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ON THE ATTRACTIVITY OF THE SOLUTIONS OF A PROBLEM INVOLVING HILFER FRACTIONAL DERIVATIVE VIA THE MEASURE OF NONCOMPACTNESS

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Abstract. In this paper, we present some results concerning the existence and attractivity of solutions for generalized Cauchy problem involving Hilfer fractional derivative using the well-known Krasnoselskii fixed point theorem and the measure of noncompactness. Finally, some examples are provided to illustrate the efficiency of the results.
Key Words and Phrases: Fractional differential equations, generalized Cauchy problem, Hilfer fractional derivative, attractivity, Krasnoselskii fixed point theorem, measure of noncompactness, Hermite-Hadamard inequality, Euler's reflection formula.
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1. Introduction

In this paper, as an initial value problem (IVP for short) we consider the general nonlinear model with composite fractional derivative:

$$\begin{cases} D^{\alpha,\beta}_{a^+}y(x) = f(x,y), \quad x > a,\ n-1 < \alpha \le n,\ 0 \le \beta \le 1, \\ \dfrac{d^k}{dx^k}(I^{n-\gamma}_{a^+}y)(a^+) = \lim\limits_{x\to a^+} \dfrac{d^k}{dx^k}(I^{n-\gamma}_{a^+}y)(x) = c_k,\ c_k \in \mathbb{R},\ (k = 0,1,\dots,n-1) \end{cases} \tag{1.1}$$

where $\gamma = \alpha + n\beta - \alpha\beta$. One may see that for the case $n = 1$, (1.1) is the problem investigated by Furati et al. [11]. From the historical background, we may see that differentiation and integration of functions of fractional order are traditionally defined utilizing Riemann-Liouville (R-L) operators $I^{\alpha}_{a^+}f$ and $D^{\alpha}_{a^+}f$, called the left-sided

Riemann-Liouville fractional integral of order α of f and left-sided Riemann-Liouville fractional derivative of order α of f, respectively, as ([3, 10, 18, 24, 29]):

$$\begin{aligned}(I^{\alpha}_{a^+}f)(x) &= \frac{1}{\Gamma(\alpha)}\int_a^x \frac{f(t)}{(x-t)^{1-\alpha}}dt, \quad (\alpha\in\mathbb{R},\ \alpha>0)\\ (D^{\alpha}_{a^+}f)(x) &= \Big(\frac{d}{dx}\Big)^n (I^{n-\alpha}_{a^+}f)(x), \quad (\alpha\in\mathbb{R},\ \alpha\geq 0,\ n=\lfloor\alpha\rfloor+1)\end{aligned} \tag{1.2}$$

where $a<x$, f is locally integrable (i.e., $f\in L^1(a,b)$). The operator $I^{\alpha}_{a+}f$ is defined on the space $L^1(a,b)$ of Lebesgue measurable functions $f(x)$ on a finite interval $[a,b]$ $(b>a)$ of the real line $\mathbb{R}$:

$$L^1(a,b)=\Big\{f\ :\ \|f\|_1=\int_a^b |f(x)|dx<\infty\Big\}.$$

Let $AC([a,b])$ be the space of real-valued functions $f(x)$ which are absolutely continuous on $[a,b]$. For $n\in\mathbb{N}$, by $AC^n([a,b])$ we mean the following:

$$AC^n([a,b])=\Big\{f\in C^{n-1}([a,b]) : f^{(n-1)}\in AC([a,b])\Big\}.$$

Very recently, in [13, 14, 15] an infinite family of fractional (R-L) derivatives having the same order were introduced as follows.

Definition 1.1. The (right-hand side) fractional derivative $D^{\alpha,\beta}_{a^+}$ of order $0<\alpha<1$ and type $0\leq\beta\leq 1$ with respect to x is defined by

$$(D^{\alpha,\beta}_{a^+}f)(x)=\Big(I^{\beta(1-\alpha)}_{a^+}\frac{d}{dx}(I^{(1-\beta)(1-\alpha)}_{a^+}f)\Big)(x) \tag{1.3}$$

whenever the right-hand side exists. This generalization gives the classical (R-L) fractional differentiation operator if $\beta=0$. For $\beta=1$ it gives the fractional differential operator introduced by Liouville ([20], page 10) but nowadays often named after Caputo. Several researchers (see [10,11]) called (1.3) the *Hilfer fractional derivative* or composite fractional derivative operator. Some applications of $D^{\alpha,\beta}_{a^+}$ can be found in [12, 14, 16, 30, 31].

Recently (Hilfer et al. [17]), this definition for $n-1<\alpha\leq n, n\in\mathbb{N}, 0\leq\beta\leq 1$, was rewritten in a more general form:

$$(D^{\alpha,\beta}_{a^+}f)(x)=\Big(I^{\beta(n-\alpha)}_{a^+}\frac{d^n}{dx^n}(I^{(1-\beta)(n-\alpha)}_{a^+}f)\Big)(x)=\Big(I^{\beta(n-\alpha)}_{a^+}D^{\alpha+n\beta-\alpha\beta}_{a^+}f\Big)(x). \tag{1.4}$$

The Hilfer fractional derivative can be considered as an interpolator between the Riemann-Liouville and Caputo derivative (see Figure 1).

This paper is organized as follows. In Section 2, we give some basic definitions and auxiliary facts together with a vital result which all will play important roles in the next sections. By presenting a Volterra integral equation as the equivalent form of (1.1), some attractivity results are established in Section 3 via the well-known Krasnoselskii fixed point theorem for the case $0<\alpha<1$. Finally, in Section 4, using the measure of noncompactness we derive some attractivity and stability results for (1.1) related to the case $n-1<\alpha\leq n$. Finally, some examples are given illustrating the obtained results in Section 5.

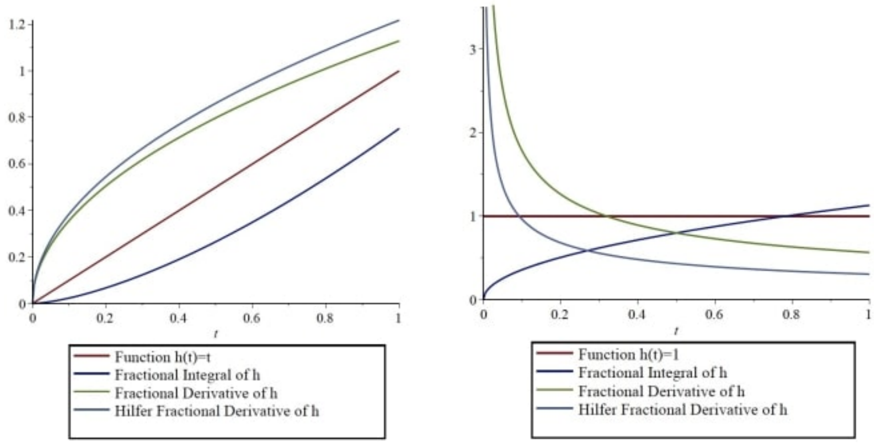

Figure 1. Assume the function $h(t) = t$, let $\alpha = 1/2$. The fractional integral and the Riemann-Liouville fractional derivative of order α are plotted. For $\beta = 1/2$, it is shown the Hilfer fractional derivative of order α and type β. The same for $h(t) = 1$ (see [12]).

2. Some auxiliary facts

This section is dedicated to the study of existence and locally attractivity of solutions of Eq. (1.1). The following fixed point theorem as the improvement of a fixed point theorem of Krasnoselskii [19] due to Burton [7, 8] will be needed further on.

Theorem 2.1 (Krasnoselskii Fixed Point Theorem). *Let S be a nonempty, closed, convex and bounded subset of the Banach space X and let $A : X \to X$ and $B : S \to X$ be two operators such that*

(a) *A is a contraction with constant $L < 1$,*
(b) *B is continuous, BS resides in a compact subset of X,*
(c) *$[x = Ax + By,\ y \in S] \Rightarrow x \in S$.*

Then the operator equation $Ax + Bx = x$ has a solution in S.

From now on, unless otherwise specified, denoting $\mathbb{R}_a = [a, \infty)$ let us assume that Ω is a nonempty subset of the space $BC(\mathbb{R}_a)$ and Q is an operator defined on Ω with values in $BC(\mathbb{R}_a)$.

Definition 2.2. The solution $u(t)$ of IVP (1.1) is *attractive* if $u(t) \to 0$ as $t \to \infty$.

Consider the following operator equation:

$$u(t) = [Qu]\,(t), \qquad \text{for all} \quad t \in \mathbb{R}_a. \tag{2.1}$$

Now we review the concept of attractivity of solutions for Eq. (2.1):

Definition 2.3. ([6]) We say that solutions of (2.1) are *locally attractive* if there exists a closed ball $B(u_0, r)$ in the space $BC(\mathbb{R}_a)$ such that for arbitrary solutions $u = u(t)$ and $v = v(t)$ of (2.1) belonging to $B(u_0, r) \cap \Omega$ we have that

$$\lim_{t\to\infty} (u(t) - v(t)) = 0. \tag{2.2}$$

In the case when limit (2.2) is uniform with respect to set $B[u_0, r] \cap \Omega$, i.e., when for each $\varepsilon > 0$ there exists $T \geq a$ such that

$$|u(t) - v(t)| \leq \varepsilon \qquad \text{for all } u, v \in B[u_0, r] \cap \Omega \quad \text{and} \quad t \geq T, \tag{2.3}$$

we will say that solutions of IVP (1.1) are *uniformly locally attractive*.

Definition 2.4. ([6]) The solution $u = u(t)$ of Eq. (2.1) is said to be *globally attractive* if (2.2) holds for each solution $v = v(t)$ of Eq. (2.1) on Ω. In other words, we may say that solutions of Eq. (2.1) are globally attractive if for arbitrary solutions $u(t)$ and $v(t)$ of Eq. (2.1) on Ω, the condition (2.2) is satisfied. In the case when the condition (2.2) is satisfied uniformly with respect to the set Ω, i.e., if for every $\epsilon > 0$ there exists $T \geq a$ such that the inequality (2.3) is satisfied for all $u, v \in \Omega$ being the solutions of Eq. (2.1) and for all $t \geq T$, we will say that solutions of Eq. (2.1) are *uniformly globally attractive* on $\mathbb{R}_a$.

Remark 2.5. Obviously, as has been noted in [6], we observe that global attractivity of solutions implies local attractivity, but the converse implication is not true.

3. Uniformly locally attractivity for the case $0 < \alpha < 1$

In this section, as an equivalent form we establish that the Cauchy type problem (1.1) can be reduced to the following nonlinear Volterra integral equation of the second kind:

$$\begin{aligned} y(x) = &\sum_{k=0}^{n-1} c_k \frac{(x-a)^{k-(n-\alpha)(1-\beta)}}{\Gamma(k-(n-\alpha)(1-\beta)+1)} \\ &+ \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t, y(t)) dt, \ x > a. \end{aligned} \tag{3.1}$$

We remark that for the case $n = 1$, one can see that (3.1) is precisely the Volterra integral equation (5) in [11].

Throughout this section we investigate the attractivity and existence of solutions for (1.1) with $n = 1$.

Lemma 3.1. ([18]) *The Riemann-Liouville fractional integral operator $I_{a^+}^{\alpha}$ of order $\alpha \in \mathbb{R}$, $\alpha > 0$, is bounded in the space $L^1(a, b)$ and*

$$\|I_{a^+}^{\alpha} \varphi\|_1 \leq A \|\varphi\|_1, \quad A = \frac{(b-a)^{\alpha}}{\alpha |\Gamma(\alpha)|}$$

for any $\varphi \in L^1(a, b)$.

Lemma 3.2. ([18]) *If $\alpha, \beta \in \mathbb{R}$ $(\alpha, \beta > 0)$, then the semigroup property*

$$I_{a^+}^{\alpha} I_{a^+}^{\beta} f = I_{a^+}^{\alpha+\beta} f \tag{3.2}$$

holds for any $f \in L^1(a, b)$.

Lemma 3.3. ([18]) *If $\alpha \in \mathbb{R}$ $(\alpha > 0)$, the compositional property*

$$D_{a^+}^{\alpha} I_{a^+}^{\alpha} f = f \tag{3.3}$$

holds for any summable function $f \in L^1(a, b)$.

If $I_{a^+}^{n-\alpha} f \in AC^k([a,b])$ $(n-1<\alpha\le n, 0\le k\le n-1)$ *then*

$$(I_{a^+}^{\alpha} D_{a^+}^{\alpha} f)(x) = f(x) - \sum_{k=0}^{n-1} \frac{(x-a)^{\alpha-n+k}}{\Gamma(\alpha-n+k+1)} \lim_{x\to a^+} \frac{d^k}{dx^k}(I_{a^+}^{n-\alpha} f)(x) \tag{3.4}$$

holds for any summable function $f \in L^1(a,b)$. *Furthermore, if*

$$f(x) \in I_{a^+}^{\alpha}(L^1(a,b)) = \{f : f = I_{a^+}^{\alpha}\varphi,\ \varphi \in L^1(a,b)\},$$

then

$$(I_{a^+}^{\alpha} D_{a^+}^{\alpha} f)(x) = f(x). \tag{3.5}$$

Proposition 3.4. *Let* U *be an open set in* $\mathbb{R}$ *and let* $f : [a,b]\times U \to \mathbb{R}$ *be a function such that* $f(x,y) \in L^1(a,b)$ *and* $f(\cdot, y(\cdot)) \in AC^n([a,b])$. *If* $y \in L^1(a,b)$, $n-1<\alpha\le n$, $n\in\mathbb{N}$, $0\le\beta\le 1$, $I_{a^+}^{(n-\alpha)(1-\beta)} y \in AC^k([a,b])$, $0\le k\le n-1$ *then* $y(x)$ *satisfies a.e. the Eq.* (1.1) *if and only if* $y(x)$ *satisfies a.e. the integral equation* (3.1).

Proof. Necessity. Suppose that $y(x) \in L^1(a,b)$ fulfills a.e. the relation (1.1). Making use of the fact that $f(x,y) \in L^1(a,b)$, by the Eq. (1.1) it follows that the fractional derivative $(D_{a^+}^{\alpha,\beta} y)(x) \in L^1(a,b)$ exists a.e. on $[a,b]$. By Lemma 3.1 the integral $I_{a^+}^{\alpha} f(x,y(x)) \in L^1(a,b)$ exists a.e. on $[a,b]$. On the other hand, using the representation (1.4) together with the compositional properties (3.2) and (3.4) we get

$$\begin{aligned}(I_{a^+}^{\alpha} D_{a^+}^{\alpha,\beta} y)(x) &= \Big(I_{a^+}^{\alpha} I_{a^+}^{\beta(n-\alpha)} D_{a^+}^{\alpha+n\beta-\alpha\beta} y\Big)(x) = \Big(I_{a^+}^{\alpha+\beta(n-\alpha)} D_{a^+}^{\alpha+\beta(n-\alpha)} y\Big)(x)\\ &= y(x) - \sum_{k=0}^{n-1} \frac{(x-a)^{k-(n-\alpha)(1-\beta)}}{\Gamma(k-(n-\alpha)(1-\beta)+1)} \lim_{x\to a^+} \frac{d^k}{dx^k}(I_{a^+}^{(n-\alpha)(1-\beta)} y)(x),\end{aligned} \tag{3.6}$$

for $x>a$. Now, applying the integral operator $I_{a^+}^{\alpha}$ to both sides of Eq. (1.1) and utilizing the relation (3.6) we obtain Eq. (3.1) and thus the necessity is easily concluded.

Sufficiency. Assume $y(x) \in L^1(a,b)$ satisfies a.e. Eq. (3.1). In view of the property of commutativity of operators integration of fractional order α and differentiation of order n, i.e.,

$$\frac{d^n}{dx^n} I_{a^+}^{\alpha}\varphi(x) = I_{a^+}^{\alpha}\varphi^{(n)}(x), \quad \Re(\alpha)>0,\ n\in\mathbb{N}$$

for any n-times differentiable function φ together with (3.2), (3.5) and $f(\cdot, y(\cdot)) \in AC^n([a,b])$ one can see that

$$\begin{aligned}(D_{a^+}^{\alpha,\beta} I_{a^+}^{\alpha} f[t,y(t)])(x) &= \Big(I_{a^+}^{\beta(n-\alpha)} \frac{d^n}{dx^n}(I_{a^+}^{(1-\beta)(n-\alpha)} I_{a^+}^{\alpha} f[t,y(t)])\Big)(x)\\ &= \Big(I_{a^+}^{\beta(n-\alpha)} \frac{d^n}{dx^n}(I_{a^+}^{n-\beta(n-\alpha)} f[t,y(t)])\Big)(x)\\ &= \Big(I_{a^+}^{\beta(n-\alpha)} (I_{a^+}^{n-\beta(n-\alpha)} f^{(n)}[t,y(t)])\Big)(x)\\ &= (I_{a^+}^{n} f^{(n)}[t,y(t)])(x)\\ &= (I_{a^+}^{n} D_{a^+}^{n} f[t,y(t)])(x)\\ &= f(x,y(x)).\end{aligned} \tag{3.7}$$

Using the fact $(D^{\alpha,\beta}_{a^+}(t-a)^{k-(n-\alpha)(1-\beta)})(x) = 0$ for $x > a$ and $0 \le k \le n-1$, and (3.7), and applying the operator $D^{\alpha,\beta}_{a^+}$ to both sides of (3.1), we obtain

$$\begin{aligned}(D^{\alpha,\beta}_{a^+}y)(x) &= \sum_{k=0}^{n-1} c_k \frac{(D^{\alpha,\beta}_{a^+}(t-a)^{k-(n-\alpha)(1-\beta)})(x)}{\Gamma(k-(n-\alpha)(1-\beta)+1)} \\ &\quad + (D^{\alpha,\beta}_{a^+} I^{\alpha}_{a^+} f[t,y(t)])(x) = f(x,y(x)). \end{aligned} \tag{3.8}$$

Now we must prove the second equality in (1.1). To do this, let us apply the operator $I^{n-\gamma}_{a^+}$ to both sides of (3.1), then

$$(I^{n-\gamma}_{a^+}y)(x) = \sum_{k=0}^{n-1} c_k \frac{(I^{n-\gamma}_{a^+}(t-a)^{k-n+\gamma})(x)}{\Gamma(k-n+\gamma+1)} + (I^{n-\gamma}_{a^+} I^{\alpha}_{a^+} f[t,y(t)])(x). \tag{3.9}$$

Now the relation

$$[I^{r}_{a^+}(t-a)^{s-1}](x) = \frac{\Gamma(s)}{\Gamma(r+s)}(t-a)^{r+s-1}, \quad x > a,\ r \ge 0,\ s > 0,$$

yields that

$$(I^{n-\gamma}_{a^+}y)(x) = \sum_{j=0}^{n-1} \frac{c_j}{j!}(x-a)^j + (I^{\alpha+n-\gamma}_{a^+} f[t,y(t)])(x). \tag{3.10}$$

Moving forward, if $0 \le k \le n-1$, then

$$\begin{aligned}\frac{d^k}{dx^k}(I^{n-\gamma}_{a^+}y)(x) &= \sum_{j=k}^{n-1} \frac{c_j}{(j-k)!}(x-a)^{j-k} + \frac{d^k}{dx^k}(I^{\alpha+n-\gamma}_{a^+} f[t,y(t)])(x) \\ &= \sum_{j=k}^{n-1} \frac{c_j}{(j-k)!}(x-a)^{j-k} + (I^{\alpha+n-\gamma-k}_{a^+} f[t,y(t)])(x) \\ &= \sum_{j=k}^{n-1} \frac{c_j}{(j-k)!}(x-a)^{j-k} \\ &\quad + \frac{1}{\Gamma(\alpha+n-\gamma-k)} \int_a^x \frac{f(t)}{(x-t)^{1-\alpha-n+\gamma+k}} dt. \end{aligned} \tag{3.11}$$

Taking $x \to a^+$ a.e., we derive the second relation of (1.1). Therefore, the sufficiency is proved, which completes the proof of theorem. □

Here, we focus on the Eq. (1.1) for $n = 1$. Our considerations are based on the following assumptions:

- (C1) The function $f(x,u(x))$ is Lebesgue measurable with respect to x on $\mathbb{R}_a$ and there exists a constant $\alpha_1 \in (0,\alpha)$ such that $f(x,u(x)) \in L^{\frac{1}{\alpha_1}}(a,b)$ for all $b \in \mathbb{R}_a$ and $u \in C(\mathbb{R}_a,\mathbb{R})$ and $f(x,u(x))$ is continuous with respect to u on $\mathbb{R}_a$.
- (C2) $|f(x,u(x))| \le M(x-a)^{-\beta_1}$, $\forall x > a$, $u(x) \in C(\mathbb{R}_a,\mathbb{R})$, $M \ge 0$ and $\alpha < \beta_1 < 1$.

By the equivalent Volterra integral equation of the second kind of (1.1), that is Eq. (3.1), we define the following operators:

$$\begin{aligned}
\left[\mathscr{F}u\right](x) &= c_0\frac{(x-a)^{\gamma-1}}{\Gamma(\gamma)} + \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1} f(t,u(t))dt,\\
\left[\mathscr{A}u\right](x) &= c_0\frac{(x-a)^{\gamma-1}}{\Gamma(\gamma)},\\
\left[\mathscr{B}u\right](x) &= \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1} f(t,u(t))dt, \quad x>a
\end{aligned}$$

for all $u \in C(\mathbb{R}_a,\mathbb{R})$.

It is obvious that $u(x)$ is a solution of (1.1) if it is a fixed point of the operator $\mathscr{F}$, and the operator $\mathscr{A}$ is a contraction with constant $L=0$.

Now we are in a position to formulate our main result as follows.

Lemma 3.5. *Suppose the assumptions (C1)–(C2) fulfill. Then the operator $\mathscr{B}$ is continuous and $\mathscr{B}\mathcal{S}$ resides in a compact subset of $C(\mathbb{R}_a,\mathbb{R})$ for $x\geq a+\theta_1$, where*

$$\mathcal{S} = \{u : u(x) \in C(\mathbb{R}_a,\mathbb{R}) \text{ and } |u(x)| \leq (x-a)^{-\gamma_1} \text{ for all } x \geq a+\theta_1\},$$

$\gamma_1 = c(\beta_1-\alpha)$ for arbitrary $0<c<1$, and θ_1 satisfies the following

$$c_0\frac{\theta_1^{\gamma-1}}{\Gamma(\gamma)} + \frac{M\Gamma(1-\beta_1)}{\Gamma(1+\alpha-\beta_1)}\theta_1^{(1-c)(\alpha-\beta_1)} \leq 1. \tag{3.12}$$

Proof. First, we prove that $\mathscr{B}$ maps $\mathcal{S}$ into $\mathcal{S}$ for $x\geq a+\theta_1$.

From the above assumption of $\mathcal{S}$, it is easy to see that $\mathcal{S}$ is a closed, bounded and convex subset of $C(\mathbb{R}_a,\mathbb{R})$. Applying condition (C1) and recalling the Euler's Beta function, for $x\geq a$, we derive

$$\begin{aligned}
|\left[\mathscr{B}u\right](x)| &\leq \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}|f(t,u(t))|dt\\
&\leq \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}M(t-a)^{-\beta_1}dt\\
&\leq \frac{M\Gamma(1-\beta_1)}{\Gamma(1+\alpha-\beta_1)}(x-a)^{-(\beta_1-\alpha)}.
\end{aligned}$$

Now since $\beta_1>\alpha$ for $x\geq a+\theta_1$, the inequality (3.12) implies that

$$\frac{M\Gamma(1-\beta_1)}{\Gamma(1+\alpha-\beta_1)}(x-a)^{(1-c)(\alpha-\beta_1)} \leq \frac{M\Gamma(1-\beta_1)}{\Gamma(1+\alpha-\beta_1)}\theta_1^{(1-c)(\alpha-\beta_1)} \leq 1,$$

which yields that

$$|\left[\mathscr{B}u\right](x)| \leq \left[\frac{M\Gamma(1-\beta_1)}{\Gamma(1+\alpha-\beta_1)}(x-a)^{(1-c)(\alpha-\beta_1)}\right](x-a)^{c(\alpha-\beta_1)} \leq (x-a)^{-\gamma_1}$$

which shows that $\mathscr{B}\mathcal{S}$ lies in $\mathcal{S}$ for $x\geq a+\theta_1$.

In the second step, we must show that $\mathscr{B}$ is continuous. To do this, let $u_k(x)$, $u(x) \in \mathcal{S}$, $k=1,2,\cdots$ with $\lim_{k\to\infty} u_k(x) = u(x)$, then from (C1) we get $\lim_{k\to\infty} f(x,u_k(x)) = f(x,u(x))$ for any $x\geq a+\theta_1$.

Let $\epsilon > 0$ be given, fix $\theta > a + \theta_1$ so that

$$\frac{M\Gamma(1-\beta_1)}{\Gamma(1+\alpha-\beta_1)}(\theta - a)^{-(\beta_1-\alpha)} < \frac{\epsilon}{2}.$$

Assume that $\rho = (\alpha-1)(1-\alpha_1)^{-1}$, then $\rho + 1 > 0$ since $0 < \alpha_1 < \alpha < 1$. For $a + \theta_1 \le x \le \theta$, using Hölder's inequality we get

$$\begin{aligned}
&|[\mathscr{B}u_k](x) - [\mathscr{B}u](x)| \le \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}|f(t,u_k(t)) - f(t,u(t))|dt \\
&\le \frac{1}{\Gamma(\alpha)}\left\{\int_a^x (x-t)^{\rho}dt\right\}^{1-\alpha_1}\left\{\int_a^x |f(t,u_k(t)) - f(t,u(t))|^{\frac{1}{\alpha_1}}dt\right\}^{\alpha_1} \\
&\le \frac{1}{\Gamma(\alpha)}\left(\frac{1}{\rho+1}(\theta-a)^{\rho+1}\right)^{1-\alpha_1}(\theta-a)^{\alpha_1}\sup_{t\in[a,\theta]}|f(t,u_k(t)) - f(t,u(t))|
\end{aligned}$$

which vanishes when $k \to \infty$. For $x > \theta$, we see that

$$\begin{aligned}
|[\mathscr{B}u_k](x) - [\mathscr{B}u](x)| &\le \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}|f(t,u_k(t)) - f(t,u(t))|dt \\
&\le \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}[|f(t,u_k(t))| + |f(t,u(t))|]dt \\
&\le \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}(2M(t-a)^{-\beta_1})dt \\
&\le \frac{2M\Gamma(1-\beta_1)}{\Gamma(1+\alpha-\beta_1)}(x-a)^{-(\beta_1-\alpha)} \\
&\le \frac{2M\Gamma(1-\beta_1)}{\Gamma(1+\alpha-\beta_1)}(\theta-a)^{-(\beta_1-\alpha)} \\
&\le \epsilon
\end{aligned}$$

which shows that for any $x \ge a + \theta_1$,

$$|[\mathscr{B}u_k](x) - [\mathscr{B}u](x)| \to 0$$

as $k \to \infty$. Hence, $\mathscr{B}$ is continuous.

Eventually, we claim that $\mathscr{B}\mathcal{S}$ is equicontinuous.

Suppose that $\epsilon > 0$ is given. Since the function $(x-a)^{\alpha-\beta_1}$ vanishes at infinity, there is a θ', sufficiently large, such that $(x-a)^{\alpha-\beta_1} < \frac{\epsilon}{2}$ for all $x > \theta'$. Let us take x_1, x_2 so that $x_2 > x_1 \ge a + \theta_1$. If $x_1, x_2 \in [a+\theta_1, \theta']$, then in view of the hypothesis

$$f(x, u(x)) \in L^{\frac{1}{\alpha_1}}(a, \theta')$$

we get

$$
\begin{aligned}
&\left|[\mathscr{B}u](x_2) - [\mathscr{B}u](x_1)\right| \\
&\leq \frac{1}{\Gamma(\alpha)}\left|\int_a^{x_2}(x_2-t)^{\alpha-1}f(t,u(t))dt - \int_a^{x_1}(x_1-t)^{\alpha-1}f(t,u(t))dt\right| \\
&\leq \frac{1}{\Gamma(\alpha)}\bigg(\int_a^{x_1}[(x_1-t)^{\alpha-1}-(x_2-t)^{\alpha-1}]|f(t,u(t))|dt \\
&\quad + \int_{x_1}^{x_2}(x_2-t)^{\alpha-1}|f(t,u(t))|dt\bigg) \\
&\leq \frac{\|f(t,x(t)\|_{L^{\frac{1}{\alpha_1}}(a,x_2)}}{\Gamma(\alpha)}\Bigg[\bigg(\int_a^{x_1}[(x_1-t)^{\alpha-1}-(x_2-t)^{\alpha-1}]^{\frac{1}{1-\alpha_1}}dt\bigg)^{1-\alpha_1} \\
&\quad + \bigg(\int_{x_1}^{x_2}(x_2-t)^{\frac{\alpha-1}{1-\alpha_1}}dt\bigg)^{1-\alpha_1}\Bigg] \\
&\leq \frac{\|f(t,x(t)\|_{L^{\frac{1}{\alpha_1}}(a,\theta')}}{\Gamma(\alpha)}\Bigg[\bigg(\frac{(x_1-a)^{1+\rho}-(x_2-a)^{1+\rho}+(x_2-x_1)^{1+\rho}}{1+\rho}\bigg)^{1-\alpha_1} \\
&\quad + \bigg(\frac{(x_2-x_1)^{1+\rho}}{1+\rho}\bigg)^{1-\alpha_1}\Bigg] \longrightarrow 0 \quad \text{as } x_2 \to x_1.
\end{aligned}
$$

If $x_1, x_2 > \theta'$, then we observe that

$$
\begin{aligned}
\left|[\mathscr{B}u](x_2) - [\mathscr{B}u](x_1)\right| &\leq \frac{1}{\Gamma(\alpha)}\int_a^{x_2}(x_2-t)^{\alpha-1}|f(t,u(t))|dt \\
&\quad + \frac{1}{\Gamma(\alpha)}\int_a^{x_1}(x_1-t)^{\alpha-1}|f(t,u(t))|dt \\
&\leq \frac{\Gamma(1-\beta_1)}{\Gamma(1+\alpha-\beta_1)}\left[(x_1-a)^{-(\beta_1-\alpha)}+(x_2-a)^{-(\beta_1-\alpha)}\right] \\
&\leq \frac{\epsilon\cdot\Gamma(1-\beta_1)}{\Gamma(1+\alpha-\beta_1)} \quad \text{as } x_2 \to x_1.
\end{aligned}
$$

Now consider the case $a+\theta_1 \leq x_1 < \theta' < x_2$ then we have the following implication:

$$(x_2 \to x_1) \quad \Longrightarrow \quad (x_2 \to \theta') \wedge (\theta' \to x_1)$$

which according to the above discussion yields that

$$\left|[\mathscr{B}u](x_2) - [\mathscr{B}u](x_1)\right| \leq \left|[\mathscr{B}u](x_2) - [\mathscr{B}u](\theta')\right| + \left|[\mathscr{B}u](\theta') - [\mathscr{B}u](x_1)\right| \longrightarrow 0$$

as $x_2 \to x_1$.

Thus, it is obvious that $\left|[\mathscr{B}u](x_2) - [\mathscr{B}u](x_1)\right| \to 0$ as $x_2 \to x_1$. Therefore $\mathscr{B}\mathcal{S}$ is equicontinuous and thus $\mathscr{B}\mathcal{S}$ is contained in a compact subset of $C(\mathbb{R}_a, \mathbb{R})$ for $x \geq a+\theta_1$. □

Now we are ready to formulate our main existence result.

Theorem 3.6. *Suppose that conditions $(C1)$–$(C2)$ are satisfied, then IVP (1.1) admits at least one attractive solution in $C(\mathbb{R}_a, \mathbb{R})$ in the sense of Definition 2.2.*

Proof. Based on Lemma 3.5, since the operator $\mathscr{A}$ is a contraction with constant $L = 0$, one can easily see that all conditions of Krasnoselskii's fixed point theorem are satisfied and so Eq. (1.1) has at least one solution belonging to $\mathcal{S}$. On the other hand, in order to prove the attractivity, using the structure of set $\mathcal{S}$ given in Lemma 3.5, we see that all functions in Lemma 3.5 vanish at infinity and thus the solution of Eq. (1.1) tends to zero as $t \to \infty$. This completes the proof. □

Remark 3.7. It is worth mentioning that conclusion of Theorem 3.6 does not imply globally attractivity of solutions in the sense of Definition 2.2.

4. Attractivity for the case $n-1 < \alpha \leq n$ via the measure of noncompactness

This section is dedicated to the study of solutions of Eq. (1.1) in Banach space $BC(\mathbb{R}_a)$ consisting of all real functions defined, continuous and bounded on the interval $\mathbb{R}_a$, via the technique of measure of noncompactness. This tool enables us to construct some sufficient conditions (quite distinct from the comparable ones in previous results) for solvability of Eq. (1.1). Indeed, we seek for assumptions concerning the functions involved in Eq. (1.1) which assure that this equation has solutions belonging to $BC(\mathbb{R}_a)$ and also being locally attractive on $\mathbb{R}_a$. In the sequel, we gather some definitions and auxiliary facts which will be needed further on.

Let E be a Banach space, $\overline{X}$ and $\operatorname{Conv} X$ stand for the closure and the convex closure of X as a subset of E, respectively. Further, denote by $\mathfrak{M}_E$ the family of all nonempty bounded subsets of E and by $\mathfrak{N}_E$ its subfamily consisting of all relatively compact sets. Also suppose that $B(x,r)$ is the closed ball centered at x with radius r and the symbol B_r stands for the ball $B(\theta, r)$ such that θ is the zero element of the Banach space E.

In the following definition we recall the notion of measure of noncompactness which has been initially introduced by Banaś and Goebel [5].

Definition 4.1. ([5]) A mapping $\mu\colon \mathfrak{M}_E \longrightarrow \mathbb{R}^+$ is said to be a *measure of noncompactness* in E if it satisfies the following conditions:

(i) The family $\ker\mu = \{X \in \mathfrak{M}_E : \mu(X) = 0\}$ is nonempty and $\ker\mu \subset \mathfrak{N}_E$.

(ii) $X \subset Y \Rightarrow \mu(X) \leq \mu(Y)$.

(iii) $\mu(\overline{X}) = \mu(X)$.

(iv) $\mu(\operatorname{Conv} X) = \mu(X)$.

(v) For all $\lambda \in [0,1]$,
$$\mu(\lambda X + (1-\lambda) Y) \leq \lambda\, \mu(X) + (1-\lambda)\, \mu(Y).$$

(vi) If $(X_n)_{n\in\mathbb{N}}$ is a sequence of closed sets from $\mathfrak{M}_E$ such that
$$X_{n+1} \subset X_n \quad \text{for all} \quad n = 1,2,\ldots \quad \text{and} \quad \lim_{n\to\infty} \mu(X_n) = 0,$$
then the intersection set
$$X_\infty = \bigcap_{n=1}^{\infty} X_n \quad \text{is nonempty.}$$

The family $\ker\mu$ described in (i) is said to be *the kernel of the measure of noncompactness* μ.

Definition 4.2. Let μ be a measure of noncompactness in E. The mapping $T\colon C \subseteq E \longrightarrow E$ is said to be a μ_E-*contraction* if there exists a constant $0 < k < 1$ such that

$$\mu(T(W)) \leq k\,\mu(W), \tag{4.1}$$

for any bounded closed subset $W \subseteq C$.

As a generalization of the well-known Schauder fixed point principle and based on measure of noncompactness, the Darbo fixed point theorem, was formulated:

Theorem 4.3 (Darbo-Sadovskii). ([5]) *Let C be a nonempty, bounded, closed, and convex subset of a Banach space E and let the continuous mapping $T\colon C \longrightarrow C$ be a μ_E-contraction. Then T has at least one fixed point in C.*

Making a historical flashback for this tool, we remark that Darbo [9] initially introduced condition (4.1) for any arbitrary measure of noncompactness μ and presented a similar result if the continuous mapping T is being a μ-contraction. Recently Aghajani and Pourhadi [1] have extended the Darbo's fixed point theorem using some control functions and presented a new result with a more complicated contraction which is applied in this section. Denote by Φ the class of functions $\phi : [0,+\infty) \to [0,+\infty)$ such that

$$\liminf_{n\to\infty} \phi(a_n) = 0, \quad \text{if } \lim_{n\to\infty} a_n = 0$$

where $\{a_n\}$ is a nonnegative sequence.

For $\phi \in \Phi$, let functions $\psi : [0,+\infty) \longrightarrow [0,+\infty)$ satisfy the following conditions:

(a) ψ is a lower semi-continuous function with $\psi(t) = 0$ if and only if $t = 0$,

(b) $\liminf_{n\to\infty} \phi(a_n) < \psi(a)$ if $\lim_{n\to\infty} a_n = a > 0$.

We denote the class of all such functions by Ψ_ϕ.

Definition 4.4. Let $T : W \subseteq E \to E$ be an arbitrary mapping. We say that T is (α,ϕ,ψ)-μ-*condensing* if there exist functions $\alpha : \mathfrak{M}_E \to [0,+\infty)$, $\phi \in \Phi$ and $\psi \in \Psi_\phi$ such that

$$\alpha(\Omega)\psi(\mu(T\Omega)) \leq \phi(\mu(\Omega)) \quad \text{for } \Omega \subseteq W,$$

where Ω and its image $T\Omega$ belong to $\mathfrak{M}_E$.

Notice that if a mapping $T : W \subseteq E \to E$ satisfies the Darbo condition with respect to a constant $k \in [0,1)$ and a measure μ, that is,

$$\mu(T\Omega) \leq k\mu(\Omega), \quad \text{for } \Omega \subseteq W \text{ and } \Omega, T\Omega \in \mathfrak{M}_E,$$

then T is an (α,ϕ,ψ)-μ-condensing operator, where $\alpha(\Omega) = 1$ for any set $\Omega \subseteq W$ such that $\Omega \in \mathfrak{M}_E$, ψ is the identity mapping and $\phi(t) = kt$ for all $t \geq 0$. For this case, T is called μ-contraction.

Definition 4.5. Let $T : W \subseteq E \to E$ and $\alpha : \mathfrak{M}_E \to [0,+\infty)$ be given mappings. We say that T is α-*admissible* if we have

$$\alpha(\Omega) \geq 1 \Longrightarrow \alpha(\mathrm{Conv}T\Omega) \geq 1, \qquad \Omega \subseteq W,\ \ \Omega, T\Omega \in \mathfrak{M}_E.$$

Theorem 4.6. ([1]) *Let $C \in \mathfrak{M}_E$ be a closed and convex subset of a Banach space E and $T : C \to C$ be a continuous (α,ϕ,ψ)-μ-condensing operator, where μ is an arbitrary measure of noncompactness. Suppose that T is α-admissible and $\alpha(C) \geq 1$. Then T has at least one fixed point which belongs to* $\ker\mu$.

By taking $\alpha = 1$, $\psi = id$ we have the following immediate consequence:

Corollary 4.7. *Let C be a nonempty, bounded, closed, and convex subset of a Banach space E and let $T : C \to C$ be a continuous function satisfying*

$$\mu(T\Omega) \leq \varphi(\mu(\Omega))$$

for each $\Omega \subseteq C$, where μ is an arbitrary measure of noncompactness and $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is a nondecreasing upper semi-continuous function with $\lim_{n\to\infty} \varphi^n(t) = 0$ for all $t \geq 0$. Then T has at least one fixed point in C.

Remark 4.8. Following the assumptions of Corollary 4.7, we note that condition (b) is obtained by $\lim_{n\to\infty} \varphi^n(t) = 0$ for $t > 0$. For the sake of enlightening the reader, if $\lim_{n\to\infty} a_n = a > 0$ then

$$\liminf_{n\to\infty} \varphi(a_n) \leq \limsup_{n\to\infty} \varphi(a_n) \leq \varphi(a) < a.$$

Measure of noncompactness has been applied in some classes of fractional differential equations in several papers. For instance, Aghajani, Pourhadi and Trujillo [2] have utilized this tool for Cauchy problem as a classic fractional differential equations in Banach spaces (see also [4, 6, 25]). It is worth mentioning that the measure of noncompactness has been also successfully employed in the study of infinite systems of differential equations in the Banach sequence spaces (see for example [1, 21, 26, 27]). Utilizing the Schauder fixed point theorem, we remark that Losada, Nieto and Pourhadi [22] applied this well-known theorem together with the measure of noncompactness to investigate the attractivity of solutions for a class of multi-term fractional functional differential equations. Very recently, Saadati, Pourhadi and Samet [28] studied the $\mathcal{PC}$-mild solutions of some abstract fractional evolution equations with non-instantaneous impulses via the measure of noncompactness.

In what follows, we will work in the Banach space $BC(\mathbb{R}_a)$, where $a \in \mathbb{R}$ is given as in (1.1). Such functional space is furnished with the standard norm $\|y\| = \sup\{|y(t)| : t \geq a\}$. For further purposes, we introduce a measure of noncompactness in the space $BC(\mathbb{R}_a)$, which is constructed by the similar reasoning process for the one in the space $BC(\mathbb{R}^+)$ (for more information see [5, Chapter 9] and references therein).

To do this, let B be a bounded subset of $BC(\mathbb{R}_a)$ and $T > a$ given. For $u \in B$ and $\varepsilon > 0$ we denote by $\omega_a^T(u, \varepsilon)$ the modulus of continuity of the function u on the interval $[a, T]$, i.e.

$$\omega_a^T(u, \varepsilon) = \sup\{|u(t) - u(s)| : t, s \in [a, T], |t - s| \leq \varepsilon\}.$$

Now, let us take

$$\begin{aligned}
\omega_a^T(B, \varepsilon) &= \sup\{\omega_a^T(u, \varepsilon) : u \in B\},\\
\omega_a^T(B) &= \lim_{\varepsilon\to 0} \omega_a^T(B, \varepsilon),\\
\omega_a(B) &= \lim_{T\to\infty} \omega_a^T(B).
\end{aligned}$$

If $t \geq a$ is a fixed number, let us denote

$$B(t) = \{u(t) : u \in B\}$$

and

$$\operatorname{diam} B(t) = \sup\{|u(t) - v(t)| : \ u, v \in B\}.$$

Finally, consider the mapping μ defined on the family $\mathfrak{M}_{BC(\mathbb{R}_a)}$ by the formula

$$\mu(B) = \omega_a(B) + \limsup_{t\to\infty} \operatorname{diam} B(t). \tag{4.2}$$

Similarly to the measure of noncompactness constructed for $BC(\mathbb{R}^+)$, one can show that the mapping μ is a measure of noncompactness in the space $BC(\mathbb{R}_a)$ (see also [5]).

It is worth mentioning that, as we will show, information about $\ker\mu$ is very helpful. In this case, the $\ker\mu$ consists of nonempty and bounded sets X of functions such that functions belonging to X are locally equicontinuous on $\mathbb{R}^+$ and the thickness of the bundle formed by functions from X tends to 0 at infinity.

In this section, we study the problem (1.1) for $n \geq 1$ and $\gamma > 1$ with the following condition:

$$n - \gamma < k \leq n - 1 \quad \Longrightarrow \quad c_k = 0. \tag{4.3}$$

Consider the following hypotheses:

(i) There exist a continuous function $h : \mathbb{R}_a \to \mathbb{R}_+$ and a nondecreasing upper semi-continuous function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ with $\varphi(0) = 0$ such that

$$|f(t,u) - f(t,v)| \leq h(t)\varphi(|u - v|), \text{ for any } u, v \in \mathbb{R},\ t \in \mathbb{R}_a,$$

$$\lambda := \sup_{x\in\mathbb{R}_a} \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} h(t)dt.$$

(ii) $\xi := \sup_{x\in\mathbb{R}_a} \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} |f(t,0)|dt < \infty.$

(iii) There exists a positive solution r_0 of the inequality

$$A_\theta + \lambda\varphi(r) + \xi \leq r \tag{4.4}$$

where $\theta > 0$ is arbitrarily fixed,

$$A_\theta := \sum_{k=0}^{\lfloor n-\gamma \rfloor} |c_k| \frac{\theta^{k-n+\gamma}}{|\Gamma(k-n+\gamma+1)|}.$$

(iv) $\lim_{n\to\infty} \lambda^n \varphi^n(t) = 0 \quad$ for all $t > 0$.

Theorem 4.9. *Under the assumptions (i)–(iv), IVP* (1.1) *has at least one solution in $BC(\mathbb{R}_{a+\theta})$ for any fixed $\theta > 0$.*

Proof. According to (4.3), we first define operator $\mathscr{F}$ for any $u \in BC(\mathbb{R}_{a+\theta})$ by

$$[\mathscr{F}u](x) = \sum_{k=0}^{\lfloor n-\gamma \rfloor} c_k \frac{(x-a)^{k-n+\gamma}}{\Gamma(k-n+\gamma+1)} + \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t,u(t))dt, \quad x \geq a + \theta.$$

By considering the conditions of theorem we infer that $\mathscr{F}u$ is continuous on $\mathbb{R}_{a+\theta}$ for any $u \in BC(\mathbb{R}_{a+\theta})$. By using conditions (i)-(iii), taking an arbitrary function

$u \in BC(\mathbb{R}_{a+\theta}, \mathbb{R})$, for a fixed $x \in \mathbb{R}_{a+\theta}$, one has

$$\begin{aligned} |[\mathscr{F}u](x)| &\leq \sum_{k=0}^{\lfloor n-\gamma \rfloor} |c_k| \frac{(x-a)^{k-n+\gamma}}{|\Gamma(k-n+\gamma+1)|} + \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} |f(t,u(t))| dt \\ &\leq \sum_{k=0}^{\lfloor n-\gamma \rfloor} |c_k| \frac{\theta^{k-n+\gamma}}{|\Gamma(k-n+\gamma+1)|} + \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} \Big[h(t)\varphi(|u(t)|) + |f(t,0)| \Big] dt \\ &\leq A_\theta + \lambda \varphi(\|u\|) + \xi \end{aligned} \tag{4.5}$$

which is clearly bounded.

Inequality (4.5) yields that $\mathscr{F}$ transforms the ball B_{r_0} into itself where r_0 is a positive solution of (4.4).

Take an arbitrary function $u \in BC(\mathbb{R}_{a+\theta})$ and fix $T > a+\theta$, $\epsilon > 0$. Next assume that $x_1, x_2 \in [a+\theta, T]$ such that $|x_1 - x_2| < \epsilon$. Without loss of generality one can assume that $x_1 < x_2$. Then, in view of imposed assumptions, one has

$$\begin{aligned} |[\mathscr{F}u](x_2) - [\mathscr{F}u](x_1)| &\leq \sum_{k=0}^{\lfloor n-\gamma \rfloor} |c_k| \frac{\Phi(\epsilon)}{|\Gamma(k-n+\gamma+1)|} \\ &+ \frac{1}{\Gamma(\alpha)} \int_a^{x_1} \Big[(x_2-t)^{\alpha-1} - (x_1-t)^{\alpha-1} \Big] |f(t,u(t))| dt \\ &+ \frac{1}{\Gamma(\alpha)} \int_{x_1}^{x_2} (x_2-t)^{\alpha-1} |f(t,u(t))| dt \\ &\leq \sum_{k=0}^{\lfloor n-\gamma \rfloor} |c_k| \frac{\Phi(\epsilon)}{|\Gamma(k-n+\gamma+1)|} \\ &+ \frac{1}{\Gamma(\alpha)} \int_a^{x_1} \Big[(x_2-t)^{\alpha-1} - (x_1-t)^{\alpha-1} \Big] \\ &\times \Big[h(t)\varphi(|u(t)|) + |f(t,0)| \Big] dt \\ &+ \frac{1}{\Gamma(\alpha)} \int_{x_1}^{x_2} (x_2-t)^{\alpha-1} \Big[h(t)\varphi(|u(t)|) + |f(t,0)| \Big] dt \\ &\leq \sum_{k=0}^{\lfloor n-\gamma \rfloor} |c_k| \frac{\Phi(\epsilon)}{|\Gamma(k-n+\gamma+1)|} \\ &+ \frac{H(t_1)\varphi(\|u\|) + F(t_1)}{\Gamma(\alpha+1)} \Big[(x_2-a)^\alpha - (x_2-x_1)^\alpha - (x_1-a)^\alpha \Big] \\ &+ \frac{H(t_2)\varphi(\|u\|) + F(t_2)}{\Gamma(\alpha+1)} (x_2-x_1)^\alpha, \end{aligned}$$

where

$$F(t) = \sup\{|f(x,0)| : \ a \leq x \leq t\}, \quad H(t) = \sup\{|h(x)| : \ a \leq x \leq t\},$$

$$\Phi(\epsilon) = \sup\{|(x_2-a)^{k-n+\gamma} - (x_1-a)^{k-n+\gamma}| : \ |x_1 - x_2| < \epsilon\}.$$

Moving forward, by the fact that

$$(x_2-a)^\alpha = (x_1-a)^\alpha + \alpha(x_2-x_1)(\eta-a)^{\alpha-1}, \quad \text{for some } \eta := \eta(x_1,x_2) \in (x_1,x_2)$$

we obtain

$$\begin{aligned}
\left|[\mathscr{F}u](x_2) - [\mathscr{F}u](x_1)\right| &\le \sum_{k=0}^{\lfloor n-\gamma \rfloor} |c_k| \frac{\Phi(\epsilon)}{|\Gamma(k-n+\gamma+1)|} \\
&\quad + \frac{H(t_1)\varphi(\|u\|)+F(t_1)}{\Gamma(\alpha+1)}\Big[\alpha\epsilon(\eta-a)^{\alpha-1}\Big] \\
&\quad + \frac{H(t_2)\varphi(\|u\|)+F(t_2)}{\Gamma(\alpha+1)}\epsilon^\alpha \\
&\le \sum_{k=0}^{\lfloor n-\gamma \rfloor} |c_k| \frac{\Phi(\epsilon)}{|\Gamma(k-n+\gamma+1)|} \\
&\quad + \frac{H(T)\varphi(\|u\|)+F(T)}{\Gamma(\alpha+1)}\Big[\alpha\epsilon(T-a)^{\alpha-1}+\epsilon^\alpha\Big].
\end{aligned} \tag{4.6}$$

Now, by the estimation (4.6) one can infer that the function $\mathscr{F}u$ is continuous on the interval $[a+\theta, T]$ for any $T > a+\theta$. This yields the continuity of $\mathscr{F}u$ on $\mathbb{R}_{a+\theta}$.
In view of (4.6), for any $X \subseteq B_{r_0}$, we see that

$$\begin{aligned}
\omega^T_{a+\theta}(\mathscr{F}X, \varepsilon) &\le \sum_{k=0}^{\lfloor n-\gamma \rfloor} |c_k| \frac{\Phi(\epsilon)}{|\Gamma(k-n+\gamma+1)|} \\
&\quad + \frac{H(T)\varphi(\sup_{u\in X}\|u\|)+F(T)}{\Gamma(\alpha+1)}\Big[\alpha\epsilon(T-a)^{\alpha-1}+\epsilon^\alpha\Big] \\
&\le \sum_{k=0}^{\lfloor n-\gamma \rfloor} |c_k| \frac{\Phi(\epsilon)}{|\Gamma(k-n+\gamma+1)|} \\
&\quad + \frac{H(T)\varphi(r_0)+F(T)}{\Gamma(\alpha+1)}\Big[\alpha\epsilon(T-a)^{\alpha-1}+\epsilon^\alpha\Big]
\end{aligned}$$

which shows that

$$\omega_{a+\theta}(\mathscr{F}X) = \lim_{T\to\infty}\lim_{\epsilon\to 0}\omega^T_{a+\theta}(\mathscr{F}X, \varepsilon) = 0. \tag{4.7}$$

On the other hand, by taking $u, v \in X \subseteq B_{r_0}$ and $x \in \mathbb{R}_{a+\theta}$ one gets

$$\begin{aligned}
\left|[\mathscr{F}u](x) - [\mathscr{F}v](x)\right| &\le \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}\Big|f(t,u(t)) - f(t,v(t))\Big| dt \\
&\le \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}h(t)\varphi\Big(\Big|u(t)-v(t)\Big|\Big) dt \\
&\le \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}h(t)\varphi\Big(\text{diam } X(t)\Big) dt
\end{aligned}$$

which implies that

$$\limsup_{t\to\infty} \operatorname{diam} [\mathscr{F}X](t) \le \lambda \cdot \limsup_{t\to\infty} \varphi\Big(\operatorname{diam} X(t)\Big) \le \lambda \cdot \varphi\Big(\limsup_{t\to\infty} \operatorname{diam} X(t)\Big). \tag{4.8}$$

Now Eqs. (4.7) and (4.8) yield the following

$$\begin{aligned}\mu(\mathscr{F}X) &= \omega_{a+\theta}(\mathscr{F}X) + \limsup_{t\to\infty} \operatorname{diam} [\mathscr{F}X](t) \\ &\le \lambda \cdot \varphi\Big(\limsup_{t\to\infty} \operatorname{diam} X(t)\Big) \le \lambda \cdot \varphi(\mu(X))\end{aligned} \tag{4.9}$$

for any $X \subseteq B_{r_0}$. Now, since μ as given by (4.2) defines a measure of noncompactness on $BC(\mathbb{R}_{a+\theta})$ then, the recent inequality together with (iv) implies that all conditions of Corollary 4.7 are fulfilled. Therefore, Eq. (1.1) has a solution in Banach space $BC(\mathbb{R}_{a+\theta})$. $\square$

Theorem 4.10. *Under the assumptions (i)–(iv), all the solutions of IVP* (1.1) *are uniformly locally attractive in $BC(\mathbb{R}_{a+\theta})$ for $\theta > 0$.*

Proof. First, linking the facts established before, IVP (1.1) has at least one solution in $BC(\mathbb{R}_{a+\theta})$. To prove that all solutions of Eq. (1.1) are uniformly locally attractive in the sense of Definition 2.3, let us consider $B^1_{r_0} = \operatorname{Conv}\mathscr{F}(B_{r_0})$, $B^2_{r_0} = \operatorname{Conv}\mathscr{F}(B^1_{r_0})$ and so on, where B_{r_0} is the ball with radius r_0 as given in (iii) and centre zero in the space $BC(\mathbb{R}_{a+\theta})$. Obviously, one can see that $B^1_{r_0} \subseteq B_{r_0}$ and $B^{n+1}_{r_0} \subseteq B^n_{r_0}$ for $n = 1, 2, \ldots$ and also the sets of this sequence are closed, convex and nonempty. Moreover, in view of the inequality (4.9) we derive that

$$\mu(B^n_{r_0}) \le \lambda^n \varphi^n(\mu(B_{r_0})), \quad \text{for any } n = 1, 2, \ldots.$$

Mixing the fact that $\mu(B_{r_0}) \ge 0$ and condition (iv) with the above inequality we get

$$\lim_{n\to\infty} \mu(B^n_{r_0}) = 0.$$

Thus, applying the definition of measure of noncompactness we infer that the

$$B^\infty_{r_0} := \bigcap_{n=1}^{\infty} B^n_{r_0}$$

is nonempty, bounded, and convex. The set $B^\infty_{r_0}$ is $\mathscr{F}$-invariant and the operator $\mathscr{F}$ is continuous on such set. Furthermore, bringing into mind that $B^\infty_{r_0} \in \ker\mu$ and the characterization of sets belonging to $\ker\mu$ we conclude that all solutions of Eq. (1.1) are uniformly locally attractive in the sense of Definition 2.3. This completes the proof. $\square$

We now study the global attractivity of the solutions of (1.1) utilizing the function μ_* defined on the family of functions included in $BC(\mathbb{R}_{a+\theta})$ by the formula

$$\mu_*(X) = \max\{\omega_{a+\theta}(X), \delta_*(X)\}$$

where

$$\delta_*(X) = \limsup_{t\to\infty} \operatorname{diam} X(t).$$

For a fixed real number c, denote

$$X(t) - c = \{x(t) - c \;\; x \in X\}, \quad \delta_c(X) = \limsup_{t\to\infty} |X(t) - c|,$$

then define function

$$\mu_c(X) = \max\{\omega_{a+\theta}(X), \delta_c(X)\}.$$

It has been shown as in Banás and Goebel [5] that the functions μ_* and μ_c are measures of noncompactness in the space $BC(\mathbb{R}_+)$, and similarly it can be concluded same facts concerning with the functions as above in the setting $BC(\mathbb{R}_{a+\theta})$.

Remark 4.11. The kernel $\ker\mu_*$ of the measure μ_* consists of nonempty and bounded subsets X of $BC(\mathbb{R}_{a+\theta})$ such that functions from X are locally equicontinuous on $\mathbb{R}_{a+\theta}$ and the thickness of the bundle formed by functions from X tends to zero at infinity. This particular characteristic of $\ker\mu_*$ is helpful in order to prove the global attractivity of class of solutions in functional spaces. Similarly, the kernel $\ker\mu_c$ of the measure μ_c consists of nonempty and bounded subsets X of $BC(\mathbb{R}_{a+\theta})$ such that functions from X are locally equicontinuous on $\mathbb{R}_{a+\theta}$ and the thickness of the bundle formed by functions from X around the line $u(x) = c$ (which come closer along a line $u(x) = c$) tends to zero at infinity. This particular characteristic of $\ker\mu_c$ is effective in establishing the global asymptotic attractivity and stability of the solutions for the considered problems.

Suppose the following hypothesis holds:

(v) There exist a real number c and a function $f_c : \mathbb{R}_a \to \mathbb{R}$ defined by

$$f_c(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t,c)dt$$

in which $f_c(x) \to c$ as $x \to \infty$.

Theorem 4.12. *Under the assumptions (i)–(v), all the solutions of IVP* (1.1) *are uniformly globally attractive in* $BC(\mathbb{R}_{a+\theta})$ *for* $\theta > 0$*. Moreover, all the solutions of the equation* (1.1) *are uniformly globally ultimately asymptotically stable to the line* $u(t) = c$ *defined on* $\mathbb{R}_{a+\theta}$*.*

Proof. For the existence of solution, the process of proof is similar to one in Theorem 4.9. To establish the uniformly globally attractivity of solutions we only need to use the last part of proof in previous result. To do this, using Eqs. (4.8) and (4.9) we apply the measure of noncompactness μ_*. That is, for any $X \subseteq B_{r_0}$, we have

$$\begin{aligned} &\delta_*(\mathscr{F}X) \le \lambda \cdot \varphi(\delta_*(X)), \quad \omega_{a+\theta}(\mathscr{F}X) = 0 \\ &\qquad \Longrightarrow \quad \mu_*(\mathscr{F}X) \le \lambda \cdot \varphi(\delta_*(X)) \le \lambda \cdot \varphi(\mu_*(X)). \end{aligned}$$

Now, following the proof of Theorem 4.10 we see that there exists $B_{r_0}^\infty$ which belongs to the family $\ker\mu_*$. Now, taking into account the description of sets belonging to $\ker\mu_*$ we deduce that all solutions of problem (1.1) are uniformly globally ultimately attractive on $\mathbb{R}_{a+\theta}$. Thus we get the required result.

Now, to prove the next claim we show that all the solutions of (1.1) are uniformly globally ultimately asymptotically stable to the line $u(x) = c$. By taking into account our assumptions, for arbitrarily fixed $x \in \mathbb{R}_{a+\theta}$ and for the solution $u \in X$ we deduce the following estimate:

$$
\begin{aligned}
&|(\mathscr{F}u)(x) - c| \\
&\leq \sum_{k=0}^{\lfloor n-\gamma \rfloor} |c_k| \frac{(x-a)^{k-n+\gamma}}{\Gamma(k-n+\gamma+1)} + \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} |f(t,u(t)) - f(t,c)| dt \\
&\quad + |f_c(x) - c| \\
&\leq \sum_{k=0}^{\lfloor n-\gamma \rfloor} |c_k| \frac{(x-a)^{k-n+\gamma}}{\Gamma(k-n+\gamma+1)} + \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} h(t) \varphi(|u(t) - c|) dt \\
&\quad + |f_c(x) - c| \\
&\leq \sum_{k=0}^{\lfloor n-\gamma \rfloor} |c_k| \frac{(x-a)^{k-n+\gamma}}{\Gamma(k-n+\gamma+1)} + \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} h(t) \varphi(\limsup_{t\to\infty} \|X(t) - c\|) dt \\
&\quad + |f_c(x) - c| \\
&= \sum_{k=0}^{\lfloor n-\gamma \rfloor} |c_k| \frac{(x-a)^{k-n+\gamma}}{\Gamma(k-n+\gamma+1)} + \lambda \cdot \varphi(\limsup_{t\to\infty} \|X(t) - c\|) + |f_c(x) - c|
\end{aligned}
$$

for each $x \in \mathbb{R}_{a+\theta}$. Taking limit superior over $x \to \infty$ and using (v) we get

$$
\delta_c(\mathscr{F}X) = \limsup_{x\to\infty} \|(\mathscr{F}X)(x) - c\| \leq \lambda \cdot \varphi(\limsup_{t\to\infty} \|X(t) - c\|) = \lambda \cdot \varphi(\delta_c(X)).
$$

This together with the fact that $\omega_{a+\theta}(\mathscr{F}X) = 0$ (see (4.7)) implies that

$$
\mu_c(\mathscr{F}X) = \max\{\omega_{a+\theta}(\mathscr{F}X), \delta_c(\mathscr{F}X)\} \leq \lambda \cdot \varphi(\delta_c(X)) \leq \lambda \cdot \varphi(\mu_c(X)).
$$

Similar to the previous part, there is a set belonging to the family $\ker\mu_c$. Now, taking into account the description of sets belonging to $\ker\mu_c$ we derive that all solutions of problem (1.1) are uniformly globally ultimately asymptotically stable to the line $u(x) = c$ and so the consequence follows. □

Remark 4.13. We notice that since the results of this section hold in the setting of $BC(\mathbb{R}_{a+\theta})$ for any arbitrarily fixed $\theta > 0$, the existence of the solutions with mentioned properties defined a.e. in $BC(\mathbb{R}_a)$ are established in the corresponding results. Furthermore, for the case $0 < \alpha < 1$, according to the hypothesis (4.3) and the Volterra integral equation (3.1) since the coefficient $c_0 = 0$ then there is no difficulty to obtain all results of this section in the setting of $BC(\mathbb{R}_a)$ without using the parameter θ; this is indeed because of the fact that $A_\theta = 0$ in condition (iii) and so all the conditions are not related to the parameter θ.

5. Examples

Below we indicate two examples for the realization of the abstract theory we have developed in this paper. First, we give an example illustrating the main result of Section 3.

Example 5.1. Consider the fractional differential equation as follows

$$\begin{cases} D_{3^+}^{\frac{1}{2},\frac{1}{2}} y(x) = f(x, y(x)), \quad x > 3, \\ (I_{3^+}^{\frac{3}{4}} y)(3^+) = r \in \mathbb{R}. \end{cases} \tag{5.1}$$

where

$$f(x, y(x)) = \frac{e^{1-x-|y(x)|} \sin y(x)}{(x+1)^{\frac{3}{4}}}.$$

By the continuity of functions involved in f, it is easy to show that condition (C1) holds. On the other hand,

$$|f(x, y(x))| \le \frac{e^{-2}}{(x+1)^{\frac{3}{4}}} \le M(x-3)^{-\beta_1}, \quad x > a,\ y(x) \in C(\mathbb{R}_3, \mathbb{R}),$$

where $M = e^{-2}$, $\beta_1 = \frac{3}{4} \in (\frac{1}{2}, 1)$ and $\mathbb{R}_3 := [3, \infty)$. Therefore, Theorem 3.6 implies that IVP (5.1) admits at least one attractive solution in $C(\mathbb{R}_3, \mathbb{R})$ in the sense of Definition 2.2.

Now, considering the imposed conditions in Section 4 we see that

$$\begin{aligned} |f(t,u) - f(t,v)| &\le \frac{e^{1-t}}{(t+1)^{\frac{3}{4}}} \left| e^{-|u|} \sin u - e^{-|v|} \sin v \right| \\ &\le \frac{e^{1-t}}{(t+1)^{\frac{3}{4}}} \left(e^{-|u|} \left| \sin u - \sin v \right| + |\sin v| \cdot \left| e^{-|u|} - e^{-|v|} \right| \right) \\ &\le \frac{e^{1-t}}{(t+1)^{\frac{3}{4}}} \left(e^{-|u|} + e^{-|\eta|} \right) |u - v| \\ &\le \frac{2e^{1-t}}{(t+1)^{\frac{3}{4}}} |u - v| \\ &:= h(t)\varphi(|u-v|), \text{ for any } u, v \in \mathbb{R},\ t \in \mathbb{R}_3 \end{aligned}$$

where φ is the identity mapping and h is defined by

$$h(t) = \frac{2e^{1-t}}{(t+1)^{\frac{3}{4}}}, \quad t \in \mathbb{R}_3.$$

To complete the verification of (i) using the Euler's Beta function we find that

$$\begin{aligned} \frac{1}{\Gamma(\frac{1}{2})} \int_3^x (x-t)^{-\frac{1}{2}} \frac{2e^{1-t}}{(t+1)^{\frac{3}{4}}} dt &\le \frac{2e^{-2}}{\Gamma(\frac{1}{2})} \int_3^x (x-t)^{-\frac{1}{2}} \frac{1}{(t+1)^{\frac{3}{4}}} dt \\ &\le \frac{2e^{-2}}{\Gamma(\frac{1}{2})} \int_0^x (x-t)^{-\frac{1}{2}} t^{\frac{-3}{4}} dt \\ &\le \frac{2e^{-2}}{\Gamma(\frac{1}{2})} x^{\frac{-1}{4}} \frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{4})}{\Gamma(\frac{3}{4})} \\ &\le \frac{2e^{-2}\Gamma(\frac{1}{4})}{\Gamma(\frac{3}{4})} x^{\frac{-1}{4}} \end{aligned}$$

and hence followed by Euler's reflection formula we get

$$\begin{aligned}\lambda &:= \sup_{x\in\mathbb{R}_3} \frac{1}{\Gamma(\frac{1}{2})}\int_3^x (x-t)^{-\frac{1}{2}}\frac{2e^{1-t}}{(t+1)^{\frac{3}{4}}}dt\\ &\leq \frac{2e^{-2}\Gamma(\frac{1}{4})}{\Gamma(\frac{3}{4})\sqrt[4]{3}}\\ &= \frac{\sqrt{2}e^{-2}\Gamma^2(\frac{1}{4})}{\sqrt[4]{3}\pi}\\ &\cong 0.6084 < 1\end{aligned}$$

which means that (i) is satisfied. Also, $f(t,0)=0$ implies that $\xi=0$ and so the condition (ii) is held. (iii) is obvious. Since $\lambda<1$ and $\varphi=id$, the condition (iv) is also clearly satisfied. Thus, all conditions of Theorems 4.9 and 4.10 are satisfied and all solutions of (5.1) are uniformly locally attractive.
By the following inequalities

$$\begin{aligned}0\leq \lim_{x\to\infty}|f_c(x)| &\leq \lim_{x\to\infty}\frac{e^{-|c|}|\sin c|}{\Gamma(\frac{1}{2})}\int_3^x (x-t)^{-\frac{1}{2}}\frac{e^{1-t}}{(t+1)^{\frac{3}{4}}}dt\\ &\leq \lim_{x\to\infty}\frac{e^{-2-|c|}|\sin c|\Gamma(\frac{1}{4})}{\Gamma(\frac{3}{4})}x^{\frac{-1}{4}}=0\end{aligned}$$

one can observe that $c=\lim_{x\to\infty} f_c(x)=0$ and all solutions of (5.1) are uniformly globally ultimately asymptotically stable to the line $u(x)=0$ (see Theorem 4.12).

Now, to show the effectiveness of main results of Section 4 we provide the following example.

Example 5.2. Consider the following initial value problem of the form

$$\begin{cases} D_{1^+}^{\frac{3}{2},\frac{1}{2}}y(x)=\dfrac{\pi(x-1)\tan^{-1}\frac{y(x)}{\pi}}{x^3+1}, \quad x>1,\\ (I_{1^+}^{\frac{1}{4}}y)(1^+)=c_0\in\mathbb{R},\\ \dfrac{d}{dx}(I_{1^+}^{\frac{1}{4}}y)(1^+)=0.\end{cases} \tag{5.2}$$

First, it is easy to see that

$$\begin{aligned}|f(t,u)-f(t,v)| &\leq \frac{\pi(t-1)}{t^3+1}|\tan^{-1}\frac{u}{\pi}-\tan^{-1}\frac{v}{\pi}|\\ &\leq \frac{\pi^2(t-1)}{(t^3+1)(\pi^2+\eta^2)}|u-v|\\ &\leq h(t)\varphi(|u-v|),\end{aligned}$$

for any $u,v\in\mathbb{R},\ t\in\mathbb{R}_1:=[1,\infty)$, and some $\eta:=\eta_{u,v}\in(u,v), u<v$. Also, we have denoted

$$h(t)=\frac{t-1}{t^3+1},\quad \varphi(x)=\frac{\pi^2x}{\pi^2+1},\quad t\in\mathbb{R}_1,\ x\in\mathbb{R}_+.$$

Moreover, by Cauchy-Schwarz inequality and the convexity of $h^2(t)$ we have

$$\begin{aligned}\int_1^x \sqrt{x-t}\cdot h(t)dt &\leq \left(\int_1^x (x-t)dt\right)^{\frac{1}{2}} \cdot \left(\int_1^x h^2(t)dt\right)^{\frac{1}{2}}\\ &\leq \left(\int_1^x (x-t)dt\right)^{\frac{1}{2}} \cdot \left((x-1)\frac{h^2(x)+h^2(1)}{2}\right)^{\frac{1}{2}}\\ &= \frac{(x-1)^{\frac{5}{2}}}{2(x^3+1)}\\ &=: g(x)\end{aligned} \tag{5.3}$$

where here the Hermite-Hadamard inequality has been utilized (see [23, Theorem 1.5.1]). Now, moving forward, we get

$$\lambda := \sup_{x\in\mathbb{R}_1} \frac{1}{\Gamma(\frac{3}{2})}\int_1^x \sqrt{x-t}\cdot h(t)dt \leq \frac{1}{\Gamma(\frac{3}{2})}\sup_{x\in\mathbb{R}_1} g(x) = \frac{g(x^*)}{\Gamma(\frac{3}{2})} \cong \frac{0.12882}{\frac{\sqrt{\pi}}{2}} < 1$$

where $x^* \cong 6.133$ is the global maximum point of g defined on $\mathbb{R}_1$. Hence, the condition (i) is satisfied. The condition (ii) is also fulfilled since $f(t,0) = 0$, then $\xi = 0$.

To verify hypothesis (iii), take an r_0 holding in

$$r_0 \geq \frac{|c_0|\theta^{\frac{-1}{4}}}{\Gamma(\frac{3}{4})(1-\lambda)}$$

where $\theta > 0$ is arbitrarily fixed. Then, the condition (iii) is satisfied. Concerning with the condition (iv), since $\lambda < 1$ by the definition of control function φ we derive the followings

$$\lim_{n\to\infty}\lambda^n\varphi^n(t) = \lim_{n\to\infty}\left(\frac{\pi^2\lambda}{\pi^2+1}\right)^n(t) = 0 \quad \text{for all } t > 0.$$

Therefore, by Theorem 4.9 the problem (5.2) has at least one solution in $BC(\mathbb{R}_{1+\theta})$ for any fixed $\theta > 0$. Moreover, based on Theorem 4.10, all the solutions of Eq. (5.2) are uniformly locally attractive.

In view of the fact that

$$0 \leq \lim_{x\to\infty}|f_c(x)| \leq \lim_{x\to\infty}\frac{1}{\Gamma(\frac{3}{2})}\int_1^x \sqrt{x-t}|f(t,c)|dt \leq \lim_{x\to\infty}\frac{\pi|\tan^{-1}\frac{c}{\pi}|}{\Gamma(\frac{3}{2})}g(x) = 0$$

one can see that $c = 0$ and then all the solutions of Eq. (5.2) are uniformly globally ultimately asymptotically stable to the line $u(x) = 0$ which is obeyed by Theorem 4.12.

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