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MILD SOLUTIONS FOR NONLINEAR IMPULSIVE FUNCTIONAL DIFFERENTIAL INCLUSIONS WITH NONLOCAL CONDITIONS

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Abstract. In this paper we investigate the existence of mild solutions for a class of first-order nonlinear impulsive functional differential inclusions with nonlocal condition in Banach spaces. Sufficient condition for the existence is obtained with the help of Martelli's fixed point theorem. We generalize some relevant results and give an example to illustrate the main result.

Key Words and Phrases: Impulsive functional differential inclusions, nonlocal conditions, mild solutions, fixed point theorem.
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1. Introduction

In this paper, we will study a class of nonlinear impulsive functional differential inclusions in Banach spaces described in the form

$$\begin{cases} \dfrac{d[x(t)-g(t,x_t)]}{dt} \in A(t)x(t)+F(t,x_t),\ t\in J\backslash\{t_1,\dots,t_m\}, \\ \Delta x(t_k)=I_k(x(t_k)),\ k=1,\dots,m, \\ x(t)+(w(x_{\eta_1},x_{\eta_2},\dots,x_{\eta_p}))(t)=\phi(t),\ t\in[-r,0], \end{cases} \tag{1.1}$$

where $J=[0,T],\ T>0,\ 0<t_1<t_2<\cdots<t_m<T,\ 0\le\eta_1<\eta_2<\cdots<\eta_p\le T$ are real numbers, $m,\ p\in N,\ A(t):X\to X$ is linear closed operator in a real separable Banach space $X,\ x_t:[-r,0]\to X,\ x_t(\theta)=x(t+\theta)\in D$, which will be defined in preliminaries, $\Delta x(t_k)=x(t_k^+)-x(t_k^-),\ x(t_k^+)=\lim_{\varepsilon\to0^+}x(t_k+\varepsilon)$, $x(t_k^-)=\lim_{\varepsilon\to0^+}x(t_k-\varepsilon),\ g:J\times D\to X,\ F:J\times D\to P(X)$ is a multivalued map, $P(X)$ is the family of all nonempty subsets of X, $I_k:X\to X\ (k=1,\dots,m)$, $w:D^p\to D\ (D^p=D\times D\times\cdots\times D,\ p-$times$)$, $\phi\in D$, $0<r<+\infty$. These mappings satisfy some conditions which will be specified later.

Recently, the theory of impulsive differential equations or inclusions has become an active area of investigation due to their applications in the fields of mechanics, electrical engineering, medicine biology, ecology, and so on, see [6, 7, 11, 19]. Nonlocal conditions are motivated by physical problems. For the importance of nonlocal conditions in different fields we refer to [4].

Fixed point theorems play a major role in discussing existence results for impulsive differential inclusions. For example, with the help of the fixed point principle regarding a condensing mapping with some measure of noncompactness, an existence result for impulsive neutral evolution differential inclusions has been given by the authors in [13]. By means of the nonlinear alternative for multivalued contractions maps in Frechet spaces due to Frigon, sufficient conditions are given to investigate the existence of mild solutions on a semi-infinite interval for impulsive neutral functional differential evolution inclusions with infinite delay by the authors in [5]. There are many other methods such as in [2, 8, 9, 14, 15, 17].

Motivated by the previous mentioned works, we will study the existence of solutions for system (1.1) by Martelli's fixed point theorem. The rest of this paper is organized as follows. In Section 2, we introduce briefly some notations and necessary preliminaries. In Section 3, we prove the existence result of solutions for system (1.1). Finally, in Section 4, an example is presented to illustrate the main result.

2. Preliminaries

In the paper, X is a real separable Banach space with norm $\|\cdot\|$. Next, we introduce notations, definitions, and preliminary facts from multivalued analysis, which are useful for the development of this paper.

Let Z be a subset of X. We denote $P(X) = \{Z \subset X : Z \neq \emptyset\}$, $P_{cv}(X) = \{Z \in P(X) : Z \text{ is convex}\}$, $P_{cp}(X) = \{Z \in P(X) : Z \text{ is compact}\}$, $P_{cv,cp}(X) = P_{cv}(X) \cap P_{cp}(X)$, and so forth.

Let $C(J, X)$ denote the Banach space of all continuous functions from J into X with the norm $\|x\|_\infty = \max\{\|x(t)\| : t \in J\}$.

Let $L^1(J, X) = \{x : J \to X |\ \|x\| : J \to [0, +\infty) \text{ is Lebesgue integrable}\}$, then $L^1(J, X)$ is a Banach space with the norm $\|x\|_{L^1} = \int_0^T \|x(t)\|\, dt$.

$$D = \{x : [-r, 0] \to X \mid x(t) \text{ is continuous everywhere except for a finite number of points } \widetilde{t}, \text{ at which } x(\widetilde{t}^-),\ x(\widetilde{t}^+) \text{ exist, and } x(\widetilde{t}^-) = x(\widetilde{t})\},$$

for any $t \in J$, we denote the element of D by x_t defined by $x_t(\theta) = x(t+\theta)$, $\theta \in [-r, 0]$. Here $x_t(\cdot)$ represents the history of the state from time $t - r$ up to the present time t. For $x_t \in D$, the norm of x_t is defined by $\|x_t\|_D = \sup\{\|x(t+\theta)\| : \theta \in [-r, 0]\}$.

$$PC([-r, T], X) = \{x : [-r, T] \to X \mid x(t) \text{ is continuous everywhere except for some } t_k \text{ at which } x(t_k^-),\ x(t_k^+) \text{ exist and } x(t_k^-) = x(t_k),\ k = 1, \dots, m\},$$

for convenience, we set $\Omega = PC([-r, T], X)$. Obviously, Ω is a Banach space with norm $\|x\|_\Omega = \sup\{\|x(t)\| : t \in [-r, T]\}$.

Let $L(X) = \{M : X \to X | M \text{ is linear bounded}\}$, and for $M \in L(X)$, we define $\|M\|_{L(X)} = \inf\{r > 0 : \forall\, x \in X,\ \|M(x)\| < r\|x\|\}$, then $(L(X), \|\cdot\|_{L(X)})$ is a Banach space.

In the paper, we assume that the part $\{A(t)\}_{t \in J}$ of (1.1) is a family of linear closed and bounded operators in X, with domain $D(A(t)) \subset X$ for all $t \in J$ and dense in X.

Definition 2.1. The operator $\{U(t,s)\}_{0\leq s\leq t\leq T}\subset L(X)$ is called the evolution operator generated by $\{A(t),\ t\in J\}$ if the following conditions hold:

(i) $U(s,s)=I$,

(ii) $U(t,r)U(r,s)=U(t,s)$ for $0\leq s\leq r\leq t\leq T$,

(iii) $(t,s)\to U(t,s)$ is strongly continuous for $0\leq s\leq t\leq T$ and

$$\frac{\partial U(t,s)}{\partial t}=A(t)U(t,s),\quad \frac{\partial U(t,s)}{\partial s}=-U(t,s)A(s).$$

Definition 2.2. The map F is called upper semicontinuous (u.s.c.) on x if for each $x_0\in X$, the set $F(x_0)$ is a nonempty, closed subset of X, and if for each open set N of X containing $F(x_0)$, there exists an open neighborhood M of x_0 such that $F(M)\subseteq N$.

Definition 2.3. The map F is said to be completely continuous if $F(U)$ is relatively compact for every bounded subset $U\subseteq X$.

Remark 2.4. If multivalued map F is completely continuous with nonempty compact values, then F is u.s.c. if and only if F has a closed graph (*i.e.* $x_n\to x^*,\ y_n\to y^*,\ y_n\in F(x_n)$ imply $y^*\in F(x^*)$).

Definition 2.5. The multivalued map $F:J\times X\to P(X)$ is said to be L^1-Carathéodory if

(i) $t\to F(t,x)$ is measurable for each $x\in X$,

(ii) $x\to F(t,x)$ is u.s.c. on X for almost all $t\in J$,

(iii) for each $\rho>0$, there exists $\varphi_\rho\in L^1(J,[0,+\infty))$ such that

$$\|F(t,x)\|_{P(X)}=\sup\{\|v\|:v\in F(t,x)\}\leq\varphi_\rho(t),\ \forall\ \|x\|\leq\rho \text{ and a.e. } t\in J.$$

Definition 2.6. Let S be a bounded subset of X. The Hausdorff measure of noncompactness of S is defined by

$$\alpha(S)=\inf\{\varepsilon>0: S \text{ has a finite cover by closed balls of radius } <\varepsilon\}.$$

Definition 2.7. A multivalued map $F:X\to P(X)$ is said to be a condensing map with respect to α (abbreviated, α-condensing) if for every bounded set $B\subset X$, $\alpha(B)>0$, $\alpha(F(B))<\alpha(B)$.

Remark 2.8. If multivalued map G is completely continuous, then G is α-condensing. For general information the reader can see [1, 10].

We give the definition of a mild solution of (1.1).

Definition 2.9. A function $x:[-r,T]\to X$ is said to be a mild solution of (1.1) if $x_t\in D$, $x(t)\in\Omega$ for $t\in J$, $x(t)=\phi(t)-(w(x_{\eta_1},x_{\eta_2},\ldots,x_{\eta_p}))(t)$ for $t\in[-r,0]$, $\Delta x(t_k)=I_k(x(t_k))$, $k=1,\ldots,m$, and $s\mapsto U(t,s)A(s)g(s,x_s)$ is integrable on $[0,t)$ such that

$$\begin{aligned}x(t)=\ & U(t,0)\Big[\phi(0)-(w(x_{\eta_1},x_{\eta_2},\ldots,x_{\eta_p}))(0)\\ & -g(0,\phi-w(x_{\eta_1},x_{\eta_2},\ldots,x_{\eta_p}))\Big]+g(t,x_t)+\int_0^t U(t,s)f(s)ds\ ,\\ & +\int_0^t U(t,s)A(s)g(s,x_s)ds+\sum_{0<t_k<t}U(t,t_k)I_k(x(t_k)),\ t\in J,\end{aligned}$$

where $f\in L^1(J,X)$ satisfying $f(t)\in F(t,x_t)$ on J.

The following lemma, which comes from [12], is useful in the proof of our main result.

Lemma 2.10. *Let X be a Banach space, and $F : J \times X \to P_{cv,cp}(X)$ be a L^1-Carathéodory multivalued map with*

$$S_{F,x} = \{f \in L^1(J, X) |\ f(t) \in F(t, x(t)) \text{ for a.e. } t \in J\} \neq \emptyset,$$

and let Γ be a linear continuous mapping from $L^1(J, X)$ to $C(J, X)$, then the operator

$$\Gamma \circ S_F : C(J, X) \to P_{cv,cp}(C(J, X)),\ u \mapsto (\Gamma \circ S_F)(x) := \Gamma(S_{F,x})$$

is a closed graph operator in $C(J, X) \times C(J, X)$.

The consideration of this paper is based on Martelli's fixed point theorem [16], we state it as follows.

Lemma 2.11. *Let X be a Banach space and let $G : X \to P_{cv,cp}(X)$ be an upper semicontinuous and condensing map. If the set*

$$\Re = \{x \in X : \lambda x \in G(x) \text{ for some } \lambda > 1\}$$

is bounded, then G has a fixed point.

In order to study system (1.1), we impose the following assumptions.

(H1) I_k are continuous, and there exist positive constants θ_k such that $\|I_k(x)\| \leq \theta_k$ for all $x \in X,\ k = 1, \dots, m$.

(H2) The operator $U(t, s)$ is compact for $t-s > 0$, and there exists positive constant M_1 such that $\|U(t, s)\|_{L(X)} \leq M_1$.

(H3) There exists positive constant M_2 such that $\|A(t)\| \leq M_2$ for all $t \in J$.

(H4) w is completely continuous and there exists a positive constant Q such that

$$\|(w(u_1, u_2, \dots, u_p))(t)\|_D \leq Q,\ (u_1, \dots, u_p) \in D^p,\ t \in [-r, 0].$$

(H5) $g : J \times D \to X$ satisfies the following conditions:

(i) g is completely continuous, the function $s \mapsto U(t, s)A(s)g(s, u)$ is integrable on $[0, t)$,

(ii) there exist constants $0 \leq c_1 < 1$ and $c_2 \geq 0$ such that

$$\|g(t, u)\| \leq c_1\|u\|_D + c_2, \text{ for all } t \in J,\ u \in D.$$

(H6) $F : J \times D \to P_{cv,cp}(X), (t, u) \mapsto F(t, u)$ is measurable with respect to t, for each $u \in D$; u.s.c. with respect to u, for each $t \in J$; and for each fixed $u \in D$, the set

$$S_{F,u} = \{f \in L^1(J, X) : f(t) \in F(t, u) \text{ for a.e. } t \in J\} \neq \emptyset.$$

(H7) There exist a continuous nondecreasing function $\psi : [0, +\infty) \to (0, +\infty)$, and a function $q \in L^1(J, [0, +\infty))$ such that

$$\|F(t, u)\| = \sup\{\|f\| : f \in F(t, u)\} \leq q(t)\psi(\|u\|_D),\ \forall\ u \in D,\ \text{a.e. } t \in J.$$

3. Main results

Now we are able to state and prove our main theorem.

Theorem 3.1. *Assume (H1)-(H7) are satisfied. Moreover, for all $t \in J$,*

$$\frac{M_1}{1-c_1}\int_0^t \max\Big(q(s), M_2(c_2+1)\Big)ds < \int_c^{+\infty} \frac{1}{\psi(s)+s}\ ds, \tag{3.1}$$

with

$$c = \frac{1}{1-c_1}\left\{ M_1[\ (1+c_1)\ (\|\phi\|_D + Q) + c_2\ ] + c_2 + M_1 \sum_{k=1}^{m} \theta_k \right\},$$

then the system (1.1) has at least one mild solution.

Proof. We transform problem (1.1) into a fixed point problem. It follows from Definition 2.9 that we consider the multivalued map $N : \Omega \to P(\Omega)$ defined by

$$N(x) = \left\{ h \in \Omega : h(t) = \left\{ \begin{array}{l} \phi(t) - (w(x_{\eta_1}, x_{\eta_2}, \dots, x_{\eta_p}))(t),\ t \in [-r, 0], \\ U(t,0)\Big[\phi(0) - (w(x_{\eta_1}, x_{\eta_2}, \dots, x_{\eta_p}))(0) \\ \quad -g(0, \phi - w(x_{\eta_1}, x_{\eta_2}, \dots, x_{\eta_p}))\Big] + g(t, x_t) \\ \quad + \int_0^t U(t,s) f(s) ds + \int_0^t U(t,s) A(s) g(s, x_s) ds \\ \quad + \sum_{0<t_k<t} U(t, t_k) I_k(x(t_k)),\ t \in J,\ f \in S_{F,x} \end{array} \right. \right\}.$$

It is clear that the fixed points of N are mild solutions of (1.1). We will show that N has a fixed point. The proof will be given in 5 steps.

Step 1. $N(x)$ is convex, for each $x \in \Omega$.

Obviously, $N(x)$ is convex for $t \in [-r, 0]$. We just need to prove the case $t \in J$. Indeed, if $h_1,\ h_2$ belong to $N(x)$, then there exist $f_1,\ f_2 \in S_{F,x}$ such that for each $t \in J$, we have

$$\begin{aligned} h_i(t) = \quad & U(t,0)\Big[\phi(0) - (w(x_{\eta_1}, x_{\eta_2}, \dots, x_{\eta_p}))(0) - g(0, \phi - w(x_{\eta_1}, x_{\eta_2}, \dots, x_{\eta_p}))\Big] \\ & + g(t, x_t) + \int_0^t U(t,s) f_i(s) ds + \int_0^t U(t,s) A(s) g(s, x_s) ds \\ & + \sum_{0<t_k<t} U(t, t_k) I_k(x(t_k)),\ i = 1, 2. \end{aligned}$$

Let $0 \le \lambda \le 1$, then for each $t \in J$, we have

$$\begin{aligned} & [\lambda h_1 + (1-\lambda) h_2](t) \\ & = U(t,0)\Big[\phi(0) - (w(x_{\eta_1}, x_{\eta_2}, \dots, x_{\eta_p}))(0) - g(0, \phi - w(x_{\eta_1}, x_{\eta_2}, \dots, x_{\eta_p}))\Big] \\ & \quad + g(t, x_t) + \int_0^t U(t,s)[\lambda f_1(s) + (1-\lambda) f_2(s)] ds + \int_0^t U(t,s) A(s) g(s, x_s) ds \\ & \quad + \sum_{0<t_k<t} U(t, t_k) I_k(x(t_k)). \end{aligned}$$

Since $S_{F,x}$ is convex (because F has convex values in (H6)), then $\lambda h_1 + (1-\lambda) h_2 \in N(x)$, so $N(x)$ is convex.

Step 2. N maps bounded sets of Ω into bounded sets.

Indeed, it is enough to show that there exists a positive constant l such that for each $x \in B_d = \{x \in \Omega : \|x\|_\Omega < d\}$, one has $\|N(x)\| = \sup\{\|h\| : h \in N(x)\} \le l$.

If $t \in J$, we have

$$\begin{aligned} \|h(t)\| \leq\ & M_1[\ \|\phi\|_D + Q + c_1\|\phi - w\|_D + c_2\ ] + c_1\|x_t\|_D + c_2 \\ & + \int_0^t M_1 q(s)\psi(\|x_s\|_D)ds + \int_0^t M_1 M_2(c_1\|x_s\|_D + c_2)ds + M_1\sum_{k=1}^m \theta_k. \end{aligned}$$

So for each $x \in B_d$, $t \in J$, we have

$$\begin{aligned} \|h(t)\| \leq\ & M_1[\ \|\phi\|_D + Q + c_1(\|\phi\|_D + Q) + c_2\ ] + c_1 d + c_2 \\ & + M_1\psi(d)\|q\|_{L^1} + M_1 M_2(c_1 d + c_2)T + M_1\sum_{k=1}^m \theta_k := l_1. \end{aligned}$$

If $t \in [-r, 0]$, since $\phi \in D$ and (H4), then $\|h(t)\| \leq \|\phi\|_D + Q$. Hence for each $h \in N(B_d)$, we have $\|N(x)\| \leq \max\{l_1, \|\phi\|_D + Q\} := l$.

Step 3. $N : \Omega \to P(\Omega)$ is a condensing mapping.

From Remark (2.8), we just need to prove that $N : \Omega \to P(\Omega)$ is completely continuous. We define two maps. Let $N_1 : \Omega \to P(\Omega)$ be defined by

$$N_1(x) = \left\{ h_0 \in \Omega : h_0(t) = \left\{ \begin{array}{l} \phi(t) - (w(x_{\eta_1}, x_{\eta_2}, \ldots, x_{\eta_p}))(t),\ t \in [-r, 0], \\ U(t,0)\Big[\phi(0) - (w(x_{\eta_1}, x_{\eta_2}, \ldots, x_{\eta_p}))(0) \\ \quad -g(0, \phi - w(x_{\eta_1}, x_{\eta_2}, \ldots, x_{\eta_p}))\Big] \\ \quad + \int_0^t U(t,s)f(s)ds + \int_0^t U(t,s)A(s)g(s,x_s)ds \\ \quad + \sum_{0<t_k<t} U(t,t_k)I_k(x(t_k)),\ t \in J,\ f \in S_{F,x} \end{array} \right. \right\},$$

and $N_2 : \Omega \to P(\Omega)$ be defined by

$$N_2(x) = g(t, x_t),\ t \in J,$$

then $N = N_1 + N_2$ on Ω. From (i) of (H5), N_2 is completely continuous, so next we prove that N_1 is also completely continuous.

Firstly, we prove that N_1 maps bounded sets into precompact set of $P(\Omega)$.

We show that N_1 sends bounded sets into equicontinuous sets of $P(\Omega)$. Let $\tau_1, \tau_2 \in J\backslash\{t_1, \ldots, t_m\}$, $\tau_1 < \tau_2$, and $\delta > 0$ such that $\{t_1, \ldots, t_m\} \cap \{t - \delta, t + \delta\} = \emptyset$, and let $x \in B_d$, $h_0 \in N_1(x)$, where B_d is a bounded set of Ω in Step 2, then we have

$$\begin{aligned} & \|h_0(\tau_2) - h_0(\tau_1)\| \\ & \leq \|U(\tau_2, 0) - U(\tau_1, 0)\|_{L(X)}[\|\phi\|_D + Q + c_1(\|\phi\|_D + Q) + c_2] \\ & \quad + \int_0^{\tau_1} \|U(\tau_2, s) - U(\tau_1, s)\|_{L(X)} q(s)\psi(d)ds + \int_{\tau_1}^{\tau_2} M_1 q(s)\psi(d)ds \\ & \quad + \int_0^{\tau_1} \|U(\tau_2, s) - U(\tau_1, s)\|_{L(X)} M_2(c_1 d + c_2)ds + \int_{\tau_1}^{\tau_2} M_1 M_2(c_1 d + c_2)ds \\ & \quad + \sum_{0<t_k\leq\tau_1} \|U(\tau_2, t_k) - U(\tau_1, t_k)\|_{L(X)}\theta_k + \sum_{\tau_1<t_k<\tau_2} M_1\theta_k. \end{aligned}$$

As $\tau_2 \to \tau_1$ and $U(t,s)$ is a strongly continuous, the righthand side of the above inequality tends to zero. This proves the equicontinuity for the case where $t \neq t_i$, $i = 1, \dots, m$. It remains to examine the equicontinuity at $t = t_i$.

Let $t = t_i^+$, and fix $\delta_1 > 0$ such that $\{t_k : k \neq i\} \cap [t_i - \delta_1, t_i + \delta_1] = \emptyset$. For $0 < \Delta t < \delta_1$, we have

$$\begin{aligned}
&\|h_0(t_i + \Delta t) - h_0(t_i)\| \\
&\leq \|U(t_i + \Delta t, 0) - U(t_i, 0)\|_{L(X)}[\|\phi\|_D + Q + c_1(\|\phi\|_D + Q) + c_2] \\
&\quad + \int_0^{t_i} \|U(t_i + \Delta t, s) - U(t_i, s)\|_{L(X)} q(s)\psi(d)ds + \int_{t_i}^{t_i+\Delta t} M_1 q(s)\psi(d)ds \\
&\quad + \int_0^{t_i} \|U(t_i + \Delta t, s) - U(t_i, s)\|_{L(X)} M_2(c_1 d + c_2)ds + \sum_{t_i < t_k < t_i + \Delta t} M_1\theta_k \\
&\quad + \sum_{0 < t_k \leq t_i} \|U(t_i + \Delta t, t_k) - U(t_i, t_k)\|_{L(X)}\theta_k + \int_{t_i}^{t_i+\Delta t} M_1 M_2(c_1 d + c_2)ds.
\end{aligned}$$

The righthand side tends to zero as $\Delta t \to 0^+$, which shows equicontinuity at $t = t_i^+$. The equicontinuity of the case $t = t_i^-$ is similar, we omit it here.

The equicontinuity of N_1 for the case $r \leq \tau_1 < \tau_2 \leq 0$ and $\tau_1 \leq 0 \leq \tau_2$ follows from the uniform continuity of $\phi(t)$ on $[-r, 0]$ and the complete continuity of w. As a consequence of Steps 2 to 3, together with the Arzela-Ascoli theorem, it suffices to show that N_1 maps B_d into a precompact set in $P(\Omega)$.

Secondly, we show that $N_1(B_d)$ is relatively compact.

Let $0 < t \leq T$ be fixed and ε be a real number satisfying $0 < \varepsilon < t$. For $x \in B_d$ we define

$$\begin{aligned}
h^\varepsilon(t) &= U(t,0)\Big[\phi(0) - (w(x_{\eta_1}, x_{\eta_2}, \dots, x_{\eta_p}))(0) - g(0, \phi - w(x_{\eta_1}, x_{\eta_2}, \dots, x_{\eta_p}))\Big] \\
&\quad + \int_0^{t-\varepsilon} U(t,s)f(s)ds + \int_0^{t-\varepsilon} U(t,s)A(s)g(s,x_s)ds \\
&\quad + \sum_{0 < t_k < t-\varepsilon} U(t,t_k)I_k(x(t_k)) \\
&= U(t,0)\Big[\phi(0) - (w(x_{\eta_1}, x_{\eta_2}, \dots, x_{\eta_p}))(0) - g(0, \phi - w(x_{\eta_1}, x_{\eta_2}, \dots, x_{\eta_p}))\Big] \\
&\quad + U(t, t-\varepsilon)\Big[\int_0^{t-\varepsilon} U(t-\varepsilon, s)f(s)ds + \int_0^{t-\varepsilon} U(t-\varepsilon, s)A(s)g(s,x_s)ds \\
&\quad + \sum_{0 < t_k < t-\varepsilon} U(t-\varepsilon, t_k)I_k(x(t_k))\Big],
\end{aligned}$$

where $f \in S_{F,x}$. Since $U(t,s)$ is a compact operator for $t - s > 0$, the set

$$H^\varepsilon(t) = \{h^\varepsilon(t) : x \in B_d\}$$

is relatively compact in Ω, for every ε, $0 < \varepsilon < t$. Moreover, for every $h_0 \in N_1(x)$, we have

$$\begin{aligned}
&\|h_0(t) - h^\varepsilon(t)\| \\
&\leq \int_{t-\varepsilon}^t M_1 q(s)\psi(\|x_s\|)ds + \int_{t-\varepsilon}^t M_1 M_2(c_1\|x_s\| + c_2)ds + \sum_{t-\varepsilon < t_k < t} M_1\theta_k.
\end{aligned}$$

Therefore there is relatively compact set arbitrarily close to the set

$$H(t) = \{h_0(t) : h_0 \in N_1(x),\ t \in J\}.$$

Hence the set $\{h_0(t) : h_0 \in N_1(B_d)\}$ is relatively compact, and $N_1 : \Omega \to P(\Omega)$ is completely continuous.

Step 4. N has a closed graph.

Obviously, we just need to discuss the case $t \in J$. Let $x_n \to x^*$, $h_n \in N(x_n)$, and $h_n \to h^*$. We will prove that $h^* \in N(x^*)$. $h_n \in N(x_n)$ means that there exists $f_n \in S_{f,x_n}$ such that

$$\begin{aligned} h_n(t) = \ & U(t,0)\Big[\phi(0) - (w((x_n)_{\eta_1}, (x_n)_{\eta_2}, \ldots, (x_n)_{\eta_p}))(0) \\ & - g(0, \phi - w((x_n)_{\eta_1}, (x_n)_{\eta_2}, \ldots, (x_n)_{\eta_p}))\Big] + g(t, (x_n)_t) + \int_0^t U(t,s) f_n(s) ds \\ & + \int_0^t U(t,s) A(s) g(s, (x_n)_s) ds + \sum_{0<t_k<t} U(t,t_k) I_k(x_n(t_k)). \end{aligned}$$

Next we must prove that there exists $f^* \in S_{f,x^*}$ such that for each $t \in J$,

$$\begin{aligned} h^*(t) = \ & U(t,0)\Big[\phi(0) - (w(x^*_{\eta_1}, x^*_{\eta_2}, \ldots, x^*_{\eta_p}))(0) \\ & - g(0, \phi - w(x^*_{\eta_1}, x^*_{\eta_2}, \ldots, x^*_{\eta_p}))\Big] + g(t, x^*_t) + \int_0^t U(t,s) f^*(s) ds \\ & + \int_0^t U(t,s) A(s) g(s, x^*_s) ds + \sum_{0<t_k<t} U(t,t_k) I_k(x^*(t_k)). \end{aligned}$$

Since $x_n \to x^*$, $h_n \to h^*$, (H1), (H4) and (H5), we have

$$\begin{aligned} & \Bigg\| h_n(t) - U(t,0)\Big[\phi(0) - (w((x_n)_{\eta_1}, (x_n)_{\eta_2}, \ldots, (x_n)_{\eta_p}))(0) \\ & - g(0, \phi - w((x_n)_{\eta_1}, (x_n)_{\eta_2}, \ldots, (x_n)_{\eta_p}))\Big] - g(t, (x_n)_t) \\ & - \int_0^t U(t,s) A(s) g(s, (x_n)_s) ds - \sum_{0<t_k<t} U(t,t_k) I_k(x_n(t_k)) \\ & \quad - \Big\{ h^*(t) - U(t,0)\Big[\phi(0) - (w(x^*_{\eta_1}, x^*_{\eta_2}, \ldots, x^*_{\eta_p}))(0) \\ & - g(0, \phi - w(x^*_{\eta_1}, x^*_{\eta_2}, \ldots, x^*_{\eta_p}))\Big] - g(t, x^*_t) \\ & - \int_0^t U(t,s) A(s) g(s, x^*_s) ds - \sum_{0<t_k<t} U(t,t_k) I_k(x^*(t_k)) \Big\} \Bigg\| \to 0, \end{aligned}$$

as $n \to \infty$. Consider the operator $\Gamma : L^1(J, X) \to C(J, X)$,

$$f \mapsto \Gamma(f)(t) = \int_0^t U(t,s) f(s) ds.$$

It is easy to see that the operator Γ is linear and continuous. Indeed, one has

$$\|(\Gamma f)\|_\infty \le M_1 \|f\|_{L^1},$$

with (H6) and (H7), all of the conditions of Lemma 2.10 are satisfied, so $\Gamma \circ S_f$ is a closed graph operator. Moreover, we have

$$h_n(t) - U(t,0)\Big[\phi(0) - (w((x_n)_{\eta_1},(x_n)_{\eta_2},\dots,(x_n)_{\eta_p}))(0)$$
$$-g(0,\phi - w((x_n)_{\eta_1},(x_n)_{\eta_2},\dots,(x_n)_{\eta_p}))\Big] - g(t,(x_n)_t)$$
$$-\int_0^t U(t,s)A(s)g(s,(x_n)_s)ds - \sum_{0<t_k<t} U(t,t_k)I_k(x_n(t_k)) \in \Gamma(S_{F,x_n}).$$

From Lemma 2.10, there exists $f^* \in S_{F,x^*}$ satisfying

$$h^*(t) - U(t,0)\Big[\phi(0) - (w(x^*_{\eta_1},x^*_{\eta_2},\dots,x^*_{\eta_p}))(0) - g(0,\phi - w(x^*_{\eta_1},x^*_{\eta_2},\dots,x^*_{\eta_p}))\Big]$$
$$-g(t,x^*_t) - \int_0^t U(t,s)A(s)g(s,x^*_s)ds - \sum_{0<t_k<t} U(t,t_k)I_k(x^*(t_k))$$
$$= \int_0^t U(t,s)f^*(s)ds.$$

Therefore, from Step 1, Step 3, Step 4 and Remark 2.4, we have that $N : \Omega \to P_{cp,cv}(\Omega)$ is upper semicontinuous.

Step 5. Now it remains to show that the set

$$\Re = \{x \in \Omega : \lambda x \in N(x) \text{ for some } \lambda > 1\}$$

is bounded. Let $x \in \Re$, then $\lambda x \in N(x)$, for some $\lambda > 1$. Thus, for each $t \in J$,

$$\begin{aligned} x(t) = \quad & \lambda^{-1}\Big\{U(t,0)\Big[\phi(0) - (w(x_{\eta_1},x_{\eta_2},\dots,x_{\eta_p}))(0) \\ & -g(0,\phi - w(x_{\eta_1},x_{\eta_2},\dots,x_{\eta_p}))\Big] + g(t,x_t) + \int_0^t U(t,s)f(s)ds \\ & + \int_0^t U(t,s)A(s)g(s,x_s)ds + \sum_{0<t_k<t} U(t,t_k)I_k(x(t_k))\Big\} \end{aligned}$$

for some $f \in S_{F,x}$. This implies that, for each $t \in J$ we have

$$\begin{aligned} \|x(t)\| \le \quad & M_1[\ \|\phi\|_D + Q + c_1\|\phi - w\|_D + c_2\ ] + c_1\|x_t\|_D + c_2 \\ & + \int_0^t M_1 q(s)\psi(\|x_s\|_D)ds + \int_0^t M_1M_2(c_1\|x_s\|_D + c_2)ds \\ & + M_1\sum_{k=1}^m \theta_k. \end{aligned} \tag{3.2}$$

We consider the function μ defined by

$$\mu(t) = \sup\{\|x(s)\| : -r \le s \le t\},\ t \in J. \tag{3.3}$$

Let $t^* \in [-r,t]$ be such that $\mu(t) = \|x(t^*)\|$. If $t^* \in J$, by (3.2), (3.3) and the increasing character of ψ, for $t \in J$ we have

$$\begin{aligned}\mu(t) \leq\ & M_1[\ \|\phi\|_D + Q + c_1(\|\phi\|_D + Q) + c_2\ ] + c_1\mu(t) + c_2 \\ & + \int_0^t M_1 q(s)\psi(\mu(s))ds + \int_0^t M_1 M_2(c_1\mu(s) + c_2)ds + M_1\sum_{k=1}^m \theta_k.\end{aligned}$$

Then,

$$\begin{aligned}\mu(t) \leq\ & \frac{1}{1-c_1}\Big\{M_1[\ (1+c_1)(\|\phi\|_D + Q) + c_2\ ] + c_2 \\ & + M_1\int_0^t q(s)\psi(\mu(s))ds + M_1M_2\int_0^t (c_1\mu(s) + c_2)ds + M_1\sum_{k=1}^m \theta_k\Big\}.\end{aligned} \tag{3.4}$$

If $t^* \in [-r,0]$, then $\mu(t) \leq \|\phi\|_D + Q$.
Let us denote the right-hand side of inequality (3.4) as $v(t)$, then we have

$$\mu(t) \leq v(t),\ \ t \in J, \tag{3.5}$$

$$v(0) = \frac{1}{1-c_1}\Big\{M_1[\ (1+c_1)(\|\phi\|_D + Q) + c_2\ ] + c_2 + M_1\sum_{k=1}^m \theta_k\Big\} \geq c_2,$$

$$v'(t) = \frac{1}{1-c_1}[M_1q(t)\psi(\mu(t)) + M_1M_2(c_1\mu(t) + c_2)] > 0. \tag{3.6}$$

So we have

$$v(t) \geq v(0) \geq c_2. \tag{3.7}$$

Using the nondecreasing character of ψ, (3.5), (3.6) and (3.7), we get

$$\begin{aligned}v'(t) &\leq \frac{1}{1-c_1}\Big[M_1q(t)\psi(v(t)) + M_1M_2(c_1v(t) + v(t))\Big] \\ &\leq \frac{M_1}{1-c_1}\max\Big(q(t), M_2(c_1+1)\Big)\Big[\psi(v(t)) + v(t)\Big].\end{aligned}$$

Then for each $t \in J$, we have

$$\int_{v(0)}^{v(t)} \frac{1}{\psi(u) + u}du \leq \frac{M_1}{1-c_1}\int_0^t \max\Big(q(s), M_2(c_1+1)\Big)ds.$$

Assumption (3.1) shows that there exists a positive constant K such that $v(t) \leq K,\ t \in J$, and hence $\mu(t) \leq K,\ t \in J$. Since for every $t \in J$, $\|x_t\| \leq \mu(t)$, we have

$$\|x\| \leq \max(\|\phi\|_D + Q, K) := K',$$

where K' depends on $T,\ \phi,\ Q$, and on the functions q and ψ. This shows that the set is bounded. As a consequence of Lemma 2.11, we deduce that N has a fixed point which is a mild solution of (1.1).

Remark 3.2. [3] only considered the case $A(t) = A$ and $g(t, x_t) = 0$ in (1.1), so our result is more general.

4. An example

In this section, as an application of our main result, an example is presented. We consider the following partial differential equation

$$\begin{cases} \dfrac{\partial[v(t,\xi)+g(t,v(t+\theta,\xi))]}{\partial t} \in h(t,\xi)\dfrac{\partial^2 v(t,\xi)}{\partial \xi^2} + F(t,v(t+\theta,\xi)), \\ \qquad t\in[0,T]\backslash\{t_1,\dots,t_m\},\ \xi\in[0,\pi],\ \theta\in[-r,0], \\ v(t,0)=v(t,\pi)=0,\ t\in[0,T], \\ v(t_k^+,\xi)-v(t_k^-,\xi)=I_k(v(t_k^-,\xi)),\ k=1,\dots,m, \\ v(\theta,\xi)+w(v(\eta_1+\theta,\xi),\dots,v(\eta_p+\theta,\xi))=\phi(\theta,\xi), \end{cases} \tag{4.1}$$

where $h(t,\xi)$ is a continuous function on $[0,T]\times[0,\pi]$ and uniformly Holder continuous in t,

$$v(t_k^+,\xi)=\lim_{\varepsilon\to 0^+} v(t_k+\varepsilon,\xi),\ v(t_k^-,\xi)=\lim_{\varepsilon\to 0^+} v(t_k-\varepsilon,\xi).$$

Let $X=L^2([0,\pi],R)$,

$$v(t)(\xi)=v(t,\xi),\ (t,\xi)\in[0,T]\times[0,\pi],$$

$$I_k(x(t_k))(\xi)=I_k(v(t_k))(\xi),\ \xi\in[0,\pi],\ k=1,\dots,m,$$

$$\phi(\theta)(\xi)=\phi(\theta,\xi),\ (\theta,\xi)\in[-r,0]\times[0,\pi],$$

$$w(v(\eta_1+\theta),\dots,v(\eta_p+\theta))(\xi)=w(v(\eta_1+\theta,\xi),\dots,v(\eta_p+\theta,\xi)).$$

Define $A(t):X\to X$ by

$$(A(t)x)(\theta)=h(t,\theta)\frac{d^2x}{d\theta^2}$$

with

$$D(A(t))=\{x\in X,\ x,\ \frac{dx}{d\theta} \text{ are absolutely continuous and } \frac{d^2x}{d\theta^2}\in X,\ x(0)=x(\pi)=0\},$$

then $A(t)$ generates an evolution operator $U(t,s)$ on X (see pages 209-210 of [18]), which is strongly continuous. Hence the partial differential inclusions (4.1) can be rewritten as the abstract form as system (1.1).

Besides the above assumptions, if we assume that the conditions stated in Theorem 3.1 is true, then system (4.1) has one mild solution.

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