{1/2}$ is less than 1, we can find an element $y_1 \in F(x_1)$ satisfying

$$[\beta(d(x_0,x_1))]^{1/2} d(y_0,y_1) \leq D(y_0, F(x_1)).$$

It appears from the above inequalities that

$$d(y_0,y_1) \leq [\beta(d(x_0,x_1))]^{1/2} d(x_0,x_1).$$

Again, there is $x_2 \in A_0$ satisfying $d(x_2,y_1) = dist(A,B)$ as $y_1 \in F(A_0) \subseteq B_0$. Thus for this $x_2 \in A_0$, we can find an element $y_2 \in F(x_2)$ with

$$d(y_1,y_2) \leq [\beta(d(x_1,x_2))]^{1/2} d(x_1,x_2).$$

In similar fashion, we derive $\{x_n\}_n \in A$ and $\{y_n\}_n \in B$ satisfying for $n \in \mathbb{N}$,

$$d(y_n,y_{n+1}) \leq [\beta(d(x_n,x_{n+1}))]^{1/2} d(x_n,x_{n+1}) \tag{3.2}$$

and $d(x_{n+1},y_n) = dist(A,B)$. Hence and taking into account that the pair (A,B) having the P-property, it occurs

$$d(x_{n+1},x_n) = d(y_n,y_{n-1}) \qquad \text{for each } n. \tag{3.3}$$

Hence and by eqn. (3.2), it appears that $\{l_n\}_n = \{d(x_n,x_{n-1})\}_n$ is monotone decreasing. Thus it is convergent to l (say). Considering the fact that $\limsup_{s\to l+} \beta(s) < 1$, there exist a natural number M and a real $q \in [0,1)$ fulfilling $\beta(l_n) < q \ \forall n \geq M$. Hence for $k = 1, 2, \cdots$ and $n \geq M$, we have from (3.2) and (3.3) that

$$\begin{aligned} d(x_n, x_{n+k}) &\leq l_{n+1} + l_{n+2} + \cdots + l_{n+k} \\ &\leq \left(\prod_{s=1}^{n} [\beta(l_s)]^{1/2} \right) l_1 + \cdots + \left(\prod_{s=1}^{n+k-1} [\beta(l_s)]^{1/2} \right) l_1 \\ &\leq E q^{n/2} [1 + q^{1/2} + \cdots + q^{(k-1)/2}] \\ &\leq E q^{n/2} / (1 - q^{1/2}). \end{aligned}$$

where E is a positive real number. Therefore $\{x_n\}_{n\in\mathbb{N}} \in A$ is Cauchy and thus convergent on account of A is closed. Assume $x_n \to x^* \in A$ when n tends to ∞. Hence and from eq (3.3), it appears that $\{y_n\}_{n\in\mathbb{N}} \in B$ will also converge to some $y^* \in B$ (say). Thus

$$\lim_{n\to\infty} d(x_{n+1}, y_n) = d(x^*, y^*).$$

Moreover for any n, $d(x_{n+1}, y_n) = dist(A, B)$ and hence one can conclude that $d(x^*, y^*) = dist(A, B)$. Since $d(x_n, y_n) \to d(x^*, y^*)$ and $y_n \in F(x_n)$, the lower semi-continuity of f implies that

$$D(y^*, F(x^*)) = f(x^*, y^*) \leq \liminf f(x_n, y_n) = \liminf D(y_n, F(x_n)) = 0.$$

Hence one has $y^* \in F(x^*)$ being as $F(x^*)$ closed. Therefore one can see

$$D(x^*, F(x^*)) \leq d(x^*, y^*) = dist(A, B) \leq D(x^*, F(x^*)).$$

Thus it appears that $x^* \in A$ satisfies $D(x^*, F(x^*)) = dist(A, B)$ and hence proved. $\square$

The subsequent examples demonstrate the above result.

Example 3.2. Consider the space l^1 consists of all real sequences

$$x = (x_n)_{n\in\mathbb{N}} = (x_1, x_2, \cdots)$$

such that

$$\sum_{j=1}^{\infty} |x_j| < \infty.$$

Let $X = l^1$ with the metric

$$d(x, y) = \sum_{j=1}^{\infty} |x_j - y_j|.$$

Then (X, d) is a complete metric space. Let

$$A = \left\{ \left(0, \frac{a}{2^0}, \frac{a}{2}, \frac{a}{2^2}, \frac{a}{2^3}, \frac{a}{2^4}, \cdots \right) : a \in [0,1] \right\}$$

and

$$B = \left\{ \left(1, \frac{b}{2^0}, \frac{b}{2}, \frac{b}{2^2}, \frac{b}{2^3}, \frac{b}{2^4}, \cdots \right) : b \in [0,1] \right\}$$

be two non-empty closed subsets of l^1. Evidently, A_0 coincides with A, B_0 coincides with B and $dist(A, B) = 1$. It is worth to mention that (A, B) holds the P-property. We form $F : A \to Cl(B)$ as follows

$$F(0, \tfrac{a}{2^0}, \tfrac{a}{2}, \tfrac{a}{2^2}, \cdots) = \begin{cases} \{(1,0,0,0,\cdots)\} & \text{when } a = 0, \\ \left\{\left(1, \frac{a^2}{2}, \frac{1}{2}\frac{a^2}{2}, \frac{1}{2^2}\frac{a^2}{2}, \cdots\right), (1,1,\frac{1}{2},\frac{1}{2^2},\cdots)\right\} & \text{otherwise.} \end{cases}$$

Define a map β from the set of non-negative reals onto $[0,1)$ by $\beta(t) = 3/4$, for all $t \geq 0$. It is easy to see that F meets the condition (3.1) of Theorem 3.1 for the above defined β. We note that the best proximity points of F are $(0,0,0,0,0,\cdots)$ and $(0,1,\frac{1}{2},\frac{1}{2^2},\frac{1}{2^3},\frac{1}{2^4},\cdots)$.

Example 3.3. Let $X = \mathbb{R}^2$ with the metric

$$d((x_1,x_2),(y_1,y_2)) = |x_1 - y_1| + |x_2 - y_2|.$$

Then (X,d) is a complete metric space. Let

$$A = \{(0,x) : x \in [0,1]\} \cup \{(3,x) : x \in [0,1]\}$$

and

$$B = \{(1,x) : x \in [0,1]\} \cup \{(4,x) : x \in [0,1]\}$$

be two non-empty closed subsets of $\mathbb{R}^2$. Evidently, A_0 coincides with A, B_0 coincides with B and $dist(A,B) = 1$. It is worth to mention that (A,B) holds the P-property. We form $F : A \to Cl(B)$ as follows

$$F(x) = \begin{cases} \left\{\left(1, \frac{x_2^2}{2}\right), (1,1)\right\} & \text{when } x = (0,x_2),\ 0 < x_2 \leq 1 \\ \{(1,0)\} & \text{when } x = (0,0) \\ \left\{\left(4, \frac{x_2^2}{2}\right), (4,1)\right\} & \text{when } x = (3,x_2),\ 0 < x_2 \leq 1 \\ \{(4,0)\} & \text{otherwise.} \end{cases}$$

Construct a map β from the set of non-negative reals onto $[0,1)$ by $\beta(t) = 3t/2$ when $t \in [0,1/2)$ and $\beta(t) = 0$, otherwise. It is easy to see that F meets the condition (3.1) of Theorem 3.1 for the above defined β. We observe that the best proximity points of F are $(0,0)$, $(0,1)$, $(3,0)$ and $(3,1)$.

By using Lemma 2.4, we can see that Theorem 3.1 yields the stated below result for maps (not necessarily self) meeting the condition (1.1).

Corollary 3.4. *Assume $A, B \subseteq X$ are non-empty closed sets. Let $A_0 \neq \emptyset$ and (A,B) be with the P-property. Let $F : A \to CB(B)$ satisfy $F(A_0) \subseteq B_0$ and for any $u, v \in A_0$*

$$H(F(u),F(v)) \leq \beta(d(u,v))d(u,v), \quad \text{where } \beta \in S.$$

Then F will have a best proximity point when (X,d) is complete.

If we take both A and B equal to the whole set X in the above Theorem 3.1, the subsequent generalization of Theorem 1.1 follows in which the map is assumed to be closed valued rather than closed and bounded valued.

Corollary 3.5. *Let $F : X \to Cl(X)$ satisfy for any $x \in X$, $y \in F(x)$*

$$D(y,F(y)) \leq \alpha(d(x,y))d(x,y) \quad \text{where } \alpha \in S.$$

Then we have $x^* \in X$ with $x^* \in F(x^*)$ when $x \to D(x,F(x))$ becomes lower semi-continuous and (X,d) is complete.

Consider $\beta(t) = c$, $\forall\, t \geq 0$ in the above Theorem 3.1. Then as a corollary, we get Theorem 1.4 proved by Sahin et al. [15].

On the other side, Edelstein [5] first studied the presence of fixed point for uniformly local contractions. Subsequently, Nadler [12] generalized this concept for set-valued maps. Recently, Dinevari and Frigon [4] deduced a simple proof for ensuring the fixed point of uniformly locally set-valued contractions using graph structure. Now, we derive a different best proximity point result for non-self set-valued maps meeting the condition (1.1) uniformly locally.

Theorem 3.6. *Let $A, B \subseteq X$ be non-empty closed sets having the P-property. Assume $F : A \to CB(B)$ satisfies $F(A_0) \subseteq B_0$ and for any $u, v \in A_0$ with $d(u, v) < \varepsilon$ (where $\varepsilon > 0$)*

$$H(F(u), F(v)) \leq \beta(d(u, v))d(u, v) \qquad \text{for some } \beta \in S. \tag{3.4}$$

Then F will have a best proximity point if (X, d) is complete and $u_0, u_1 \in A_0$ occurs fulfilling $d(u_0, u_1) < \varepsilon$ and $d(u_1, v_0) = dist(A, B)$ for some $v_0 \in F(u_0)$.

Proof. Since $v_0 \in F(u_0)$, there is $v_1 \in F(u_1)$ meeting the condition

$$d(v_1, v_0) \leq H(F(u_1), F(u_0)) + \beta^{l_1}(d(u_1, u_0)),$$

where $l_1 \in \mathbb{N}$ satisfying $\beta^{l_1}(d(u_1, u_0)) < (1 - \beta(d(u_1, u_0)))d(u_1, u_0)$. As we know $d(u_0, u_1) < \varepsilon$, it appears from the above inequalities that

$$d(v_1, v_0) \leq \beta(d(u_1, u_0))d(u_1, u_0) + \beta^{l_1}(d(u_1, u_0)) < d(u_1, u_0).$$

As $v_1 \in F(A_0) \subseteq B_0$, we have $d(u_2, v_1) = dist(A, B)$ for some $u_2 \in A_0$. As the P-property holds for the pair (A, B), it occurs $d(u_2, u_1) = d(v_1, v_0)$. Thus we observe $d(u_2, u_1) < \varepsilon$ and hence we can find $v_2 \in F(u_2)$ meeting

$$d(v_2, v_1) \leq \beta(d(u_2, u_1))d(u_2, u_1) + \beta^{l_2}(d(u_2, u_1)) < d(u_2, u_1),$$

where $l_2 > l_1$. In a similar fashion, we construct $\{u_n\}_n \in A$ and $\{v_n\}_n \in B$ with $v_n \in F(u_n)$, $d(u_{n+1}, v_n) = dist(A, B)$ for any $n \geq 1$ and it meets the condition

$$d(v_n, v_{n-1}) \quad \leq \quad \beta(c_n)c_n + \beta^{l_n}(c_n) < c_n, \tag{3.5}$$

where l_n is a natural number with $l_n > l_{n-1}$ and $c_n = d(u_n, u_{n-1})$. As $d(u_{n+1}, v_n) = dist(A, B)$, $n \in \mathbb{N}$, according to the P-property of (A, B) it occurs

$$c_{n+1} = d(u_{n+1}, u_n) = d(v_n, v_{n-1}) \qquad \forall\, n \in \mathbb{N}. \tag{3.6}$$

Hence by eqn. (3.5), it appears that $\{c_n\}_n$ is monotone decreasing. Thus $\lim_{n\to\infty} c_n$ exists and let the limit be c. Considering the fact that $\limsup_{s\to c+} \beta(s) < 1$, there exist a real $h \in [0, 1)$ and a natural number N satisfying $\beta(c_n) < h$, for any $n \geq N$.

Therefore by eqns. (3.5) and (3.6) it appears for any $n \geq N$,

$$\begin{aligned} c_{n+1} &\leq \beta(c_n)c_n + \beta^{l_n}(c_n) \\ &\leq \prod_{q=0}^{n} \beta(c_q)c_0 + \sum_{s=1}^{n} \left(\prod_{q=s}^{n} \beta(c_q) \right) \beta^{l_s}(c_{s-1}) + \beta^{l_n}(c_n) \\ &\leq E_1 h^n + \sum_{s=1}^{n} \left(\prod_{q=\max\{s,N\}}^{n} \beta(c_q) \right) \beta^{l_s}(c_{s-1}) + h^n \\ &\leq E_1 h^n + E_2 h^n + \sum_{s=N+1}^{n} h^{n-s+l_s} + h^n \\ &\leq E_1 h^n + E_2 h^n + E_3 h^n + h^n \leq E h^n, \end{aligned}$$

in which E_1, E_2, E_3, E are positive reals. Thus for $m \in \mathbb{N}$ and $n \geq N$,

$$\begin{aligned} d(u_n, u_{n+m}) &\leq c_{n+1} + c_{n+2} \cdots + c_{n+m} \\ &\leq Eh^n + Eh^{n+1} + \cdots + Eh^{n+m-1} \leq Eh^n/(1-h). \end{aligned}$$

Therefore $\{u_n\}_n \in A$ is Cauchy and thus convergent. Let $\lim_{n\to\infty} u_n = u^*$. Hence and from eqn. (3.6) it can be concluded that $\lim_{n\to\infty} v_n$ exists and let it be v^*. Moreover for $n \geq 1$, $d(u_{n+1}, v_n) = dist(A, B)$ and hence it occurs $d(u^*, v^*) = \lim_{n\to\infty} d(u_{n+1}, v_n) = dist(A, B)$. If we are able to prove that $v^* \in F(u^*)$, then we are done. Since $u_n \to u^*$, $d(u_n, u^*) < \varepsilon \; \forall\, n \geq M$ (where M is a natural number). Thus for $n \geq M$,

$$D(v_n, F(u^*) \leq H(F(u_n), F(u^*) \leq \beta(d(u_n, u^*))d(u_n, u^*).$$

Thus we have $\lim_{n\to\infty} D(v_n, F(u^*) = 0$ and hence $v^* \in F(u^*)$. Therefore it appers $D(u^*, F(u^*)) = dist(A, B)$. □

Remark 3.7. We have noticed that the above stated theorem extends Theorem 1.3 proved by Akbar and Gabeleh [1]. Indeed, let us suppose that F meets the conditions of Theorem 1.3. Then $\exists\, x_0 \in A_0$ because of $A_0 \neq \emptyset$. Since $F(A_0) \subseteq B_0$, the elements $y_0 \in F(x_0), x_1 \in A_0$ occur with $d(x_1, y_0) = dist(A, B)$. Let ε be a real where $0 < \varepsilon < d(x_1, y_0)$. For this chosen ε, the map F meets the condition (3.4). At the end, by taking $\beta(t) = k$ for all $t \geq 0$, we are able to see F fulfills the assumptions of the above Theorem 3.6. The converse may not true which we can observe below.

Example 3.8. Let $X = l^1$ with the metric

$$d(x, y) = \sum_{j=1}^{\infty} |x_j - y_j|.$$

Take

$$\begin{aligned} A = &\left\{ (0, \frac{1}{2}, 0, 0, \cdots), (0, 0, \frac{1}{2^2}, 0, \cdots), (0, 0, 0, \frac{1}{2^3}, 0, \cdots), \cdots \right\} \\ &\cup \{(0, 0, 0, \cdots), (0, 1, 0, 0, \cdots)\} \end{aligned}$$

and

$$B = \left\{(1, \frac{1}{2}, 0, 0, \cdots, 0, \cdots), (1, 0, \frac{1}{2^2}, 0, \cdots, 0, \cdots), (1, 0, 0, \frac{1}{2^3}, 0, \cdots, 0, \cdots), \cdots\right\}$$
$$\cup \{(1, 0, 0, 0 \cdots, 0, \cdots), (1, 1, 0, 0 \cdots, 0, \cdots)\}.$$

It is evident that A, B are closed sets where $dist(A, B) = 1$, A_0 coincides with A and B_0 coincides with B. Construct a map $F : A \to CB(B)$ as

$$F(x) = \begin{cases} \{(1, 0, 0, 0 \cdots), (1, \frac{1}{2}, 0, 0, \cdots)\} & \text{for } x = (0, 0, 0, 0, \cdots) \\ \{(1, 0 \cdots, \frac{1}{2^{n+1}}, 0, \cdots), (1, \frac{1}{2}, 0, 0 \cdots)\} & \text{for } x = (0, \cdots, \frac{1}{2^n}, 0, \cdots), n \geq 1 \\ \{(1, 1, 0, 0, \cdots)\} & \text{for } x = (0, 1, 0, 0, \cdots). \end{cases}$$

For choosing $\varepsilon = \frac{1}{2}$, it is easy to see that F meets the condition (3.4) of Theorem 3.6 for the map β defined by $\beta(s) = 1/2$, $s \geq 0$. We observe that best proximity points of F are

$$\left\{(0, 0, 0, 0, \cdots, 0, \cdots), (0, \frac{1}{2}, 0, 0, \cdots, 0, \cdots), (0, 1, 0, 0, \cdots, 0 \cdots)\right\}.$$

Indeed,

$$H(F((0, 0, 0, 0, \cdots)), F((0, 1, 0, 0, \cdots)) = 1 = d((0, 0, 0, 0, \cdots), (0, 1, 0, 0, \cdots)).$$

Hence the condition (1.3) of Theorem 1.3 fails to occur.

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