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ENDPOINTS OF GENERALIZED BERINDE NONEXPANSIVE MAPPINGS IN HYPERBOLIC SPACES

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Abstract. In this paper, we introduce the notion of generalized Berinde nonexpansive mappings in metric spaces and show that it is weaker than the notion of Berinde nonexpansive mappings and stronger than the notion of semi-nonexpansive mappings. We also obtain the semiclosed principle, an endpoint theorem, and a common endpoint theorem for generalized Berinde nonexpansive mappings. Strong and $\Delta-$convergence theorems of the Ishikawa iteration process for semi-nonexpansive mappings are also discussed.
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1. Introduction

Let E be a nonempty subset of a metric space (X, d). For $x \in X$, we set

$$\operatorname{dist}(x, E) := \inf\{d(x, y) : y \in E\},$$

and

$$R(x, E) := \sup\{d(x, y) : y \in E\}.$$

We denote by $\mathcal{K}(X)$ the family of nonempty compact subsets of X. Let $H(\cdot, \cdot)$ be the Pompeiu-Hausdorff distance on $\mathcal{K}(X)$, that is,

$$H(A, B) = \max\left\{\sup_{a\in A} \operatorname{dist}(a, B), \sup_{b\in B} \operatorname{dist}(b, A)\right\} \text{ for all } A, B \in \mathcal{K}(X).$$

A mapping T from E to $\mathcal{K}(X)$ is called a multi-valued mapping. In particular, if $T(x)$ is a singleton for every x in E then T is called a single-valued mapping. A point $x \in E$ is called a fixed point of T if $x \in T(x)$. We denote by $Fix(T)$ the set of all fixed points of T. A multi-valued mapping $T : E \to \mathcal{K}(X)$ is said to be a contraction if there exists a constant $\lambda \in [0, 1)$ such that

$$H(T(x), T(y)) \leq \lambda d(x, y) \text{ for all } x, y \in E. \tag{1.1}$$

If (1.1) is valid when $\lambda = 1$, then T is called nonexpansive.

Fixed point theory for multi-valued mappings has many useful applications in applied sciences, for instance, in game theory and optimization theory. Thus, it is natural to study the extensions of the known fixed point results for single-valued mappings to the setting of multi-valued mappings. One of the fundamental results in fixed point theory for multi-valued mappings which extends the well-known Banach contraction principle was proved by Nadler [16]. He showed that every multi-valued contraction on a complete metric space always has a fixed point.

In 2007, Berinde and Berinde [2] extended the concept of multi-valued contractions to a general concept of mappings in the following way: a mapping $T : E \to \mathcal{K}(X)$ is called a weak contraction if there exist two constants $\lambda \in [0, 1)$ and $L \in [0, \infty)$ such that

$$H(T(x), T(y)) \leq \lambda d(x, y) + L\text{dist}(x, T(x)) \text{ for all } x, y \in E.$$

In 2019, Bunlue and Suantai [3] introduced the concept of Berinde nonexpansive mappings in the following manner: a multi-valued mapping $T : E \to \mathcal{K}(X)$ is called a Berinde nonexpansive mapping if there exists a constant $\mu \geq 0$ such that

$$H(T(x), T(y)) \leq d(x, y) + \mu\text{dist}(x, T(x)) \text{ for all } x, y \in E.$$

In [3], the authors also obtained fixed point theorems and convergence theorems for Berinde nonexpansive mappings in uniformly convex Banach spaces and Banach spaces which satisfy the Opial's condition.

The concept of endpoints (or strict fixed points) for multi-valued mappings is an important concept which lies between the concept of fixed points for single-valued mappings and the concept of fixed points for multi-valued mappings. In 1986, Corley [6] proved that a maximization with respect to a cone is equivalent to the problem of finding an endpoint of a certain multi-valued mapping. Moreover, Tarafdar and Yuan [29] proved an endpoint theorem and applied it to obtain the existence of Pareto optima for multi-valued mappings in ordered Banach spaces. For more details and further applications of the endpoint theory, the reader is referred to [10, 22, 28].

In 2015, Panyanak [20] and Espínola et al. [8] proved the existence of endpoints for multi-valued nonexpansive mappings in certain classes of Banach spaces. After that, Saejung [23] obtained endpoint theorems for some generalized multi-valued nonexpansive mappings in uniformly convex Banach spaces and Banach spaces which satisfy the Opial's condition. Since then endpoint results for some generalized multi-valued nonexpansive mappings in several classes of metric and Banach spaces have been developed and many papers have appeared (see, e.g., [4, 5, 12, 13, 17, 21]). But, there is no result regarding the existence of endpoints for Berinde nonexpansive mappings.

In this paper, we introduce the class of generalized Berinde nonexpansive mappings and show that it contains the class of Berinde nonexpansive mappings as a proper subclass. We also give sufficient conditions for the existence of endpoints of a generalized Berinde nonexpansive mapping in a uniformly convex hyperbolic space. Moreover, we also prove strong and $\Delta-$convergence theorems of the Ishikawa iteration process for the class of semi-nonexpansive mappings which includes the class of generalized Berinde nonexpansive mappings as well. Our results extend and improve the results in [13, 18, 30] and many others.

2. Preliminaries

Throughout this paper, $\mathbb{N}$ stands for the set of natural numbers and $\mathbb{R}$ stands for the set of real numbers. Let E be a nonempty subset of a metric space (X, d) and $T : E \to \mathcal{K}(X)$ be a multi-valued mapping. A point x in E is called an endpoint (or a strict fixed point) of T if $T(x) = \{x\}$. We denote by $End(T)$ the set of all endpoints of T. A multi-valued mapping $T : E \to \mathcal{K}(X)$ is said to satisfy the endpoint condition [26] if $End(T) = Fix(T)$. A sequence $\{x_n\}$ in E is called an approximate endpoint sequence for T [1] if $\lim_{n\to\infty} R(x_n, T(x_n)) = 0$.

Definition 2.1. A multi-valued mapping $T : E \to \mathcal{K}(X)$ is said to be
(i) quasi-nonexpansive if $Fix(T) \neq \emptyset$ and

$$H(T(x), T(p)) \leq d(x, p) \text{ for all } x \in E \text{ and } p \in Fix(T);$$

(ii) semi-nonexpansive if $End(T) \neq \emptyset$ and

$$H(T(x), T(q)) \leq d(x, q) \text{ for all } x \in E \text{ and } q \in End(T).$$

The following proposition can be found in [19].

Proposition 2.2. *The following statements hold.*
(i) If T is nonexpansive and $Fix(T) \neq \emptyset$, then T is quasi-nonexpansive.
(ii) If T is quasi-nonexpansive and $End(T) \neq \emptyset$, then T is semi-nonexpansive.
(iii) The converse of (ii) is true if T satisfies the endpoint condition.

Now, we give the definition of generalized Berinde nonexpansive mapping.

Definition 2.3. A multi-valued mapping $T : E \to \mathcal{K}(X)$ is said to be generalized Berinde nonexpansive if there exists $\mu \geq 0$ such that

$$H(T(x), T(y)) \leq d(x, y) + \mu R(x, T(x)) \text{ for all } x, y \in E. \tag{2.1}$$

The following proposition is easy to establish.

Proposition 2.4. *The following statements hold.*
(i) If T is Berinde nonexpansive, then T is generalized Berinde nonexpansive.
(ii) If T is generalized Berinde nonexpansive and $End(T) \neq \emptyset$, then T is semi-nonexpansive.

The folowing examples show that the converses of (i) and (ii) in Proposition 2.4 are not true.

Example 2.5. Let $X = \mathbb{R}$, $E = [1, 2]$ and $T : E \to \mathcal{K}(X)$ be defined by

$$T(x) = [x, x + \sqrt{x}] \text{ for all } x \in E.$$

Put $x = 1$ and $y = 2$. Then $T(x) = [1, 2]$ and $T(y) = [2, 2 + \sqrt{2}]$. This implies that

$$H(T(x), T(y)) = \sqrt{2} > 1 = |x - y| + \mu \text{dist}(x, T(x)) \text{ for all } \mu \geq 0.$$

Hence T is not Berinde nonexpansive. Next, we shows that T is generalized Berinde nonexpansive. Choose $\mu = 1$ and let $x, y \in E$. Without loss of generality, we may assume that $x < y$. This implies that

$$\begin{aligned} H(T(x),T(y)) &= (y+\sqrt{y})-(x+\sqrt{x}) \\ &= |x-y|+(\sqrt{y}-\sqrt{x}) \\ &\leq |x-y|+\sqrt{y} \\ &= |x-y|+\mu R(y,T(y)), \end{aligned}$$

and

$$\begin{aligned} H(T(x),T(y)) &= (y+\sqrt{y})-(x+\sqrt{x}) \\ &= |x-y|+(\sqrt{y}-\sqrt{x}) \\ &\leq |x-y|+\sqrt{x} \\ &= |x-y|+\mu R(x,T(x)). \end{aligned}$$

Hence T is generalized Berinde nonexpansive.

Example 2.6. Let $X = \mathbb{R}$, $E = [0,1]$ and $T : E \to \mathcal{K}(X)$ be defined by

$$T(x) = \begin{cases} \left[\left|x(1-x)\sin(\frac{1}{x})\right|, \left|\frac{x}{1+x}\sin(\frac{1}{x})\right|\right] & \text{if } x \neq 0; \\ \{0\} & \text{if } x = 0. \end{cases}$$

It is easy to see that $End(T) = \{0\}$. For $x \in (0,1]$, we have

$$H(T(x),T(0)) = \left|\frac{x}{1+x}\sin(\frac{1}{x})\right| \leq \left|\frac{x}{1+x}\right| \leq |x-0|. \tag{2.2}$$

This implies that T is a semi-nonexpansive mapping. For each $n \in \mathbb{N}$, we set

$$x_n := \frac{1}{2\pi n + \pi/2} \text{ and } y_n := \frac{1}{2\pi n}.$$

From (2.2), we get that $R(x_n, T(x_n)) = x_n - x_n(1-x_n) = x_n^2$. Thus

$$\begin{aligned} \frac{H(T(x_n),T(y_n)) - |x_n - y_n|}{R(x_n,T(x_n))} &= \frac{\frac{x_n}{1+x_n} - (y_n - x_n)}{x_n^2} \\ &= \frac{1}{(1+x_n)x_n} - \frac{(y_n-x_n)}{x_n^2} \\ &= \frac{(2\pi n+\pi/2)^2}{2\pi n+\pi/2+1} - \frac{2\pi n+\pi/2}{4n} \to \infty. \end{aligned}$$

This implies that T is not generalized Berinde nonexpansive.

In 1970, Takahashi [27] introduced the concept of convex metric spaces which is more general than the concept of convexity in Banach spaces. His concept was specialized to hyperbolic spaces by Leuştean [14] in 2007.

Definition 2.7. A hyperbolic space is a metric space (X,d) together with a function $W : X \times X \times [0,1] \to X$ such that for all $x,y,z,w \in X$ and $s,t \in [0,1]$, we have

(i) $d(z, W(x,y,t)) \leq (1-t)d(z,x) + td(z,y)$;
(ii) $d\left(W(x,y,s), W(x,y,t)\right) = |s-t|d(x,y)$;
(iii) $W(x,y,t) = W(y,x,1-t)$;
(iv) $d(W(x,z,t), W(y,w,t)) \leq (1-t)d(x,y) + td(z,w)$.

If $x, y \in X$ and $t \in [0,1]$, then we use the notation $(1-t)x \oplus ty$ for $W(x, y, t)$. A nonempty subset E of X is said to be convex if $(1-t)x \oplus ty \in E$ for all $x, y \in E$ and $t \in [0,1]$. The hyperbolic space (X, d, W) is said to be uniformly convex if for any $r \in (0, \infty)$ and $\varepsilon \in (0,2]$ there exists $\delta \in (0,1]$ such that for all $x, y, z \in X$ with $d(x, z) \leq r$, $d(y, z) \leq r$ and $d(x, y) \geq r\varepsilon$, we have

$$d\left(\frac{1}{2}x \oplus \frac{1}{2}y, z\right) \leq (1-\delta)r.$$

A function $\eta : (0, \infty) \times (0, 2] \to (0, 1]$ providing such a $\delta := \eta(r, \varepsilon)$ for given $r \in (0, \infty)$ and $\varepsilon \in (0,2]$ is called a modulus of uniform convexity. Moreover, we call η monotone if it is a nonincreasing function of r for every fixed ε.

Definition 2.8. Let (X, d) be a uniformly convex hyperbolic space. For each $r \in (0, \infty)$ and $\varepsilon \in (0, 2]$, we define

$$\Psi(r, \varepsilon) := \inf\left\{\frac{1}{2}d^2(x, z) + \frac{1}{2}d^2(y, z) - d^2(\frac{1}{2}x \oplus \frac{1}{2}y, z)\right\},$$

where the infimum is taken over all $x, y, z \in X$ such that $d(x, z) \leq r$, $d(y, z) \leq r$, and $d(x, y) \geq r\varepsilon$. We say that (X, d) is 2-uniformly convex if

$$c_M := \inf\left\{\frac{\Psi(r, \varepsilon)}{r^2\varepsilon^2} : r \in (0, \infty), \varepsilon \in (0, 2]\right\} > 0.$$

In [13], the authors prove that

$$d^2((1-t)x \oplus ty, z) \leq (1-t)d^2(x, z) + td^2(y, z) - 4c_M t(1-t)d^2(x, y), \tag{2.3}$$

for all $x, y, z \in X$ and $t \in [0, 1]$.

Example 2.9. (1) If X is a CAT(0) space, then it is a 2-uniformly convex hyperbolic space with $c_M = \frac{1}{4}$ (see [9]).

(2) Every uniformly convex Banach space is a 2-uniformly convex hyperbolic space. To see this, we suppose that X is a uniformly convex Banach space. Let $r \in (0, \infty)$ and $\varepsilon \in (0, 2]$ and $x, y, z \in X$ be such that $\|x - z\| \leq r$, $\|y - z\| \leq r$ and $\|x - y\| \geq r\varepsilon$. Set $u = \frac{x-z}{r\varepsilon}$ and $v = \frac{y-z}{r\varepsilon}$. Then $\|u\| \leq \frac{1}{\varepsilon}$, $\|v\| \leq \frac{1}{\varepsilon}$ and $\|u - v\| \geq 1$. By Theorem 2 of [31], there exists a continuous, strictly increasing and convex function $g : [0, \infty) \to [0, \infty)$ such that $g(0) = 0$ and

$$\left\|\frac{1}{2}u + \frac{1}{2}v\right\|^2 \leq \frac{1}{2}\|u\|^2 + \frac{1}{2}\|v\|^2 - \frac{1}{4}g(\|u - v\|).$$

This implies

$$\frac{g(1)}{4} \leq \frac{g(\|u - v\|)}{4} \leq \frac{1}{2}\|u\|^2 + \frac{1}{2}\|v\|^2 - \left\|\frac{1}{2}u + \frac{1}{2}v\right\|^2,$$

which yields

$$\frac{g(1)}{4} \leq \frac{1}{2}\left\|\frac{x - z}{r\varepsilon}\right\|^2 + \frac{1}{2}\left\|\frac{y - z}{r\varepsilon}\right\|^2 - \left\|\frac{x - z}{2r\varepsilon} + \frac{y - z}{2r\varepsilon}\right\|^2.$$

Thus, $\frac{g(1)}{4} \leq \frac{\Psi(r,\varepsilon)}{r^2\varepsilon^2}$ and hence $0 < \frac{g(1)}{4} \leq c_M$. Therefore, X is a 2-uniformly convex hyperbolic space.

From now on, X stands for a complete 2-uniformly convex hyperbolic space with a monotone modulus of uniform convexity. Let E be a nonempty subset of X and $\{x_n\}$ be a bounded sequence in X. The asymptotic radius of $\{x_n\}$ relative to E is defined by

$$r(E,\{x_n\}) := \inf\big\{ \limsup_{n\to\infty} d(x_n,x) : x \in E\big\}.$$

The asymptotic center of $\{x_n\}$ relative to E is the set

$$A(E,\{x_n\}) := \big\{x \in E : \limsup_{n\to\infty} d(x_n,x) = r(E,\{x_n\})\big\}.$$

It is known from [15] that if E is a nonempty closed convex subset of X, then $A(E,\{x_n\})$ consists of exactly one point.

Now, we give the concept of $\Delta-$convergence and collect some of its basic properties.

Definition 2.10. Let E be a nonempty closed convex subset of X and $x \in E$. Let $\{x_n\}$ be a bounded sequence in X. We say that $\{x_n\}$ $\Delta-$converges to x if $A(E,\{u_n\}) = \{x\}$ for every subsequence $\{u_n\}$ of $\{x_n\}$. In this case we write $x_n \stackrel{\Delta}{\longrightarrow} x$ and call x the $\Delta-$limit of $\{x_n\}$.

It is known from [11] that every bounded sequence in X has a $\Delta-$convergent subsequence. The following fact can be found in [7].

Lemma 2.11. *Let E be a nonempty closed convex subset of X and $\{x_n\}$ be a bounded sequence in X. If $A(E,\{x_n\}) = \{x\}$ and $\{u_n\}$ is a subsequence of $\{x_n\}$ with $A(E,\{u_n\}) = \{u\}$ and the sequence $\{d(x_n,u)\}$ converges, then $x = u$.*

Definition 2.12. Let E be a nonempty closed convex subset of X and $T : E \to \mathcal{K}(E)$. Let I be the identity mapping on E. We say that

(i) T is continuous if $H(T(x_n),T(x)) \to 0$ whenever $x_n \to x$;

(ii) $I - T$ is semiclosed if for any sequence $\{x_n\}$ in E such that $x_n \stackrel{\Delta}{\longrightarrow} x$ and $R(x_n,T(x_n)) \to 0$, one has $T(x) = \{x\}$.

Lemma 2.13. *Let E be a nonempty subset of X and $T : E \to \mathcal{K}(E)$. Then the following statements hold.*

(i) If E is convex and T is semi-nonexpansive, then $End(T)$ is convex.

(ii) If E is closed and convex and $I - T$ is semiclosed, then $End(T)$ is closed.

Proof. (i) Let $x, y \in End(T)$ and $z = \alpha x \oplus (1-\alpha)y$ for some $\alpha \in (0,1)$. We show that $z \in End(T)$. Suppose there exists $w \in T(z)$ such that $w \neq z$. Let $u = \frac{1}{2}z \oplus \frac{1}{2}w$. Then by (2.3) we have

$$\begin{aligned} d^2(x,u) &\leq \frac{1}{2}d^2(x,z) + \frac{1}{2}d^2(x,w) - c_M d^2(z,w) \\ &< \frac{1}{2}d^2(x,z) + \frac{1}{2}H^2(T(x),T(z)) \\ &\leq d^2(x,z). \end{aligned}$$

Thus, $d(x,u) < d(x,z)$. Similarly, we can show that $d(y,u) < d(y,z)$. These imply that

$$d(x,y) \leq d(x,u) + d(u,y) < d(x,z) + d(z,y) = d(x,y),$$

a contradiction. Hence, $T(z) = \{z\}$ and therefore $End(T)$ is convex.

(ii) Let $\{x_n\}$ be a sequence in $End(T)$ such that $\lim_{n\to\infty} x_n = x$. Then $R(x_n, T(x_n)) = 0$ for all $n \in \mathbb{N}$. It follows from the semiclosedness of $I - T$ that $T(x) = \{x\}$, and hence $x \in End(T)$. This shows that $End(T)$ is closed. □

3. Endpoint theorems

This section is begun by proving the semiclosed principle for generalized Berinde nonexpansive mappings in uniformly convex hyperbolic spaces. Notice that it is an extension of Lemma 4.6 in [20].

Theorem 3.1. *Let E be a nonempty closed convex subset of X, and I the identity mapping on E, and $T : E \to \mathcal{K}(E)$ a generalized Berinde nonexpansive mapping. Then $I - T$ is semiclosed.*

Proof. Let $\{x_n\}$ be a sequence in E such that $x_n \xrightarrow{\Delta} x$ and $R(x_n, T(x_n)) \to 0$. For each $n \in \mathbb{N}$, we can choose $y_n \in T(x_n)$ and $z_n \in T(x)$ such that

$$d(x_n, y_n) = R(x_n, T(x_n)) \text{ and } d(y_n, z_n) = \operatorname{dist}(y_n, T(x)).$$

Since $T(x)$ is compact, there exists a subsequence $\{z_{n_k}\}$ of $\{z_n\}$ such that $\lim_{k\to\infty} z_{n_k} = v$ for some $v \in T(x)$. By (2.1), we have

$$\begin{aligned} d(x_{n_k}, v) &\leq d(x_{n_k}, y_{n_k}) + d(y_{n_k}, z_{n_k}) + d(z_{n_k}, v) \\ &\leq R(x_{n_k}, T(x_{n_k})) + H(T(x_{n_k}), T(x)) + d(z_{n_k}, v) \\ &\leq (1+\mu)R(x_{n_k}, T(x_{n_k})) + d(x_{n_k}, x) + d(z_{n_k}, v). \end{aligned}$$

This implies that $\limsup_{k\to\infty} d(x_{n_k}, v) \leq \limsup_{k\to\infty} d(x_{n_k}, x)$. Therefore, $v \in A(E, \{x_{n_k}\}) = \{x\}$ and hence $x = v \in T(x)$. Let $w \in T(x)$. For each k, there exists u_{n_k} in $T(x_{n_k})$ such that $d(w, u_{n_k}) = \operatorname{dist}(w, T(x_{n_k}))$. By (2.1), we have

$$\begin{aligned} d(x_{n_k}, w) &\leq d(x_{n_k}, u_{n_k}) + d(u_{n_k}, w) \\ &\leq R(x_{n_k}, T(x_{n_k})) + H(T(x_{n_k}), T(x)) \\ &\leq (1+\mu)R(x_{n_k}, T(x_{n_k})) + d(x_{n_k}, x). \end{aligned}$$

This implies that $\limsup_{k\to\infty} d(x_{n_k}, w) \leq \limsup_{k\to\infty} d(x_{n_k}, x)$, and hence $w \in A(E, \{x_{n_k}\}) = \{x\}$. Therefore $w = x$ for all $w \in T(x)$. Thus $T(x) = \{x\}$. □

As a consequence of Theorem 3.1, we can obtain the following result. Notice that it is an extension of Theorem 3.4 in [17].

Theorem 3.2. *Let E be a nonempty closed convex subset of X and $T : E \to \mathcal{K}(E)$ a generalized Berinde nonexpansive mapping. Then T has an endpoint if and only if T has a bounded approximate endpoint sequence in E.*

Proof. The necessity is clear. For the sufficiency, we suppose that T has a bounded approximate endpoint sequence $\{x_n\}$ in E. As we have observed, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \xrightarrow{\Delta} x \in E$. By Theorem 3.1, x is an endpoint of T. □

We also obtain a common endpoint theorem as the following result.

Theorem 3.3. *Let E be a nonempty closed convex subset of X and let $S, T : E \to \mathcal{K}(E)$ be two generalized Berinde nonexpansive mappings. If S has a bounded approximate endpoint sequence in $End(T)$, then S and T have a common endpoint.*

Proof. Let $\{x_n\}$ be a bounded approximate endpoint sequence for S in $End(T)$. Then, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \xrightarrow{\Delta} x \in E$. By Theorem 3.1, $x \in End(S)$. Let $w \in T(x)$. Then

$$\begin{aligned} d(w, x_{n_k}) &= \operatorname{dist}(w, T(x_{n_k})) \\ &\leq H(T(x), T(x_{n_k})) \\ &\leq d(x, x_{n_k}) + \mu R(x_{n_k}, T(x_{n_k})). \end{aligned}$$

This implies that $\limsup_{k\to\infty} d(w, x_{n_k}) \leq \limsup_{k\to\infty} d(x, x_{n_k})$, and hence $w \in A(E, \{x_{n_k}\}) = \{x\}$. Therefore $w = x$ for all $w \in T(x)$. Thus $x \in End(T)$. □

The following example shows that Theorem 3.3 may not be true if S does not have an approximate endpoint sequence in $End(T)$.

Example 3.4. Let $X = \mathbb{R}$, $E = [0, 1]$ and $S : E \to \mathcal{K}(E)$ be defined by

$$S(x) = \left[0, \frac{x}{2}\right] \text{ for all } x \in E.$$

Let $T : E \to \mathcal{K}(E)$ be defined by

$$T(x) = [x, 1] \text{ for all } x \in E.$$

Then S and T are generalized Berinde nonexpansive with $End(S) = \{0\}$ and $End(T) = \{1\}$. Notice that S does not have an approximate endpoint sequence in $End(T)$, and vice versa. Obviously, S and T do not have a common endpoint.

4. Convergence theorems

In this section, we prove strong and $\Delta-$convergence theorems of the Ishikawa iteration process for semi-nonexpansive mappings. Let E be a nonempty convex subset of X, and $\{\alpha_n\}$, $\{\beta_n\}$ be sequences in $[0, 1]$, and $T : E \to \mathcal{K}(E)$ be a multi-valued mapping. The sequence of Ishikawa iteration is defined by $x_1 \in E$,

$$y_n = (1 - \beta_n)x_n \oplus \beta_n z_n, \ \ n \in \mathbb{N},$$

where $z_n \in T(x_n)$ such that $d(x_n, z_n) = R(x_n, T(x_n))$, and

$$x_{n+1} = (1 - \alpha_n)x_n \oplus \alpha_n z'_n, \ \ n \in \mathbb{N}, \tag{4.1}$$

where $z'_n \in T(y_n)$ such that $d(y_n, z'_n) = R(y_n, T(y_n))$.

A sequence $\{x_n\}$ in X is said to be Fejér monotone with respect to E if

$$d(x_{n+1}, p) \leq d(x_n, p) \text{ for all } p \in E \text{ and } n \in \mathbb{N}.$$

The following lemma shows that the sequence of Ishikawa iteration defined by (4.1) is Fejér monotone with respect to the endpoint set of semi-nonexpansive mapping.

Lemma 4.1. *Let E be a nonempty convex subset of X and $T : E \to \mathcal{K}(E)$ a semi-nonexpansive mapping. Let $\{x_n\}$ be the sequence of Ishikawa iteration defined by (4.1). Then $\{x_n\}$ is Fejér monotone with respect to $End(T)$.*

Proof. Let $p \in End(T)$. For each $n \in \mathbb{N}$, we have

$$\begin{aligned} d(y_n, p) &\leq (1-\beta_n)d(x_n, p) + \beta_n d(z_n, p) \\ &\leq (1-\beta_n)d(x_n, p) + \beta_n H(T(x_n), T(p)) \\ &\leq d(x_n, p). \end{aligned}$$

This implies that

$$\begin{aligned} d(x_{n+1}, p) &\leq (1-\alpha_n)d(x_n, p) + \alpha_n d(z_n', p) \\ &\leq (1-\alpha_n)d(x_n, p) + \alpha_n H(T(y_n), T(p)) \\ &\leq (1-\alpha_n)d(x_n, p) + \alpha_n d(y_n, p) \\ &\leq d(x_n, p). \end{aligned}$$

Thus $\{x_n\}$ is Fejér monotone with respect to $End(T)$. □

The following fact can be found in [5, 24].

Lemma 4.2. *Let E be a nonempty closed subset of X and $\{x_n\}$ a Fejér monotone sequence with respect to E. Then $\{x_n\}$ converges strongly to an element of E if and only if $\lim_{n\to\infty} dist(x_n, E) = 0$.*

The following fact is also needed.

Lemma 4.3. *Let E be a nonempty closed convex subset of X and $T : E \to \mathcal{K}(E)$ a mapping such that $I - T$ is semiclosed. If $\{x_n\}$ is a bounded sequence in E such that $\lim_{n\to\infty} R(x_n, T(x_n)) = 0$ and $\{d(x_n, v)\}$ converges for all $v \in End(T)$, then $\omega_w(x_n) \subseteq End(T)$. Here $\omega_w(x_n) := \bigcup A(E, \{u_n\})$ where the union is taken over all subsequences $\{u_n\}$ of $\{x_n\}$. Moreover, $\omega_w(x_n)$ consists of exactly one point.*

Proof. Let $u \in \omega_w(x_n)$, then there exists a subsequence $\{u_n\}$ of $\{x_n\}$ such that $A(E, \{u_n\}) = \{u\}$. Since $\{u_n\}$ is bounded, there exists a subsequence $\{v_n\}$ of $\{u_n\}$ such that $v_n \xrightarrow{\Delta} v \in E$. It follows from Lemma 2.11 and the semiclosedness of $I - T$ that $u = v \in End(T)$, which implies $\omega_w(x_n) \subseteq End(T)$. Next, we show that $\omega_w(x_n)$ consists of exactly one point. Let $\{u_n\}$ be a subsequence of $\{x_n\}$ with $A(E, \{u_n\}) = \{u\}$ and let $A(E, \{x_n\}) = \{x\}$. Since $u \in \omega_w(x_n) \subseteq End(T)$, $\{d(x_n, u)\}$ converges. By Lemma 2.11, $x = u$. This completes the proof. □

Now, we prove $\Delta-$convergence theorem.

Theorem 4.4. *Let E be a nonempty closed convex subset of X and $T : E \to \mathcal{K}(E)$ a semi-nonexpansive mapping such that $I - T$ is semiclosed. Let $\alpha_n, \beta_n \in [a, b] \subset (0, 1)$ and $\{x_n\}$ be the sequence of Ishikawa iteration defined by (4.1). Then $\{x_n\}$ $\Delta-$converges to an endpoint of T.*

Proof. Let $p \in End(T)$. It follows from (2.3) that

$$\begin{aligned} d^2(y_n,p) &\leq (1-\beta_n)d^2(x_n,p)+\beta_n d^2(z_n,p)-4c_M\beta_n(1-\beta_n)d^2(x_n,z_n) \\ &\leq (1-\beta_n)d^2(x_n,p)+\beta_n H^2(T(x_n),T(p))-4c_M\beta_n(1-\beta_n)d^2(x_n,z_n) \\ &\leq d^2(x_n,p)-4c_M\beta_n(1-\beta_n)d^2(x_n,z_n), \end{aligned}$$

which yields

$$\begin{aligned} d^2(x_{n+1},p) &\leq (1-\alpha_n)d^2(x_n,p)+\alpha_n d^2(z'_n,p)-4c_M\alpha_n(1-\alpha_n)d^2(x_n,z'_n) \\ &\leq (1-\alpha_n)d^2(x_n,p)+\alpha_n H^2(T(y_n),T(p))-4c_M\alpha_n(1-\alpha_n)d^2(x_n,z'_n) \\ &\leq (1-\alpha_n)d^2(x_n,p)+\alpha_n d^2(y_n,p) \\ &\leq d^2(x_n,p)-4c_M\alpha_n\beta_n(1-\beta_n)d^2(x_n,z_n). \end{aligned}$$

Thus

$$\sum_{n=1}^{\infty} a^2(1-b)d^2(x_n,z_n) \leq \sum_{n=1}^{\infty}\alpha_n\beta_n(1-\beta_n)d^2(x_n,z_n) < \infty. \tag{4.2}$$

This implies that $\lim_{n\to\infty} d^2(x_n,z_n)=0$, and hence

$$\lim_{n\to\infty} R(x_n,T(x_n)) = \lim_{n\to\infty} d(x_n,z_n)=0. \tag{4.3}$$

By Lemma 4.1, $\{d(x_n,v)\}$ converges for all $v \in End(T)$. By Lemma 4.3, $\omega_w(x_n)$ consists of exactly one point and is contained in $End(T)$. This shows that $\{x_n\}$ $\Delta-$converges to an element of $End(T)$. □

As a consequence of Theorems 3.1 and 4.4, we can obtain the following result.

Corollary 4.5. *Let E be a nonempty closed convex subset of X and $T : E \to \mathcal{K}(E)$ a generalized Berinde nonexpansive mapping such that $End(T) \neq \emptyset$. Let $\alpha_n, \beta_n \in [a,b] \subset (0,1)$ and $\{x_n\}$ be the sequence of Ishikawa iteration defined by (4.1). Then $\{x_n\}$ $\Delta-$converges to an endpoint of T.*

Next, we prove strong convergence theorems. Recall that a mapping $T : E \to \mathcal{K}(E)$ is said to satisfy condition (J) if $End(T) \neq \emptyset$ and there exists a nondecreasing function $g : [0,\infty) \to [0,\infty)$ with $g(0)=0$, $g(r)>0$ for $r \in (0,\infty)$ such that

$$R(x,T(x)) \geq g(\text{dist}(x,End(T))) \text{ for all } x \in E.$$

The mapping T is said to be semicompact if for any sequence $\{x_n\}$ in E such that

$$\lim_{n\to\infty} R(x_n,T(x_n))=0,$$

there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\lim_{k\to\infty} x_{n_k} = q \in E$.

The following fact is also needed.

Lemma 4.6. ([25]) *Let $\{\alpha_n\},\{\beta_n\}$ be two real sequences in $[0,1)$ such that $\beta_n \to 0$ and $\sum \alpha_n\beta_n = \infty$. Let $\{\gamma_n\}$ be a nonnegative real sequence such that*

$$\sum \alpha_n\beta_n(1-\beta_n)\gamma_n < \infty.$$

Then $\{\gamma_n\}$ has a subsequence which converges to zero.

Theorem 4.7. *Let E be a nonempty closed convex subset of X and $T : E \to \mathcal{K}(E)$ a semi-nonexpansive mapping such that $I - T$ is semiclosed. Let $\alpha_n, \beta_n \in [a, b] \subset (0, 1)$ and $\{x_n\}$ be the sequence of Ishikawa iteration defined by (4.1). If T satisfies condition (J), then $\{x_n\}$ converges strongly to an endpoint of T.*

Proof. By Lemma 2.13, $End(T)$ is closed. Since T satisfies condition (J), by (4.3) we get that $\lim_{n\to\infty} \operatorname{dist}(x_n, End(T)) = 0$. By Lemma 4.1, $\{x_n\}$ is Fejér monotone with respect to $End(T)$. The conclusion follows from Lemma 4.2. □

Theorem 4.8. *Let E be a nonempty convex subset of X and $T : E \to \mathcal{K}(E)$ a semi-nonexpansive mapping. Let $\alpha_n, \beta_n \in [0, 1)$ be such that $\beta_n \to 0$ and $\sum \alpha_n\beta_n = \infty$ and let $\{x_n\}$ be the sequence of Ishikawa iteration defined by (4.1). If T is semicompact and continuous, then $\{x_n\}$ converges strongly to an endpoint of T.*

Proof. From (4.2), we get that

$$\sum_{n=1}^{\infty} \alpha_n\beta_n(1 - \beta_n)d^2(x_n, z_n) < \infty.$$

By Lemma 4.6, there exists a subsequence $\{d^2(x_{n_k}, z_{n_k})\}$ of $\{d^2(x_n, z_n)\}$ such that $\lim_{k\to\infty} d^2(x_{n_k}, z_{n_k}) = 0$, and hence

$$\lim_{k\to\infty} R(x_{n_k}, T(x_{n_k})) = \lim_{k\to\infty} d(x_{n_k}, z_{n_k}) = 0. \tag{4.4}$$

Since T is semicompact, by passing to a subsequence, we may assume that $x_{n_k} \to q \in E$. Since T is continuous,

$$\operatorname{dist}(q, T(q)) \le d(q, x_{n_k}) + \operatorname{dist}(x_{n_k}, T(x_{n_k})) + H(T(x_{n_k}), T(q)) \to 0 \text{ as } k \to \infty.$$

This implies that $q \in T(q)$. Let $v \in T(q)$. For each k, there exists v_{n_k} in $T(x_{n_k})$ such that $d(v, v_{n_k}) = \operatorname{dist}(v, T(x_{n_k}))$. It follows from (4.4) and the continuity of T that

$$\begin{aligned} d(q, v) &\le d(q, x_{n_k}) + d(x_{n_k}, v_{n_k}) + d(v_{n_k}, v) \\ &\le d(q, x_{n_k}) + R(x_{n_k}, T(x_{n_k})) + H(T(x_{n_k}), T(q)) \to 0 \text{ as } k \to \infty. \end{aligned}$$

Thus $v = q$ for all $v \in T(q)$. Therefore $q \in End(T)$. By Lemma 4.1, $\lim_{n\to\infty} d(x_n, q)$ exists and hence q is the strong limit of $\{x_n\}$. □

The following result shows that if T is generalized Berinde nonexpansive, then the continuity of T in Theorem 4.8 can be omitted.

Theorem 4.9. *Let E be a nonempty convex subset of X and $T : E \to \mathcal{K}(E)$ a generalized Berinde nonexpansive mapping such that $End(T) \neq \emptyset$. Let $\alpha_n, \beta_n \in [0, 1)$ be such that $\beta_n \to 0$ and $\sum \alpha_n\beta_n = \infty$ and let $\{x_n\}$ be the sequence of Ishikawa iteration defined by (4.1). If T is semicompact, then $\{x_n\}$ converges strongly to an endpoint of T.*

Proof. As in the proof of Theorem 4.8, we can show that there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\lim_{k\to\infty} R(x_{n_k}, T(x_{n_k})) = 0 \text{ and } \lim_{k\to\infty} x_{n_k} = q \in E. \tag{4.5}$$

Since T is generalized Berinde nonexpansive, there exists $\mu \geq 0$ such that

$$H(T(x_{n_k}), T(q)) \leq d(x_{n_k}, q) + \mu R(x_{n_k}, T(x_{n_k})) \text{ for all } k \in \mathbb{N}. \tag{4.6}$$

This implies that

$$\begin{aligned} \operatorname{dist}(q, T(q)) &\leq d(q, x_{n_k}) + \operatorname{dist}(x_{n_k}, T(x_{n_k})) + H(T(x_{n_k}), T(q)) \\ &\leq 2d(x_{n_k}, q) + (1+\mu)R(x_{n_k}, T(x_{n_k})) \ \to 0 \ \text{ as } \ k \to \infty. \end{aligned}$$

Thus $q \in T(q)$. Let $w \in T(q)$. For each k, there exists w_{n_k} in $T(x_{n_k})$ such that $d(w, w_{n_k}) = \operatorname{dist}(w, T(x_{n_k}))$. By (4.5) and (4.6), we have

$$\begin{aligned} d(x_{n_k}, w) &\leq d(x_{n_k}, w_{n_k}) + d(w_{n_k}, w) \\ &\leq R(x_{n_k}, T(x_{n_k})) + H(T(x_{n_k}), T(q)) \\ &\leq (1+\mu)R(x_{n_k}, T(x_{n_k})) + d(x_{n_k}, q) \to 0 \text{ as } k \to \infty. \end{aligned}$$

Thus $w = q$, and hence $q \in End(T)$. By Lemma 4.1, $\lim_{n\to\infty} d(x_n, q)$ exists and hence q is the strong limit of $\{x_n\}$. □

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