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UNIQUE POSITIVE DEFINITE SOLUTION OF NON-LINEAR MATRIX EQUATION ON RELATIONAL METRIC SPACES

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Abstract. In this study, we consider a non-linear matrix equation of the form

$$\mathcal{X} = \mathcal{Q} + \sum_{i=1}^{m}\sum_{j=1}^{t} \mathcal{A}_i^* \mathcal{F}_j(\mathcal{X})\mathcal{A}_i,$$

where $\mathcal{Q}$ is a Hermitian positive definite matrix, $\mathcal{A}_i^*$ stands for the conjugate transpose of an $n \times n$ matrix $\mathcal{A}_i$ and $\mathcal{F}_j$ are order-preserving continuous mappings from the set of all Hermitian matrices to the set of all positive definite matrices such that $\mathcal{F}(O) = O$. We discuss sufficient conditions that ensure the existence of a unique positive definite solution of the given matrix equation. For this, we derive some fixed point results for Suzuki-implicit type mappings on metric spaces (not necessarily complete) endowed with arbitrary binary relation (not necessarily a partial order). We provide adequate examples to validate the fixed-point results and the importance of related work, and the convergence analysis of non-linear matrix equations.
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1. Introduction

The study of nonlinear matrix equations (NME) appeared first in the literature concerned with algebraic Riccati equation. These equations occur in large number of problems in control theory, dynamical programming, ladder network, stochastic filtering, queuing theory, statistics and many other applicable areas.

Let $\mathcal{H}(n)$ (resp. $\mathcal{K}(n)$, $\mathcal{P}(n)$) denote the set of all $n \times n$ Hermitian (resp. positive semi-definite, positive definite) matrices over $\mathbb{C}$ and $\mathcal{M}(n)$ the set of all $n \times n$ matrices over $\mathbb{C}$. In [18], Ran and Reurings discussed the existence of solutions of the following equation:

$$\mathcal{X} + \mathcal{B}^* F(\mathcal{X})\mathcal{B} = \mathcal{Q} \tag{1.1}$$

in $\mathcal{K}(n)$, where $\mathcal{B} \in \mathcal{M}(n)$, $\mathcal{Q}$ is positive definite and F is a mapping from $\mathcal{K}(n)$ into $\mathcal{M}(n)$. Note that $\mathcal{X}$ is a solution of (1.1) if and only if it is a fixed point of the mapping $\mathcal{G}(\mathcal{X}) = \mathcal{Q} - \mathcal{B}^* F(\mathcal{X})\mathcal{B}$. In [19], they used the notion of partial ordering and established a modification of Banach Contraction Principle, which they applied for solving a class of NMEs of the form $\mathcal{X} = \mathcal{Q} + \sum_{i=1}^{m} \mathcal{B}_i^* F(\mathcal{X})\mathcal{B}_i$ using the Ky Fan norm in $\mathcal{M}(n)$.

Theorem 1.1. [19] *Let $F : \mathcal{H}(n) \to \mathcal{H}(n)$ be an order-preserving, continuous mapping which maps $\mathcal{P}(n)$ into itself and $\mathcal{Q} \in \mathcal{P}(n)$. If $\mathcal{B}_i, \mathcal{B}_i^* \in \mathcal{P}(n)$ and*

$$\sum_{i=1}^{m} \mathcal{B}_i \mathcal{B}_i^* < M \cdot \mathcal{I}_n$$

for some $M > 0$ ($\mathcal{I}_n$ – the unit matrix in $\mathcal{M}(n)$) and if

$$|\operatorname{tr}(F(\mathcal{Y}) - F(\mathcal{X}))| \leq \frac{1}{M} |\operatorname{tr}(\mathcal{Y} - \mathcal{X})|,$$

for all $\mathcal{X}, \mathcal{Y} \in \mathcal{H}(n)$ with $\mathcal{X} \leq \mathcal{Y}$, then the equation

$$\mathcal{X} = \mathcal{Q} + \sum_{i=1}^{m} \mathcal{B}_i^* F(\mathcal{X})\mathcal{B}_i$$

has a unique positive definite solution (PDS).

On the other hand, throughout the last decades, many authors have obtained a huge number of fixed point and common fixed point results and applied these results to obtain solutions of different kinds of equations arising in different situations in many mathematical problems. In 2008, Suzuki [21] defined yet another new contraction, often referred as Suzuki contraction (a self-mapping $\mathcal{T}$ defined on a metric space (Ξ, d) is said to be a Suzuki contraction if there exists a nondecreasing function $\theta : [0, 1) \to (\frac{1}{2}, 1]$ defined by

$$\theta(k) = \begin{cases} 1, & \text{if } 0 \leq k \ \leq (\sqrt{5} - 1)/2 \\ (1 - k)k^{-2}, & \text{if } (\sqrt{5} - 1)/2 \leq k \leq 2^{-\frac{1}{2}} \\ (1 + k)^{-1}, & \text{if } 2^{-\frac{1}{2}} \leq k < 1. \end{cases}$$

such that for all $\nu, \ \vartheta \in \Xi$ and ($k \in [0, 1)$),

$$\theta(k)d(\nu, \ \mathcal{T}\nu) \leq d(\nu, \ \vartheta) \text{ implies } d(\mathcal{T}\nu, \ \mathcal{T}\vartheta) \leq k \ d(\nu, \ \vartheta)).$$

He utilized the same to prove a fixed point result which is another important generalization of Banach Contraction Principle. Suzuki's result was generalized by Popescu [17] (see also [16]). He considered the following more general contraction condition known as generalized Suzuki contraction.

$$\begin{aligned} &\theta(k)d(\nu, \ \mathcal{T}\nu) \leq d(\nu, \ \vartheta) \text{ implies} \\ &d(\mathcal{T}\nu, \ \mathcal{T}\vartheta) \leq \ k \max\left\{ d(\nu, \vartheta), d(\nu, \mathcal{T}\nu), d(\vartheta, \mathcal{T}\vartheta), \frac{1}{2}[d(\nu, \mathcal{T}\vartheta) + d(\vartheta, \mathcal{T}\nu)] \right\}. \end{aligned}$$

In recent years, a number of mathematicians have obtained fixed point results for contraction type mappings in metric spaces equipped with partial order. Some early results in this direction were established by Turinici in [22, 23]; one may note that their starting points were "amorphous" contributions in the area due to Matkowski [12, 13]. These types of results have been reinvestigated by Ran and Reurings [18] and also by Nieto and Ródríguez-López [14, 15]. The results of Turinici were further extended and refined in papers [14, 15]. Recently, Samet and Turinici [20] established fixed point theorems for nonlinear contraction under symmetric closure of an arbitrary relation. Most recently, Ahmadullah et al. [1, 2, 3, 4] and Alam and Imdad [5] employed an amorphous relation to prove a relation-theoretic analogue of Banach Contraction Principle which in turn unifies a lot of well known relevant order-theoretic fixed point theorems.

Motivated by the above mentioned work, in this paper, we introduce the notion of Suzuki-implicit type mapping on metric spaces endowed with an arbitrary binary relation (not necessarily partial order) and then we prove existence and uniqueness fixed point results under weaker conditions. We justify our work by some illustrative examples and demonstrate the genuineness of Suzuki-implicit type contraction over Suzuki contraction, generalized Suzuki contraction and implicit type contraction mapping. Further, we apply this result to NMEs and discuss its convergence behaviour with respect to three different initializations with graphical representations using MATLAB.

2. Preliminaries

Throughout this article, the notations $\mathbb{Z}$, $\mathbb{N}$, $\mathbb{R}$, $\mathbb{R}^+$ have their usual meanings, and $\mathbb{N}^* = \mathbb{N} \cup \{0\}$.

We call $(\Xi, \mathfrak{R})$ a relational set if (i) $\Xi \neq \emptyset$ is a set and (ii) $\mathfrak{R}$ is a binary relation on Ξ. In addition, if (Ξ, d) is a metric space, we call $(\Xi, d, \mathfrak{R})$ a relational metric space (RMS, for short).

The following are some standard terms used in the theory of relational sets (see, e.g., [5, 9, 10, 11, 20]).

Let $(\Xi, \mathfrak{R})$ be a relational set, $(\Xi, d, \mathfrak{R})$ be an RMS, and let $\mathcal{T}$ be a self-mapping on Ξ. Then:

(1) $\nu \in \Xi$ is $\mathfrak{R}$-related to $\vartheta \in \Xi$ if and only if $(\nu, \vartheta) \in \mathfrak{R}$.
(2) The set $(\Xi, \mathfrak{R})$ is said to be comparable if for all $\nu, \vartheta \in \Xi$, $[\nu, \vartheta] \in \mathfrak{R}$, where $[\nu, \vartheta] \in \mathfrak{R}$ means that either $(\nu, \vartheta) \in \mathfrak{R}$ or $(\vartheta, \nu) \in \mathfrak{R}$.
(3) A sequence (ν_n) in Ξ is said to be $\mathfrak{R}$-preserving if $(\nu_n,\ \nu_{n+1}) \in \mathfrak{R}$, $\forall n \in \mathbb{N} \cup \{0\}$.
(4) $(\Xi, d, \mathfrak{R})$ is said to be $\mathfrak{R}$-complete if every $\mathfrak{R}$-preserving Cauchy sequence converges in Ξ.
(5) $\mathfrak{R}$ is said to be $\mathcal{T}$-closed if $(\nu, \vartheta) \in \mathfrak{R} \Rightarrow (\mathcal{T}\nu, \mathcal{T}\vartheta) \in \mathfrak{R}$. It is said to be weakly $\mathcal{T}$-closed if $(\nu, \vartheta) \in \mathfrak{R} \Rightarrow [\mathcal{T}\nu, \mathcal{T}\vartheta] \in \mathfrak{R}$.
(6) $\mathfrak{R}$ is said to be d-self-closed if for every $\mathfrak{R}$-preserving sequence with $\nu_n \to \nu$, there is a subsequence (ν_{n_k}) of (ν_n), such that $[\nu_{n_k}, \nu] \in \mathfrak{R}$, for all $k \in \mathbb{N} \cup \{0\}$.
(7) A subset $\mathfrak{Z}$ of Ξ is called $\mathfrak{R}$-directed if for each $\nu, \vartheta \in \mathfrak{Z}$, there exists $\mu \in \Xi$ such that $(\nu, \mu) \in \mathfrak{R}$ and $(\vartheta, \mu) \in \mathfrak{R}$. It is called $(\mathcal{T}, \mathfrak{R})$-directed if for each $\nu, \vartheta \in \mathfrak{Z}$, there exists $\mu \in \Xi$ such that $(\nu, \mathcal{T}\mu) \in \mathfrak{R}$ and $(\vartheta, \mathcal{T}\mu) \in \mathfrak{R}$.

(8) $\mathcal{T}$ is said to be $\mathfrak{R}$-continuous at ν if for every $\mathfrak{R}$-preserving sequence (ν_n) converging to ν, we get $\mathcal{T}(\nu_n) \to \mathcal{T}(\nu)$ as $n \to \infty$. Moreover, $\mathcal{T}$ is said to be $\mathfrak{R}$-continuous if it is $\mathfrak{R}$-continuous at every point of Ξ.

(9) For $\nu, \vartheta \in \Xi$, a path of length k (where k is a natural number) in $\mathfrak{R}$ from ν to ϑ is a finite sequence $\{\mu_0,\ \mu_1,\ \mu_2,\ \ldots,\ \mu_k\} \subset \Xi$ satisfying the following conditions:
  (i) $z_0 = \nu$ and $\mu_k = \vartheta$,
  (ii) $(\mu_i,\ \mu_{i+1}) \in \mathfrak{R}$ for each i $(0 \leq i \leq k-1)$,
then this finite sequence is called a path of length k joining ν to ϑ in $\mathfrak{R}$.

(10) If for a pair of $\nu, \vartheta \in \Xi$, there is a finite sequence $\{\mu_0,\ \mu_1,\ \mu_2,\ \ldots,\ \mu_k\} \subset \Xi$ satisfying the following conditions:
  (i) $\mathcal{T}\mu_0 = \nu$ and $\mathcal{T}\mu_k = \vartheta$,
  (ii) $(\mathcal{T}\mu_i,\ \mathcal{T}\mu_{i+1}) \in \mathfrak{R}$ for each i $(0 \leq i \leq k-1)$.
then this finite sequence is called a $\mathcal{T}$-path of length k joining ν to ϑ in $\mathfrak{R}$. Notice that a path of length k involves $k+1$ elements of Ξ, although they are not necessarily distinct.

In what follows, we define a modified version of implicit relation discussed in [4, 6]. Denote by Φ the set of all functions $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying the following conditions:

(i) φ is increasing and $\varphi(0) = 0$;
(ii) $\sum_{n=1}^{\infty} \varphi^n(t) < \infty$, for $t > 0$; where φ^n denotes the n-th iterate.

It should be noted that $\varphi(\zeta) < \zeta$ and the family $\Phi \neq \emptyset$.

Example 2.1. Consider (Ξ, d) with usual metric, where $\Xi = [0, 1]$. Define the mapping $\varphi(\zeta) = \frac{2\lambda\zeta}{9}$, where $0 < \lambda < 1$. Then we have $\varphi^n(\zeta) \leq \frac{2^n\lambda^n\zeta}{9^n}$. Therefore,

$$\sum_{n=1}^{\infty} \varphi^n(\zeta) = \sum_{n=1}^{\infty} \frac{2^n\lambda^n\zeta}{9^n} < \infty$$

and hence $\Phi \neq \emptyset$.

Let $\mathfrak{G}$ be the set of all lower semi-continuous (l.s.c.) functions $\mathcal{G} : \mathbb{R}_+^6 \to \mathbb{R}$ that satisfy the following conditions:

($\mathcal{G}_1$) $\mathcal{G}(\zeta, \xi, \xi, \zeta, \mu, 0) \leq 0$, $\mathcal{G}(\zeta, \xi, \xi, \zeta, 0, \mu) \leq 0$ for all $\zeta, \xi, \mu \geq 0$, implies that there exists $\varphi \in \Phi$ such that $\zeta \leq \varphi(\xi)$;
($\mathcal{G}_2$) $\mathcal{G}(\zeta, 0, \zeta, 0, 0, \zeta) > 0$, for all $\zeta > 0$.

Let $\mathfrak{F} \subseteq \mathfrak{G}$ where functions $\mathcal{G} \in \mathfrak{F}$ satisfy the following additional conditions:

($\mathcal{G}_3$) $\mathcal{G}(\zeta, 0, 0, \zeta, \zeta, 0) > 0$, for all $\zeta > 0$;
($\mathcal{G}_4$) $\mathcal{G}(\zeta, \zeta, 0, 0, \zeta, \zeta) > 0$, for all $\zeta > 0$.

We fix the following notation for a relational space $(\Xi, \mathfrak{R})$, a self-mapping $\mathcal{T}$ on Ξ and an $\mathfrak{R}$-directed subset $\mathfrak{D}$ of Ξ:

(i) $Fix(\mathcal{T})$:= the set of all fixed points of $\mathcal{T}$,
(ii) $\mathfrak{X}(\mathcal{T}, \mathfrak{R}) := \{\nu \in \Xi : (\nu, \mathcal{T}\nu) \in \mathfrak{R}\}$,
(iii) $\mathfrak{P}(\nu, \vartheta, \mathfrak{R})$:= the class of all paths in $\mathfrak{R}$ from ν to ϑ in $\mathfrak{R}$, where $\nu, \vartheta \in \Xi$.

3. Suzuki-implicit type results

Definition 3.1. Let $(\Xi, d, \mathfrak{R})$ be an RMS and $\mathcal{T}\colon \Xi \to \Xi$ be a given mapping. A mapping $\mathcal{T}$ is said to be a Suzuki-implicit type mapping if there exists $\mathcal{G} \in \mathfrak{G}$ such that for $(\nu, \vartheta) \in \Xi$ with $(\nu, \vartheta) \in \mathfrak{R}^*$,

$$\begin{cases} \frac{1}{2} d(\nu, \mathcal{T}\nu) \leq d(\nu, \vartheta) \text{ implies} \\ \mathcal{G}(d(\mathcal{T}\nu, \mathcal{T}\vartheta), d(\nu, \vartheta), d(\nu, \mathcal{T}\nu), d(\vartheta, \mathcal{T}\vartheta), d(\nu, \mathcal{T}\vartheta), d(\vartheta, \mathcal{T}\nu)) \leq 0, \end{cases} \tag{3.1}$$

where

$$\mathfrak{R}^* = \{(\nu, \vartheta) \in \mathfrak{R} \mid \mathcal{T}\nu \neq \mathcal{T}\vartheta\}.$$

We denote by $\mathfrak{T}$ the collection of all Suzuki-implicit type mappings on $(\Xi, d, \mathfrak{R})$.

Now, we are equipped to state and prove our first main result as follows:

Theorem 3.2. *Let $(\Xi, d, \mathfrak{R})$ be an RMS and $\mathcal{T}\colon \Xi \to \Xi$. Suppose that the following conditions hold:*

- (C_1) $\mathfrak{X}(\mathcal{T}, \mathfrak{R}) \neq \emptyset$;
- (C_2) $\mathfrak{R}$ *is $\mathcal{T}$-closed and $\mathcal{T}$-transitive;*
- (C_3) Ξ *is $\mathcal{T}$-$\mathfrak{R}$-complete;*
- (C_4) $\mathcal{T} \in \mathfrak{T}$;
- (C_5) $\mathcal{T}$ *is $\mathfrak{R}$-continuous or*
- (C_5') $\mathfrak{R}$ *is d-self-closed with $\mathcal{G} \in \mathfrak{F}$.*

Then there exists a point $\omega \in Fix(\mathcal{T})$.

Proof. Starting with $\nu_0 \in \mathfrak{X}(\mathcal{T}, \mathfrak{R})$ given by (C_1), we construct a sequence $\{\nu_n\}$ of Picard iterates $\nu_{n+1} = \mathcal{T}^n(\nu_0)$ for all $n \in \mathbb{N}^*$. Using (C_2), we have that $(\mathcal{T}\nu_0, \mathcal{T}^2\nu_0) \in \mathfrak{R}$. Continuing this process inductively, we obtain

$$(\mathcal{T}^n\nu_0, \mathcal{T}^{n+1}\nu_0) \in \mathfrak{R} \tag{3.2}$$

for any $n \in \mathbb{N}^*$. Hence, $\{\nu_n\}$ is an $\mathfrak{R}$-preserving sequence.

Now, if there exists some $n_0 \in \mathbb{N}_0$ such that $d(\nu_{n_0},\ \mathcal{T}\nu_{n_0}) = 0$ then the result follows immediately. Otherwise, for all $n \in \mathbb{N}^*$, $d(\nu_n,\ \mathcal{T}\nu_n) > 0$ so that $\mathcal{T}\nu_{n-1} \neq \mathcal{T}\nu_n$ which implies that $(\nu_{n-1},\ \nu_n) \in \mathfrak{R}^*$ and $\dfrac{1}{2} d(\nu_{n-1},\ \nu_n) \leq d(\nu_{n-1},\ \nu_n)$. Since $\mathcal{T}$ is a Suzuki-implicit type contraction, we have

$$\mathcal{G}\left(\begin{array}{c} d(\mathcal{T}\nu_{n-1}, \mathcal{T}\nu_n), d(\nu_{n-1}, \mathcal{T}\nu_{n-1}), d(\nu_{n-1}, \mathcal{T}\nu_{n-1}), \\ d(\mathcal{T}\nu_{n-1}, \mathcal{T}\nu_n), d(\nu_{n-1}, \mathcal{T}\nu_n), d(\mathcal{T}\nu_{n-1}, \mathcal{T}\nu_{n-1}) \end{array} \right) \leq 0,$$

$$\mathcal{G}\left(\begin{array}{c} d(\nu_n, \nu_{n+1}), d(\nu_{n-1}, \nu_n), d(\nu_{n-1}, \nu_n), \\ d(\nu_n, \nu_{n+1}), d(\nu_{n-1}, \nu_{n+1}), 0 \end{array} \right) \leq 0.$$

It follows from $(\mathcal{G}_1)$ that there is $\varphi \in \Phi$, such that

$$d(\nu_n, \nu_{n+1}) \leq \varphi(d(\nu_{n-1}, \nu_n)). \tag{3.3}$$

Following [7], $\{\nu_n\}$ is an $\mathfrak{R}$-preserving Cauchy sequence in Ξ. The $\mathfrak{R}$-completeness of Ξ ensures the existence of $\omega \in \Xi$ with $\lim_{n\to\infty} \nu_n = \omega$.

Firstly, assume that $\mathcal{T}$ is $\mathfrak{R}$-continuous. Then we have

$$\omega = \lim_{n\to\infty} \nu_{n+1} = \lim_{n\to\infty} \mathcal{T}\nu_n = \mathcal{T}(\lim_{n\to\infty} \nu_n) = \mathcal{T}\omega,$$

and hence ω is a fixed point of $\mathcal{T}$.

Alternatively, suppose that $\mathfrak{R}$ is d-self-closed. Then, there exists a subsequence $\{\nu_{n(\ell)}\}$ of $\{\nu_n\}$ with $[\nu_{n(\ell)},\ \omega] \in \mathfrak{R}$ for all $\ell \in \mathbb{N}_0$. Now, we assert that

$$\frac{1}{2}d(\nu_{n(\ell)},\ \mathcal{T}\nu_{n(\ell)}) < d(\nu_{n(\ell)},\ \omega) \text{ or } \frac{1}{2}d(\mathcal{T}\nu_{n(\ell)},\ \mathcal{T}^2\nu_{n(\ell)}) < d(\mathcal{T}\nu_{n(\ell)},\ \omega) \tag{3.4}$$

for all $\ell \in \mathbb{N}_0$. Let, to the contrary, there exists $\varsigma \in \mathbb{N}$ such that

$$\frac{1}{2}d(\nu_{n(\varsigma)},\ \mathcal{T}\nu_{n(\varsigma)}) \geq d(\nu_{n(\varsigma)},\ \omega) \text{ and } \frac{1}{2}d(\mathcal{T}\nu_{n(\varsigma)},\ \mathcal{T}^2\nu_{n(\varsigma)}) \geq d(\mathcal{T}\nu_{n(\varsigma)},\ \omega), \tag{3.5}$$

so that

$$2d(\nu_{n(\varsigma)},\ \omega) \leq d(\nu_{n(\varsigma)},\ \mathcal{T}\nu_{n(\varsigma)}) \leq d(\nu_{n(\varsigma)},\ \omega) + d(\omega,\ \mathcal{T}\nu_{n(\varsigma)}),$$

and

$$d(\nu_{n(\varsigma)},\ \omega) \leq d(\omega,\ \mathcal{T}\nu_{n(\varsigma)}) \leq \frac{1}{2}d(\mathcal{T}\nu_{n(\varsigma)},\ \mathcal{T}^2\nu_{n(\varsigma)}). \tag{3.6}$$

Now, from (3.6), we have

$$\begin{aligned} d(\mathcal{T}\nu_{n(\varsigma)},\ \mathcal{T}^2\nu_{n(\varsigma)}) &< d(\nu_{n(\varsigma)},\ \mathcal{T}\nu_{n(\varsigma)}) \\ &\leq d(\nu_{n(\varsigma)},\ \omega) + d(\omega,\ \mathcal{T}\nu_{n(\varsigma)}) \\ &\leq \frac{1}{2}d(\mathcal{T}\nu_{n(\varsigma)},\ \mathcal{T}^2\nu_{n(\varsigma)}) + \frac{1}{2}d(\mathcal{T}\nu_{n(\varsigma)},\ \mathcal{T}^2\nu_{n(k)}) \\ &= d(\mathcal{T}\nu_{n(\varsigma)},\ \mathcal{T}^2\nu_{n(\varsigma)}), \end{aligned}$$

a contradiction and therefore (3.4) remains true.

Now, we distinguish two cases for $\Lambda = \{\ell \in \mathbb{N} : \mathcal{T}\nu_{n(\ell)} = \mathcal{T}\omega\}$. If Λ is finite, then there exists $\ell_0 \in \mathbb{N}$ such that $\mathcal{T}\nu_{n(\ell)} \neq \mathcal{T}\omega$ for all $\ell > \ell_0$. It follows from (3.4), (for all $\ell > \ell_0$) that either

$$\mathcal{G}\left(\begin{array}{l} d(\mathcal{T}\nu_{n(\ell)}, \mathcal{T}\omega), d(\nu_{n(\ell)}, \omega), d(\nu_{n(\ell)}, \mathcal{T}\nu_{n(\ell)}), \\ d(\omega, \mathcal{T}\omega), d(\nu_{n(\ell)}, \mathcal{T}\omega), d(\omega, \mathcal{T}\nu_{n(\ell)}) \end{array} \right) \leq 0 \tag{3.7}$$

or

$$\mathcal{G}\left(\begin{array}{l} d(\mathcal{T}^2\nu_{n(\ell)}, \mathcal{T}\omega), d(\mathcal{T}\nu_{n(\ell)}, \omega), d(\mathcal{T}\nu_{n(\ell)}, \mathcal{T}^2\nu_{n(\ell)}), \\ d(\omega, \mathcal{T}\omega), d(\mathcal{T}\nu_{n(\ell)}, \mathcal{T}\omega), d(\omega, \mathcal{T}^2\nu_{n(\ell)}) \end{array} \right) \leq 0. \tag{3.8}$$

If (3.7) holds for infinitely many values of $\ell \in \mathbb{N}$, then passing $\ell \to \infty$ and using $\lim_{\ell\to\infty} d(\nu_{n(\ell)}, \omega) = 0$, we get

$$\mathcal{G}(d(\omega, \mathcal{T}\omega), 0, 0, d(\omega, \mathcal{T}\omega), d(\omega, \mathcal{T}\omega), 0) \leq 0. \tag{3.9}$$

Using $(\mathcal{G}_3)$, we obtain $d(\omega, \mathcal{T}\omega) = 0$. Hence ω is a fixed point of $\mathcal{T}$.

If (3.8) holds for infinitely many values of $\ell \in \mathbb{N}$, then passing $\ell \to \infty$ and using $\lim_{\ell\to\infty} d(\nu_{n(k)}, \omega) = 0$, we get (3.9), and hence similar conclusion holds.

Otherwise, if Λ is not finite, then there is a subsequence $\{\nu_{n(\ell(\varsigma))}\}$ of $\{\nu_{n(\ell)}\}$ such that

$$\nu_{n(\ell(\varsigma))+1} = \mathcal{T}\nu_{n(\ell(\varsigma))} = \mathcal{T}\omega,\ \forall \varsigma \in \mathbb{N}.$$

As $\nu_{n(\ell)} \to^d \omega$, therefore $\mathcal{T}\omega = \omega$. □

Theorem 3.3. *In addition to the assumptions of Theorem 3.2, let* $\mathfrak{P}(\nu, \vartheta; \mathfrak{R}|_{\mathcal{T}\Xi}) \neq \emptyset$ *for all* $\nu, \vartheta \in \mathcal{T}(\Xi)$*. Then* $\mathcal{T}$ *has a unique fixed point, provided* $\mathcal{G} \in \mathfrak{F}$*.*

Proof. In view of Theorem 3.2, $Fix(\mathcal{T}) \neq \emptyset$. Suppose that $Fix(\mathcal{T})$ is not a singleton, and let $\omega, \varpi \in Fix(\mathcal{T})$ with $\omega \neq \varpi$. Since $\mathfrak{P}(\nu, \vartheta; \mathfrak{R}|_{\mathcal{T}\Xi}) \neq \emptyset$, for all $u,\ v \in \mathcal{T}(\Xi)$, there exists a path $\{\mathcal{T}z_0,\ \mathcal{T}z_1,\ \cdots,\ \mathcal{T}z_\ell\}$ of some length ℓ in $\mathfrak{R}|_{\mathcal{T}\Xi}$ such that $\mathcal{T}z_0 = \omega,\ \mathcal{T}z_\ell = \varpi$ and $(\mathcal{T}z_j,\ \mathcal{T}z_{j+1}) \in \mathfrak{R}|_{\mathcal{T}\Xi}$ for each $j = 0, 1, 2,\ \cdots,\ \ell - 1$. Since $\mathfrak{R}$ is $\mathcal{T}$-transitive, we have

$$(\omega,\ \mathcal{T}z_1) \in \mathfrak{R},\ (\mathcal{T}z_1,\ \mathcal{T}z_2) \in \mathfrak{R},\ \cdots,\ (\mathcal{T}z_{\ell-1},\ \varpi) \in \mathfrak{R} \Rightarrow (\omega,\ \varpi) \in \mathfrak{R}.$$

Also due to the fact $\frac{1}{2}d(\omega,\ \mathcal{T}\omega) < d(\omega,\ \varpi)$, we have

$$\mathcal{G}(d(\mathcal{T}\omega, \mathcal{T}\varpi), d(\omega, \varpi), d(\omega, \mathcal{T}\omega), d(\varpi, \mathcal{T}\varpi), d(\omega, \mathcal{T}\varpi), d(\varpi, \mathcal{T}\omega)) \leq 0,$$

that is

$$\mathcal{G}(d(\omega, \varpi), d(\omega, \varpi), 0, 0, d(\omega, \varpi), d(\varpi, \omega)) \leq 0,$$

a contradiction due to $(\mathcal{G}_4)$. Thus $\mathcal{T}$ has a unique fixed point. $\square$

Example 3.4. Let $\Xi = [0, \infty)$ be equipped with the usual metric d defined by $d(\nu, \vartheta) = |\nu - \vartheta|$. Define a binary relation $\mathfrak{R}$ on Ξ by $(\nu, \vartheta) \in \mathfrak{R}$ if and only if either $\nu, \vartheta < 1$ or $\nu, \vartheta \geq 1$. Then $(\Xi, d, \mathfrak{R})$ is an RMS. Next, we define a mapping $\mathcal{T} \colon \Xi \to \Xi$ by

$$\mathcal{T}\nu = \begin{cases} \frac{1+\nu}{2}, & \text{if } \nu < 1; \\ 1, & \text{if } \nu \geq 1. \end{cases}$$

Then it is easy to verify that $\mathfrak{X}(\mathcal{T}, \mathfrak{R}) \neq \emptyset$; $\mathfrak{R}$ is $\mathcal{T}$-closed and $\mathcal{T}$-transitive; Ξ is $\mathcal{T}$-$\mathfrak{R}$-complete and $\mathcal{T}$ is $\mathfrak{R}$-continuous.

Now we take $\mathcal{G} \in \mathfrak{G}$ defined by

$$\mathcal{G}(\hbar_1, \hbar_2, \hbar_3, \hbar_4, \hbar_5, \hbar_6) = \hbar_1 - a \max\{\hbar_2, \hbar_3, \hbar_4\} + b\,(\hbar_5 + \hbar_6)$$

$0 \leq a < 1$ and $b > 0$.

Let $\nu, \vartheta \in \Xi$ be such that $(\nu, \vartheta) \in \mathfrak{R}^*$ and $\frac{1}{2}d(\nu, \mathcal{T}\nu) \leq d(\nu, \vartheta)$. Take $a = 8/9$ and $b = 1/9$. Then the following cases may arise:

Case I: Let $\nu, \vartheta < 1$ and $\nu \geq \vartheta$. Then

$$\begin{aligned} d(\mathcal{T}\nu, \mathcal{T}\vartheta) &= \frac{1}{2}d(\nu, \vartheta) \\ &\leq \frac{8}{9}|\nu - \vartheta| - \frac{1}{9}\left[|\nu - \frac{1+\vartheta}{2}| + |\vartheta - \frac{1+\nu}{2}|\right]. \end{aligned}$$

Therefore,

$$d(\mathcal{T}\nu, \mathcal{T}\vartheta) \leq \frac{8}{9}\max\{d(\nu, \vartheta), d(\nu, \mathcal{T}\nu), d(\vartheta, \mathcal{T}\vartheta)\} - \frac{1}{9}\left(d(\nu, \mathcal{T}\vartheta) + d(\vartheta, \mathcal{T}\nu)\right)$$

implies

$$\mathcal{G}(d(\mathcal{T}\nu, \mathcal{T}\vartheta), d(\nu, \vartheta), d(\nu, \mathcal{T}\nu), d(\vartheta, \mathcal{T}\vartheta), d(\nu, \mathcal{T}\vartheta), d(\vartheta, \mathcal{T}\nu)) \leq 0.$$

Case II: Let $\nu, \vartheta \geq 1$. In this case, it is obvious that

$$\mathcal{G}(d(\mathcal{T}\nu, \mathcal{T}\vartheta), d(\nu, \vartheta), d(\nu, \mathcal{T}\nu), d(\vartheta, \mathcal{T}\vartheta), d(\nu, \mathcal{T}\vartheta), d(\vartheta, \mathcal{T}\nu)) \leq 0.$$

Thus $\mathcal{T} \in \mathfrak{T}$. Therefore by Theorem 3.2, it follows that $Fix(\mathcal{T}) \neq \emptyset$. Moreover, $\mathfrak{R}|_{\mathcal{T}\Xi}$ is transitive and for all $\nu,\ \vartheta \in \mathcal{T}(\Xi)$, we have $(\nu,\ \vartheta) \in \mathfrak{R}$, so $\mathfrak{P}(\nu, \vartheta, \mathfrak{R}|_{\mathcal{T}(\Xi)})$

is nonempty for all $\nu,\ \vartheta \in \mathcal{T}(\Xi)$. Thus, by Theorem 3.2, it follows that $Fix(\mathcal{T})$ is a singleton; namely, $1 \in Fix(\mathcal{T})$.

Example 3.5. Let $\Xi = [0, +\infty)$ be endowed with the usual metric d. Define a binary relation $\mathfrak{R}$ on Ξ by $(\nu, \vartheta) \in \mathfrak{R}$ if and only if $\nu, \vartheta \leq 1$. Then $(\Xi, d, \mathfrak{R})$ is an RMS. Define a mapping $\mathcal{T} : \Xi \to \Xi$ by

$$\mathcal{T}\nu = \begin{cases} \frac{\nu^2}{2}, & \nu \in [0, 1]; \\ 2\nu, & \nu \in (1, +\infty). \end{cases}$$

Then it is easy to verify that $\mathfrak{X}(\mathcal{T}, \mathfrak{R}) \neq \emptyset$; $\mathfrak{R}$ is $\mathcal{T}$-closed, $\mathcal{T}$-transitive; Ξ is $\mathcal{T}$-$\mathfrak{R}$-complete and $\mathcal{T}$ is $\mathfrak{R}$-continuous.

Similar to previous example, we take $\mathcal{G} \in \mathfrak{G}$ as

$$\mathcal{G}(\hbar_1, \hbar_2, \hbar_3, \hbar_4, \hbar_5, \hbar_6) = \hbar_1 - \frac{9}{10} \max\{\hbar_2, \hbar_3, \hbar_4\} + \frac{1}{5}[\hbar_5 + \hbar_6].$$

Let $\nu, \vartheta \in \Xi$ be such that $(\nu, \vartheta) \in \mathfrak{R}^*$ and $\frac{1}{2}d(\nu, \mathcal{T}\nu) \leq d(\nu, \vartheta)$. Then

$$\left|\frac{\nu^2}{2} - \frac{\vartheta^2}{2}\right| \leq \frac{9}{10} \max\left\{|\nu - \vartheta|, \left|\nu - \frac{\nu^2}{2}\right|, \left|\vartheta - \frac{\vartheta^2}{2}\right|\right\} - \frac{1}{5}\left[\left|\nu - \frac{\vartheta^2}{2}\right| + \left|\vartheta - \frac{\nu^2}{2}\right|\right]$$

which is true for all $\nu, \vartheta \in [0, 1]$ with $(\nu, \vartheta) \in \mathfrak{R}^*$. At the same time, for $\nu = 0, \vartheta = 2$ with $(\nu, \vartheta) \notin \mathfrak{R}$, we have $\frac{1}{2}d(\nu, \mathcal{T}\nu) = 0 < 2 = d(\nu, \vartheta)$ and

$$d(\mathcal{T}\nu, \mathcal{T}\vartheta) = 4 \nleq 4a - 6b = a\ \max\{d(\nu, \vartheta), d(\nu, \mathcal{T}\nu), d(\vartheta, \mathcal{T}\vartheta)\} - b[d(\nu, \mathcal{T}\vartheta) + d(\vartheta, \mathcal{T}\nu)]$$

for any $a \in [0, 1)$ and $b > 0$. Also

$$d(\mathcal{T}\nu, \mathcal{T}\vartheta) = 4 \nleq 4k = k\ \max\{d(\nu, \vartheta), d(\nu, \mathcal{T}\nu), d(\vartheta, \mathcal{T}\vartheta), d(\nu, \mathcal{T}\vartheta), d(\vartheta, \mathcal{T}\nu)\}$$

for any $k \in (0, 1)$.

It is clear from the above that $\mathcal{T} \in \mathfrak{T}$ only for $(\nu, \vartheta) \in \mathfrak{R}^*$ and not in the whole domain where $(\nu, \vartheta) \notin \mathfrak{R}$. Hence, Suzuki-implicit type contraction is a proper generalization of generalized Suzuki contraction [17] and of quasi contraction [8]. By Theorem 3.2, it follows that $Fix(\mathcal{T}) \neq \emptyset$. Moreover, $\mathfrak{R}|_{\mathcal{T}\Xi}$ is transitive and for all $\nu,\ \vartheta \in \mathcal{T}(\Xi)$, we have $(\nu,\ \vartheta) \in \mathfrak{R}$, so $\mathfrak{P}(\nu, \vartheta, \mathfrak{R}|_{\mathcal{T}(\Xi)})$ is nonempty for all $\nu,\ \vartheta \in \mathcal{T}(\Xi)$. Thus, by Theorem 3.2, it follows that $Fix(\mathcal{T})$ is a singleton. Indeed, we see that $0 \in Fix(\mathcal{T})$.

A modified version of example given in [2] is the following.

Example 3.6. Let $\Xi\ = [0, 6)$ be equipped with usual metric d. Consider the binary relation on Ξ as follows:

$$\mathfrak{R} = \{(0, 1),\ (2, 1),\ (2, 2),\ (2, 4),\ (3, 1),\ (3, 2),\ (3, 3),\ (3, 4),\ (4, 1),\ (4, 2),\ (4, 4)\}.$$

Define a mapping $\mathcal{T} : \Xi \to \Xi$ by

$$\mathcal{T}\nu = \begin{cases} 1, & 0 \leq \nu < 1; \\ 3, & \nu = 1; \\ 4, & 1 < \nu < 6. \end{cases}$$

Then $\mathcal{T}$ is not continuous while $\mathcal{T}$ is $\mathfrak{R}$-continuous, $\mathcal{R}$ is $\mathcal{T}$-closed, and $\mathcal{T}$-transitive; Ξ is $\mathcal{T}$-$\mathfrak{R}$-complete. Also $\mathfrak{R}^* = \{(0, 1),\ (4, 1)\}$ and $\mathfrak{X}(\mathcal{T}; \mathfrak{R}) \neq \emptyset$ as $(4,\ \mathcal{T}4) = (4, 4) \in \mathfrak{R}$.

Following Example 3.4, we take $\mathcal{G} \in \mathfrak{G}$ as

$$\mathcal{G}(\hbar_1, \hbar_2, \hbar_3, \hbar_4, \hbar_5, \hbar_6) = \hbar_1 - \frac{4}{5}\max\{\hbar_2, \hbar_3, \hbar_4\} - \frac{1}{12}[\hbar_5 + \hbar_6].$$

Consider $(\nu, \vartheta) = (4, 1) \in \mathfrak{R}^*$ with $\frac{1}{2}d(\nu,\ \mathcal{T}\nu) = 0 < 3 = d(\nu, \vartheta)$ and

$$d(\mathcal{T}\nu, \mathcal{T}\vartheta) = 1 \leq \frac{4}{5}\ \max\{d(\nu, \vartheta), d(\nu, \mathcal{T}\nu), d(\vartheta, \mathcal{T}\vartheta)] - \frac{1}{12}\,[d(\nu, \mathcal{T}\vartheta) + d(\vartheta, \mathcal{T}\nu)]$$

implies that $\mathcal{T} \in \mathfrak{T}$. Thus all the conditions of Theorem 3.2 are satisfied, hence $\mathcal{T}$ has a fixed point. Moreover, $\mathfrak{R}|_{\mathcal{T}\Xi}$ is transitive while $\mathfrak{R}$ is not and for all $\nu,\ \vartheta \in \mathcal{T}(\Xi)$, we have $(\nu,\ \vartheta) \in \mathfrak{R}$, so $\mathfrak{P}(\nu, \vartheta, \mathfrak{R}|_{\mathcal{T}(\Xi)})$ is nonempty for all $\nu,\ \vartheta \in \mathcal{T}(\Xi)$. Following Theorem 3.3, $\mathcal{T}$ has a unique fixed point which is $\omega = 4$.

Now for $(0, 1) \in \mathfrak{R}$,

$$d(\mathcal{T}\nu, \mathcal{T}\vartheta) = 2 \nleq 2a - 3b = a\ \max\{d(\nu, \vartheta), d(\nu, \mathcal{T}\nu), d(\vartheta, \mathcal{T}\vartheta)\} - b[d(\nu, \mathcal{T}\vartheta) + d(\vartheta, \mathcal{T}\nu)]$$

for any $a \in [0, 1)$ and $b > 0$. Thus $\mathcal{T}$ is not implicit type mapping on $(\Xi, d, \mathfrak{R})$. Also

$$d(\mathcal{T}\nu, \mathcal{T}\vartheta) = 2 \nleq 2k = k\ \max\left\{d(\nu, \vartheta), d(\nu, \mathcal{T}\nu), d(\vartheta, \mathcal{T}\vartheta), \frac{1}{2}[d(\nu, \mathcal{T}\vartheta) + d(\vartheta, \mathcal{T}\nu)]\right\}$$

which is not true for any $k \in (0, 1)$. Hence Theorem 1 and Theorem 2 [1] cannot be applied to the present example.

Also, as $1, 0 \in \Xi$, $(1, 0) \notin \mathfrak{R}$ with $\mathcal{T}1 = 3 \neq 1 = \mathcal{T}0$ such that $\frac{1}{2}d(1,\ \mathcal{T}1) = d(1, 0)$ but $d(\mathcal{T}1,\ \mathcal{T}0) \nleq k\ d(1, 0))$ and

$$d(\mathcal{T}\nu, \mathcal{T}\vartheta) = 2 \nleq 2k = k\ \max\left\{d(\nu, \vartheta), d(\nu, \mathcal{T}\nu), d(\vartheta, \mathcal{T}\vartheta), \frac{1}{2}[d(\nu, \mathcal{T}\vartheta) + d(\vartheta, \mathcal{T}\nu)]\right\},$$

which shows that $\mathcal{T}$ is neither Suzuki-contraction nor generalized Suzuki-contraction for any $k \in [0, 1)$. Hence results of Suzuki [21] and Popescu[17] cannot be applied to the present example, while our Theorems 3.2- 3.3 are applicable. This shows that our results are genuine improvements over the corresponding results contained in Suzuki [21], Popescu[17] and Ahmadullah et al. [1, Theorem 1-Theorem 2].

If we take $\mathfrak{R} = \{(\nu,\ \vartheta) \in \Xi \times \Xi \mid \nu \preceq \vartheta\}$, then we have more new results as consequences of Theorem 3.2.

Corollary 3.7. *Let $(\Xi, d, \preceq)$ be an ordered complete metric space. Let $\mathcal{T} \colon \Xi \to \Xi$ be increasing and Suzuki-implicit type mapping on $\Xi_{\preceq}$ w.r.t. some $\mathcal{G} \in \mathfrak{G}$. Suppose there exists $\nu_0 \in \Xi$ such that $\nu_0 \preceq \mathcal{T}\nu_0$. If $\mathcal{T}$ is $\Xi_{\preceq}$-continuous or $\Xi_{\preceq}$ is d-self-closed satisfying $(\mathcal{G}_3) \in \mathfrak{F}$, then $\omega \in Fix(\mathcal{T})$. Moreover, for each $\nu_0 \in \Xi$ with $\nu_0 \preceq \mathcal{T}\nu_0$, the Picard sequence $\mathcal{T}^n(\nu_0)$ for all $n \in \mathbb{N}$ converges to a $\omega \in Fix(\mathcal{T})$.*

Corollary 3.8. *Let $(\Xi, d, \mathfrak{R})$ be an RMS and $\mathcal{T} \colon \Xi \to \Xi$. Suppose that the following conditions hold:*

(I) $\mathfrak{X}(\mathcal{T}, \mathfrak{R}) \neq \emptyset$*;*
(II) $\mathfrak{R}$ *is* $\mathcal{T}$*-closed and* $\mathcal{T}$*-transitive;*
(III) Ξ *is* $\mathcal{T}$*-*$\mathfrak{R}$*-complete;*

(IV) *$\mathcal{T}$ is implicit type contraction, that is, there exists $\mathcal{G} \in \mathfrak{G}$ such that for $(\nu,\vartheta) \in \Xi$ with $(\nu,\vartheta) \in \mathfrak{R}$*

$$\mathcal{G}(d(\mathcal{T}\nu,\mathcal{T}\vartheta), d(\nu,\vartheta), d(\nu,\mathcal{T}\nu), d(\vartheta,\mathcal{T}\vartheta), d(\nu,\mathcal{T}\vartheta), d(\vartheta,\mathcal{T}\nu)) \leq 0; \tag{3.10}$$

(V) *$\mathcal{T}$ is $\mathfrak{R}$-continuous or*

(V′) *$\mathfrak{R}$ is d-self-closed with $\mathcal{G} \in \mathfrak{F}$.*

Then there exists a point $\omega \in Fix(\mathcal{T})$.

4. Application to nonlinear matrix equations

Let $\mathcal{H}(n)$ stand for the set of all $n \times n$ Hermitian matrices over $\mathbb{C}$, $\mathcal{K}(n)$ $\big(\subset \mathcal{H}(n)\big)$ stand for the set of all $n \times n$ positive semi-definite matrices, $\mathcal{P}(n)$ $\big(\subset \mathcal{K}(n)\big)$ stand for the set of $n \times n$ positive definite matrices, $\mathcal{M}(n)$ stand for the set of all $n \times n$ matrices over $\mathbb{C}$.

For a matrix $\mathcal{B} \in \mathcal{H}(n)$, we will denote by $s(\mathcal{B})$ any of its singular values and by $tr(\mathcal{B})$ the sum of all of its singular values, that is, its the trace norm $tr(\mathcal{B}) = \|\mathcal{B}\|_{tr}$. For $\mathcal{C}, \mathcal{D} \in \mathcal{H}(n)$, $\mathcal{C} \succeq \mathcal{D}$ (resp. $\mathcal{C} \succ \mathcal{D}$) will mean that the matrix $\mathcal{C} - \mathcal{D}$ is positive semi-definite (resp. positive definite).

The following lemmas are needed in the subsequent discussion.

Lemma 4.1. [18] *If $\mathcal{A} \succeq O$ and $\mathcal{B} \succeq O$ are $n \times n$ matrices, then*

$$0 \leq tr(\mathcal{AB}) \leq \|\mathcal{A}\| tr(\mathcal{B}).$$

Lemma 4.2. [18] *If $\mathcal{A} \in \mathcal{H}(n)$ such that $\mathcal{A} \prec I_n$, then $\|\mathcal{A}\| < 1$.*

We establish the existence and uniqueness of the solution of the nonlinear matrix equation

$$\mathcal{X} = \mathcal{Q} + \sum_{i=1}^{m} \sum_{j=1}^{t} \mathcal{A}_i^* \mathcal{F}_j(\mathcal{X}) \mathcal{A}_i, \tag{4.1}$$

where $\mathcal{Q}$ is a Hermitian positive definite matrix, $\mathcal{A}_i^*$ stands for the conjugate transpose of an $n \times n$ matrix $\mathcal{A}_i$ and $\mathcal{F}_j$ are order-preserving continuous mappings from the set of all Hermitian matrices to the set of all positive definite matrices such that $\mathcal{F}(O) = O$.

Theorem 4.3. *Consider the problem described by (4.1). Assume that there exists a positive real number η such that*

(H_1) *there exists $\mathcal{Q} \in \mathcal{P}(n)$ such that $\sum_{i=1}^{m} \sum_{j=1}^{t} \mathcal{A}_i^* \mathcal{F}_j(\mathcal{Q}) \mathcal{A}_i \succ 0$;*

(H_2) *$\sum_{i=1}^{m} \mathcal{A}_i \mathcal{A}_i^* \prec \eta I_n$;*

(H_3) *for every $\mathcal{X}$, $\mathcal{Y} \in \mathcal{P}(n)$ such that $\mathcal{X} \preceq \mathcal{Y}$ with*

$$\sum_{i=1}^{m} \sum_{j=1}^{t} \mathcal{A}_i^* \mathcal{F}_j(\mathcal{X}) \mathcal{A}_i \neq \sum_{i=1}^{m} \sum_{j=1}^{t} \mathcal{A}_i^* \mathcal{F}_j(\mathcal{Y}) \mathcal{A}_i$$

and if

$$tr\Bigg(\mathcal{X}-\mathcal{Q}-\sum_{i=1}^{m}\sum_{j=1}^{t}\mathcal{A}_i^*\mathcal{F}_j(\mathcal{Y})\mathcal{A}_i\Bigg) \; < 2|tr(\mathcal{X}-\mathcal{Y})|,$$

holds, then for $0<a,b<1$ *and* $a+2b<1,$ *we have*

$$\begin{aligned}&\max_j(tr(\mathcal{F}_j(\mathcal{Y})-\mathcal{F}_j(\mathcal{X})))\\ &\leq \frac{1}{\eta t}\left[\begin{array}{c} a.(tr(\mathcal{Y}-\mathcal{X}))^2+ \\ b\frac{\left(tr\left(\mathcal{X}-\mathcal{Q}-\sum_{i=1}^{m}\sum_{j=1}^{t}\mathcal{A}_i^*\mathcal{F}_j(\mathcal{X})\mathcal{A}_i\right)\right)^2+\left(tr\left(\mathcal{Y}-\mathcal{Q}-\sum_{i=1}^{m}\sum_{j=1}^{t}\mathcal{A}_i^*\mathcal{F}_j(\mathcal{Y})\mathcal{A}_i\right)\right)^2}{tr\left(\mathcal{X}-\mathcal{Q}-\sum_{i=1}^{m}\sum_{j=1}^{t}\mathcal{A}_i^*\mathcal{F}_j(\mathcal{Y})\mathcal{A}_i\right)+tr\left(\mathcal{Y}-\mathcal{Q}-\sum_{i=1}^{m}\sum_{j=1}^{t}\mathcal{A}_i^*\mathcal{F}_j(\mathcal{X})\mathcal{A}_i\right)+1}\end{array}\right]^{1/2}.\end{aligned}$$

Then the matrix equation (4.1) *has a unique solution. Moreover, the iteration*

$$\mathcal{X}_n=\mathcal{Q}+\sum_{i=1}^{m}\sum_{j=1}^{t}\mathcal{A}_i^*\mathcal{F}_j(\mathcal{X}_{n-1})\mathcal{A}_i \tag{4.2}$$

where $\mathcal{X}_0\in\mathcal{P}(n)$ *satisfies*

$$\mathcal{X}_0\preceq\mathcal{Q}+\sum_{i=1}^{m}\sum_{j=1}^{t}\mathcal{A}_i^*\mathcal{F}_j(\mathcal{X}_0)\mathcal{A}_i,$$

converges in the sense of trace norm $\|.\|_{tr}$ *to the solution of the matrix equation* (4.1).

Proof. Define a mapping $\mathcal{T}:\mathcal{P}(n)\to\mathcal{P}(n)$ by

$$\mathcal{T}(\mathcal{X})=\mathcal{Q}+\sum_{i=1}^{m}\sum_{j=1}^{t}\mathcal{A}_i^*\mathcal{F}_j(\mathcal{X})\mathcal{A}_i,\ \text{for all } \mathcal{X}\in\mathcal{P}(n),$$

and a binary relation

$$\mathfrak{R}=\{(\mathcal{X},\ \mathcal{Y})\in\mathcal{P}(n)\times\mathcal{P}(n):\mathcal{X}\preceq\mathcal{Y}\}.$$

Then fixed point of the mapping $\mathcal{T}$ is a solution of the matrix equation (4.1). Notice that $\mathcal{T}$ is well defined, $\mathfrak{R}$-continuous and $\mathfrak{R}$ is $\mathcal{T}$-closed. Since

$$\sum_{i=1}^{m}\sum_{j=1}^{t}\mathcal{A}_i^*\mathcal{F}_j(\mathcal{Q})\mathcal{A}_i\succ 0,$$

for some $\mathcal{Q}\in\mathcal{P}(n)$, we have $(\mathcal{Q},\mathcal{T}(\mathcal{Q}))\in\mathfrak{R}$ and hence $\mathcal{P}(n)(\mathcal{T};\mathfrak{R})\neq\emptyset$.
Now, let $(\mathcal{X},\ \mathcal{Y})\in\mathfrak{R}^*=\{(\mathcal{X},\ \mathcal{Y})\in\mathfrak{R}:\mathcal{T}(\mathcal{X})\neq\mathcal{T}(\mathcal{Y})\}$ such that

$$\frac{1}{2}\|\mathcal{X}-\mathcal{T}(\mathcal{Y})\|_{tr}<\|\mathcal{X}-\mathcal{Y}\|_{tr}.$$

Then

$$\begin{aligned}
&\|\mathcal{T}(\mathcal{X})-\mathcal{T}(\mathcal{Y})\|_{tr}\\
&= tr(\mathcal{T}(\mathcal{X})-\mathcal{T}(\mathcal{Y}))\\
&= tr(\sum_{i=1}^{m}\sum_{j=1}^{t}\mathcal{A}_i^*(\mathcal{F}_j(\mathcal{X})-\mathcal{F}_j(\mathcal{Y}))\mathcal{A}_i)\\
&= \sum_{i=1}^{m}\sum_{j=1}^{t} tr(\mathcal{A}_i^*(\mathcal{F}_j(\mathcal{X})-\mathcal{F}_j(\mathcal{Y}))\mathcal{A}_i)\\
&= \sum_{i=1}^{m}\sum_{j=1}^{t} tr(\mathcal{A}_i\mathcal{A}_i^*(\mathcal{F}_j(\mathcal{X})-\mathcal{F}_j(\mathcal{Y})))\\
&= tr((\sum_{i=1}^{m}\mathcal{A}_i\mathcal{A}_i^*)\sum_{j=1}^{t}(\mathcal{F}_j(\mathcal{X})-\mathcal{F}_j(\mathcal{Y})))\\
&\le \|\sum_{i=1}^{m}\mathcal{A}_i\mathcal{A}_i^*\|\times t\times \max\|(\mathcal{F}_j(\mathcal{X})-\mathcal{F}_j(\mathcal{Y}))\|_{tr}\\
&\le t\times\frac{\|\sum_{i=1}^{m}\mathcal{A}_i\mathcal{A}_i^*\|}{\eta t}\times\left[\ a\|\mathcal{X}-\mathcal{Y}\|_{tr}^2+b\tfrac{\|\mathcal{X}-\mathcal{T}\mathcal{X}\|_{tr}^2+\|\mathcal{Y}-\mathcal{T}\mathcal{Y}\|_{tr}^2}{\|\mathcal{X}-\mathcal{T}\mathcal{Y}\|_{tr}+\|\mathcal{Y}-\mathcal{T}\mathcal{X}\|_{tr}+1}\ \right]^{1/2}\\
&\le \left[\ a\|\mathcal{X}-\mathcal{Y}\|_{tr}^2+b\tfrac{\|\mathcal{X}-\mathcal{T}\mathcal{X}\|_{tr}^2+\|\mathcal{Y}-\mathcal{T}\mathcal{Y}\|_{tr}^2}{\|\mathcal{X}-\mathcal{T}\mathcal{Y}\|_{tr}+\|\mathcal{Y}-\mathcal{T}\mathcal{X}\|_{tr}+1}\ \right]^{1/2},
\end{aligned}$$

that is,

$$\|\mathcal{T}(\mathcal{X})-\mathcal{T}(\mathcal{Y})\|_{tr}^2\le\ \ a\|\mathcal{X}-\mathcal{Y}\|_{tr}^2+b\tfrac{\|\mathcal{X}-\mathcal{T}\mathcal{X}\|_{tr}^2+\|\mathcal{Y}-\mathcal{T}\mathcal{Y}\|_{tr}^2}{\|\mathcal{X}-\mathcal{T}\mathcal{Y}\|_{tr}+\|\mathcal{Y}-\mathcal{T}\mathcal{X}\|_{tr}+1}.$$

Consider $\mathcal{G}\in\mathfrak{G}$ given by

$$\mathcal{G}(\hbar_1,\hbar_2,\hbar_3,\hbar_4,\hbar_5,\hbar_6)=\hbar_1^2-a\hbar_2^2-b\frac{\hbar_3^2+\hbar_4^2}{\hbar_5+\hbar_6+1},$$

where $0<a,b<1$ and $a+2b<1$. Thus all the hypotheses of Theorem 3.2 are satisfied, therefore there exists $\hat{\mathcal{X}}\in\mathcal{P}(n)$ such that $\mathcal{T}(\hat{\mathcal{X}})=\hat{\mathcal{X}}$, and hence the matrix equation (4.1) has a solution in $\mathcal{P}(n)$. Furthermore, due to the existence of least upper bound and greatest lower bound for each $\mathcal{X},\mathcal{Y}\in\mathcal{T}(\mathcal{P}(n))$, we have $\mathfrak{P}(\mathcal{X},\mathcal{Y};\mathfrak{R}|_{\mathcal{T}(\mathcal{P}(n))})\neq\emptyset$ for all $\mathcal{X},\mathcal{Y}\in\mathcal{T}(\mathcal{P}(n))$. Hence, on using Theorem 3.3, $\mathcal{T}$ has a unique fixed point, and hence we conclude that the matrix equation (4.1) has a unique solution in $\mathcal{P}(n)$. □

Example 4.4. Consider the NME (4.1) for $m=3$, $t=3$, $n=3$, with $\mathcal{F}_1(\mathcal{X})=\mathcal{X}^{1/3}$, $\mathcal{F}_2(\mathcal{X})=\mathcal{X}^{1/5}$, $\mathcal{F}_3(\mathcal{X})=\mathcal{X}^{1/7}$, i.e.,

$$\mathcal{X}=\mathcal{Q}+\mathcal{A}_1^*\mathcal{X}^{1/3}\mathcal{A}_1+\mathcal{A}_2^*\mathcal{X}^{1/5}\mathcal{A}_2+\mathcal{A}_3^*\mathcal{X}^{1/7}\mathcal{A}_3, \tag{4.3}$$

where

$$\mathcal{Q}=\begin{bmatrix}11.115067021600000 & 0.600077140000000 & 0.988864768800000\\ 0.600077140000000 & 10.546782276400000 & 0.819752189620000\\ 0.988864768800000 & 0.819752189620000 & 11.364895128121001\end{bmatrix},$$

$$\mathcal{A}_1 = \begin{bmatrix} 0.072850000000000 & 0.097960000000000 & 0.193440000000000 \\ 0.078740000000000 & 0.047740000000000 & 0.197780000000000 \\ 0.202740000000000 & 0.079980000000000 & 0.037820000000000 \end{bmatrix},$$

$$\mathcal{A}_2 = \begin{bmatrix} 0.022400000000000 & 0.029000000000000 & 0.033000000000000 \\ 0.047000000000000 & 0.031400000000000 & 0.036800000000000 \\ 0.049000000000000 & 0.047800000000000 & 0.031800000000000 \end{bmatrix},$$

$$\mathcal{A}_3 = \begin{bmatrix} 0.550000000000000 & 0.860000000000000 & 0.270000000000000 \\ 0.460000000000000 & 0.240000000000000 & 0.520000000000000 \\ 0.960000000000000 & 0.360000000000000 & 0.560000000000000 \end{bmatrix}.$$

The conditions of Theorem 4.3 can be checked numerically, taking various special values for matrices involved. For example, they can be tested (and verified to be true) for

$$\mathcal{X} = \begin{bmatrix} 1.115000000000000 & 0.599800000000000 & 0.988800000000000 \\ 0.599800000000000 & 0.539600000000000 & 0.819200000000000 \\ 0.988800000000000 & 0.819200000000000 & 1.364800000000000 \end{bmatrix},$$

$$\mathcal{Y} = \begin{bmatrix} 10.000067021600000 & 0.000277140000000 & 0.000064768800000 \\ 0.000277140000000 & 10.007182276400000 & 0.000552189620000 \\ 0.000064768800000 & 0.000552189620000 & 10.000095128121000 \end{bmatrix}.$$

We take $\eta = 0.89$, $a = 0.9$, $b = 0.04$. To see the convergence of the sequence $\{\mathcal{X}_n\}$ defined in (4.2), we start with three different initial values

$$\mathcal{U}_0 = \begin{bmatrix} 0.003173000000000 & 0.007557000000000 & 0.002530000000000 \\ 0.007557000000000 & 0.019038000000000 & 0.006370000000000 \\ 0.002530000000000 & 0.006370000000000 & 0.002308000000000 \end{bmatrix}$$

with $\|\mathcal{U}_0\| = 0.024206671245210$, $\mathcal{V}_0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ with $\|\mathcal{V}_0\| = 1$, $\mathcal{W}_0 = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$ with $\|\mathcal{W}_0\| = 5$.

After 10 iterations, we have the following approximation of the unique positive definite solution of the system (4.1) as

$$\widehat{\mathcal{U}} \approx \mathcal{U}_{10} = \begin{bmatrix} 13.390482772806859 & 2.078090573177774 & 2.475074932112452 \\ 2.078090573177774 & 11.977120513625879 & 1.755499292483718 \\ 2.475074932112452 & 1.755499292483718 & 12.539088509873810 \end{bmatrix}$$

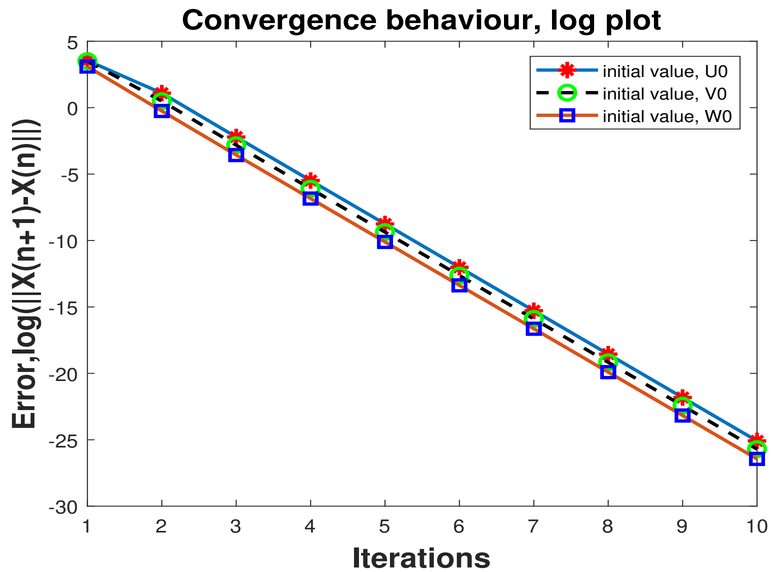
with Error $= 1.244 \times 10^{-12}$.

$$\widehat{\mathcal{V}} \approx \mathcal{V}_{10} = \begin{bmatrix} 13.390482772806962 & 2.078090573177856 & 2.475074932112529 \\ 2.078090573177857 & 11.977120513625945 & 1.755499292483779 \\ 2.475074932112529 & 1.755499292483779 & 12.539088509873872 \end{bmatrix}$$

with Error $= 6.887 \times 10^{-13}$.

$$\widehat{\mathcal{W}} \approx \mathcal{W}_{10} = \begin{bmatrix} 13.390482772807026 & 2.078090573177907 & 2.475074932112577 \\ 2.078090573177907 & 11.977120513625984 & 1.755499292483816 \\ 2.475074932112578 & 1.755499292483816 & 12.539088509873912 \end{bmatrix}$$

with Error $= 3.286 \times 10^{-13}$. Also, the elements of each sequence are order preserving. The graphical representation of convergence is shown below:



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References

[1] M. Ahmadullah, J. Ali, M. Imdad, *Unified relation-theoretic metrical fixed point theorems under an implicit contractive condition with an application*, Fixed Point Theory Appl., **42**(2016).

[2] M. Ahmadullah, M. Imdad, *Unified relation-theoretic fixed point results via F_R-Suzuki-contractions with an application*, Fixed Point Theory, **21**(2020), no. 1, 19-34.

[3] M. Ahmadullah, M. Imdad, M. Arif, *Relation-theoretic metrical coincidence and common fixed point theorems under nonlinear contractions*, Appl. Gen. Topol., **19**(2018), no. 1, 65-84.

[4] M. Ahmadullah, M. Imdad, M. Arif, *Common fixed point theorems under an implicit contractive condition on metric spaces endowed with an arbitrary binary relation and an application*, Asian Europian J. Math., **14**(2021), no. 1, 2050146.

[5] A. Alam, M. Imdad, *Relation-theoretic contraction principle*, J. Fixed Point Theory Appl., **17**(2015), no. 4, 693-702.

[6] I. Altun, A. Erduran, *A Suzuki type fixed-point theorem*, Inter. J. Math. Math. Sci., **2011**, Art. ID 736063, 9 pages.

[7] F. Browder, *On the convergence of successive approximations for nonlinear functional equations*, Indag. Math., **30**(1968), 27-35.

[8] Lj. B. Ćirić, *A generalization of Banach's contraction principle spaces*, Proc. Am. Math. Soc., **45**(1974), no. 2, 267-273.

[9] B. Kolman, R.C. Busby, S. Ross, *Discrete Mathematical Structures*, 3rd ed., PHI Pvt. Ltd., New Delhi, 2000.

[10] S. Lipschutz, *Schaum's Outlines of Theory and Problems of Set Theory and Related Topics*, McGraw-Hill, New York, 1964.

[11] R.D. Maddux, *Relation Algebras. Studies in Logic and Foundations of Mathematics*, 150, Elsevier B.V., Amsterdam, 2006.

[12] J. Matkowski, *Integrable solutions of functional equations*, Dissertationes Math., **127**(1975), 1-68.

[13] J. Matkowski, *Fixed point theorems for mappings with a contractive iterate at a point*, Proc. Amer. Math. Soc., **62**(1977), 344-348.

[14] J.J. Nieto, R.R. López, *Contractive mapping theorems in partially ordered sets and applications to ordinary differential equations*, Order, **22**(2005), 223-239.

[15] J.J. Nieto, R.R. López, *Fixed point theorems in ordered abstract spaces*, Proc. Amer. Math. Soc., **135**(2007), 2505-2517.

[16] R. Pant, *Fixed point theorems for generalized semi-quasi contractions*, J. Fixed Point Theory Appl., **19**(2017), no. 2, 1581-1590.

[17] O. Popescu, *Two fixed point theorems for generalized contractions with constants in complete metric space*, Cent. Eur. J. Math., **7**(2009), no. 3, 529-538.

[18] A.C.M. Ran, M.C.B. Reurings, *On the matrix equation $X + A^*F(X)A = Q$: Solutions and perturbation theory*, Linear Algebr. Appl., **346**(2002), 15-26.

[19] A.C.M. Ran, M.C.B. Reurings, *A fixed point theorem in partially ordered sets and some applications to matrix equations*, Proc. Amer. Math. Soc., **132**(2004), 1435-1443.

[20] B. Samet, M. Turinici, *Fixed point theorems on a metric space endowed with an arbitrary binary relation and applications*, Commun. Math. Anal., **13**(2012), 82-97.

[21] T. Suzuki, *A generalized Banach contraction principle which characterizes metric completness*, Proc. Amer. Math. Soc., **136**(2008), 1861-1869.

[22] M. Turinici, *Abstract comparison principles and multivariable Gronwall-Bellman inequalities*, J. Math. Anal. Appl., **117**(1986), 100-127.

[23] M. Turinici, *Fixed points for monotone iteratively local contractions*, Demonstratio Math., **19**(1986), 171-180.

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