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A FIXED POINT THEOREM IN ABSTRACT SPACES WITH APPLICATION TO CAUCHY PROBLEM

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Abstract. We establish a new fixed point theorem in abstract spaces. We then derive two main consequences in topological spaces for mappings admitting precompact images or leading to a nonempty ω-limit set. The study is carried out by introducing a cone of special functions which enables us to extend, unify and improve fixed point results due to Bailey, Ćirić, Dass-Gupta, Edelstein, Hardy-Rogers, Jaggi, Karapınar, Liepiņš, Nemytskii, Popa, Popescu, Reich, Suzuki and Wardowski. Finally, we introduce the notion of ξ-Lipschitz property and we investigate the existence of solutions to a class of Cauchy problems.

Key Words and Phrases: Fixed points, precompact, omega-limit sets, Cauchy problems.
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1. Introduction

Fixed points theorems are fundamental tools for studying problems of the existence of equilibrium points which arise for example in dynamical systems or in economic models. It is important to establish new results adapted to the increasing complexity of the problems encountered. One of most famous fixed point theorem was proved in 1922 by Banach [2]. Later, Nemytskii [10] and Edelstein [5] obtained some fixed point theorems for more general contractive condition. Since then, a number of substantial generalizations and improvements of Nemytskii's and Edelstein's results has been appeared. The purpose of this paper is to establish new fixed point theorems, which extend, unify and improve a large class of contractive type mappings existing in the literature.

In Section 2, we recall some fixed point theorems appearing in [1, 5, 6, 8, 9, 10, 11, 12, 15]. Then, we observe that the sufficient conditions used in this theorems belong to a more general class of conditions. We characterize this class by a specific convex cone in the space of real valued functions. In Section 3, we establish a fixed point theorem in abstract spaces involving these cones. Next, we derive two main consequences for mappings satisfying appropriate continuity conditions. The first consequence extends fixed point theorems defined on compact metric spaces of Section 2, as well as those of Ćirić [3, Theorems 1 & 2], Dass-Gupta [4, Theorem 1], Reich [14, Theorem 3] and Wardowski [16, Theorem 2.1]. The second consequence extends all the results of Section 2, but for mappings that have a nonempty ω-limit set, with the exception of Bailey's theorem. Section 4 is devoted to derive an extension of results of Section 2. In the last section, we introduce the concept of ξ-Lipschitz functions and then we give sufficient conditions for the existence of unique solution to a Cauchy problem. Finally, we present an example to support our results.

2. Preliminaries

In this section, we provide some fixed point theorems for contractive mappings in order to make this paper self-containing. In 1936, M. Nemytskii proved the following theorem:

Theorem 2.1. (Nemytskii [10]). *Let (X,d) be a compact metric space and $T\colon X \to X$ be a given mapping. Assume that for all $x,y \in X$,*

$$x \neq y \implies d(Tx,Ty) < d(x,y).$$

Then T has a unique fixed point.

This theorem was then generalized in the compact framework by several authors as follows:

Theorem 2.2. (Bailey [1]). *Let (X,d) be a compact metric space and $T\colon X \to X$ be a given mapping. Assume that for all $x,y \in X$ there exists an integer $m = m(x,y)$ such that*

$$x \neq y \implies d(T^m x, T^m y) < d(x,y).$$

Then T has a unique fixed point.

Theorem 2.3. (Suzuki [15]). *Let (X,d) be a compact metric space and $T\colon X \to X$ be a mapping. Assume that for all $x,y \in X$,*

$$\frac{1}{2}d(x,Tx) < d(x,y) \implies d(Tx,Ty) < d(x,y),$$

Then T has a unique fixed point.

Theorem 2.4. (Popescu [12]). *Let (X,d) be a compact metric space and $T\colon X \to X$ be a continuous mapping. Assume that for all $x,y \in X$,*

$$ad(x,Tx) + bd(y,Tx) < d(x,y) \implies d(Tx,Ty) < d(x,y),$$

where a,b are non-negative reals and $2a+b<1$. Then T has a unique fixed point.

In the previous results the authors modified either sufficient condition of the contraction or the left side of its necessary condition. However, Hardy and Rogers modified the right side of the necessary condition as follows:

Theorem 2.5. (Hardy-Rogers [6]). *Let (X,d) be a compact metric space and $T\colon X \to X$ be a continuous mapping. Assume that for all $x,y \in X$,*

$$x \neq y \implies d(Tx,Ty) < ad(x,Tx) + bd(y,Ty) + cd(x,Ty) + ed(y,Tx) + fd(x,y),$$

where a, b, c, e, f are nonnegative reals with $a + b + c + e + f = 1$. Then T a unique fixed point.

This result has been improved by Karapinar in the following way:
Theorem 2.6. (Karapınar [8]). *Let (X, d) be a compact metric space and $T\colon X \to X$ be a mapping. Assume that for all $x, y \in X$,*

$$\frac{1}{2}d(x, Tx) < d(x, y) \implies d(Tx, Ty) < M(x, y),$$

where $M(x, y) = \max\left\{d(x, y), d(x, Tx), d(y, Ty), \frac{1}{2}d(x, Ty), \frac{1}{2}d(y, Tx)\right\}$. Then T has a unique fixed point.

Using the concept of ω-limit sets, Edelstein [5] extended and unified both the Banach contraction principle [2] and Theorem 2.1.
Definition 2.7. Let X be a topological space and $T\colon X \to X$ be a mapping. For $x_0 \in X$ the ω-limit set is the set

$$\omega_T(x_0) := \bigcap_{n \in \mathbb{N}} \overline{\left\{T^k x_0 : k \geq n\right\}},$$

where $\mathbb{N}$ is the set of all positive integers.

Theorem 2.8. (Edelstein [5]). *Let (X, d) be a metric space and $T\colon X \to X$ be a given mapping such that for all $x, y \in X$,*

$$x \neq y \implies d(Tx, Ty) < d(x, y).$$

If there exists $x_0 \in X$ such that $\omega_T(x_0) \neq \emptyset$, then T has a unique fixed point.

In [9], by using continuous mapping instead of metric distance, Liepiņš generalized Edelstein's theorem in the context of topological spaces.
Theorem 2.9. (Liepiņš [9]). *Let X be a topological space and T be a continuous selfmap of X. Suppose there exists a continuous mapping $g : X \times X \to \mathbb{R}_+$ satisfying*

$$x \neq y \implies \left|g(Tx, Ty)\right| < \left|g(x, y)\right|.$$

If there exists $x_0 \in X$ such that $\omega_T(x_0) \neq \emptyset$, then T has a unique fixed point.

Next, Popa [11] generalized Jaggi's fixed point theorem [7] for rational contractive mappings with nonempty ω-limit set.
Theorem 2.10. (Popa [11]). *Let X be a Hausdorff space and $f : X \times X \to \mathbb{R}_+$ be a continuous function such that:*

(i) *$f(x, y) \neq 0$, for all $x \neq y$.*
(ii) *$f(x, y)^2 \geq f(x, x)f(y, y)$, for all $x \neq y$.*

Assume that $T\colon X \to X$ is a continuous mapping and satisfies

$$x \neq y \implies f(Tx, Ty) \leq a\frac{f(x, Tx)\, f(y, Ty)}{f(x, y)} + bf(x, y),$$

where $a, b \in \mathbb{R}_+$ and $a + b < 1$. If there exists $x_0 \in X$ such that $\omega_T(x_0) \neq \emptyset$, then T has a unique fixed point.

Throughout this paper, the diagonal set is denoted by $\Delta := \{(x,x) : x \in X\}$, where X is nonempty set, and the set of all fixed points of a mapping $T\colon X \to X$ is denoted by $\mathrm{Fix}(T) := \{x \in X : Tx = x\}$.

Definition 2.11. Let X be a nonempty set, p be a non-negative integer and $T\colon X \to X$ be a given mapping. A mapping $\delta : X \times X \to \mathbb{R}$ is called T-Suzuki-Edelstein function of level p if it satisfies the following conditions

$$T^p\big(Gr(T)\big) \setminus \Delta \subseteq \big\{(T^p x, T^p y) : \delta(x,y) > 0\,\big\} \text{ and } \Delta \subseteq \big\{(x,y) : \delta(x,y) \leq 0\,\big\},$$

where $T^p\big(Gr(T)\big) := \big\{(T^p x, T^{p+1}x) : x \in X\big\}$. The set of T-Suzuki-Edelstein functions of level p is denoted $S_T^p(X)$.

Remark 2.12. *We present some properties of $S_T^p(X)$:*

(i) *Any function $\delta : X \times X \to \mathbb{R}_+$ such that $\delta(x,y) > 0$ for all $x \neq y$ and $\delta(x,x) = 0$ for all $x \in X$ (for example $\delta_0(x,y) = 1$ for all $x \neq y$ and 0 otherwise), is in $S_T^0(X)$. In particular, $S_T^0(X)$ is nonempty.*

(ii) *$S_T^p(X) \subseteq S_T^{p+1}(X)$ for all $p \geq 0$. Moreover, the mapping δ defined by*

$$\delta(x,y) = \begin{cases} 1 & \text{if } (T^{p+1}x, T^{p+1}y) \in T^{p+1}(Gr(T)) \setminus \Delta, \\ 0 & \text{otherwise.} \end{cases}$$

is an element of $S_T^{p+1}(X)$, but if there exists $x \in X$ such that $T^p x \neq T^{p+1}x$ and $T^{p+1}x = T^{p+2}x$ then $\delta(x,Tx) = 0$, which means that $\delta \notin S_T^p(X)$.

(iii) *The set $S_T^p(X)$ is a salient convex cone for every $p \geq 0$.*

Example 2.13. Here we give examples of elements of $S_T^p(X)$. Some of them has been introduced in the literature.

- [Nymetskii-Edelstein]: $\delta_1(x,y) = d(x,y)$ and $p = 0$.
- [Suzuki]: $\delta_2(x,y) = d(x,y) - \dfrac{1}{2}d(x,Tx)$ and $p = 0$.
- [Popescu]: $\delta_3(x,y) = d(x,y) - ad(x,Tx) - bd(y,Tx)$, where $a > 0, b > 0, 2a+b < 1$ and $p = 0$.
- [Wardowski]: $\delta_4(x,y) = d(Tx,Ty)$ and $p = 1$.
- $\delta_5(x,y) = f(d(x,y)) - g(d(x,Tx))$, with $f,g : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying $f(t) > g(t)$ for all $t > 0$, $f(0) = 0$ and $p = 0$.
- $\delta_6(x,y) = d(T^p x, T^p y)^n - ad(T^p x, T^{p+1}x)^n$, $n \geq 1$ and $0 < a < 1$.
- $\delta_7(x,y) = \rho(T^p x, T^p y) - a\rho(T^p x, T^{p+1}x) - b\rho(T^{r+1}x, T^r y)$, where $0 \leq a,b < 1$, $r \in \mathbb{N}$ and $\rho : X \times X \to \mathbb{R}_+$ such that $\rho(x,x) = 0$ for all $x \in X$.

Remark 2.14. *Later, we will see that the question of uniqueness is related to the appropriate choice of δ.*

Definition 2.15. Let X be a nonempty space, $\psi, \varphi : X \times X \to \mathbb{R}_+$, $m : X \times X \to \mathbb{N}$ be three functions, $T\colon X \to X$ be a given mapping and $\delta \in S_T^p(X)$ for some $p \in \mathbb{N}$. We say that T is m-δ-(ψ,φ)-contractive if for all $x, y \in X$, we have

(i) $\delta(x,Tx) > 0 \implies \varphi(x,Tx) \leq \psi(x,Tx)$.

(ii) $\delta(x,y)>0 \implies \psi(T^{m(x,y)}x, T^{m(x,y)}y) < \varphi(x,y)$.

In particular, if $m(x,y)=1$ for all $x,y\in X$, we say that T is δ-(ψ,φ)-contractive.

Example 2.16. Here we present a table of m-δ-(ψ,φ)-contractive type mappings involving the functions δ of Remark 2.12 and Example 2.13.

Authors	m-δ-(ψ,φ)-contractive condition	m, ψ, φ
X is a topological space		
Liepiņš [9]	$\delta_0(x,y)>0 \Rightarrow \lvert g(Tx,Ty)\rvert < \lvert g(x,y)\rvert$, where g is a continuous function	$m(x,y)=1$, $\psi=\varphi=g$
Popa [11]	$\delta_0(x,y)>0 \Rightarrow f(Tx,Ty)\le a\frac{f(x,Tx)\,f(y,Ty)}{f(x,y)}+bf(x,y)$, where f is a continuous function	$m(x,y)=1$, $\psi=f$, $\varphi(x,y)=a\frac{f(x,Tx)\,f(y,Ty)}{f(x,y)}+bf(x,y)$
(X,d) is a metric space		
Nymetskii-Edelstein [10, 5]	$\delta_1(x,y)>0 \Rightarrow d(Tx,Ty)<d(x,y)$	$m(x,y)=1$, $\psi=\varphi=d$
Bailey [1]	$\delta_1(x,y)>0 \Rightarrow d(T^{m(x,y)}x,T^{m(x,y)}y)<d(x,y)$	$\psi=\varphi=d$
Suzuki [15]	$\delta_2(x,y)>0 \Rightarrow d(Tx,Ty)<d(x,y)$	$m(x,y)=1$, $\psi=\varphi=d$
Popescu [12]	$\delta_3(x,y)>0 \Rightarrow d(Tx,Ty)<d(x,y)$, where $a>0$, $b>0$ and $2a+b<1$	$m(x,y)=1$, $\psi=\varphi=d$
Wardowski type [16]	$\delta_4(x,y)>0 \Rightarrow c\exp\big(F(d(Tx,Ty))\big) < \exp\big(F(d(x,y))\big)$, where $c\ge 1$ and F is a continuous real function	$m(x,y)$=1, $\psi(x,y)=c\exp\big(F(d(x,y))\big)$, $\varphi(x,y)=\exp\big(F(d(x,y))\big)$

3. Main results

The main result of the paper is the following:

Theorem 3.1. *Let X be a nonempty set, $\psi,\varphi : X\times X\to\mathbb{R}_+$, $m : X\times X\to\mathbb{N}$ be three functions, $T\colon X\to X$ be a given mapping and $\delta\in S_T^p(X)$ for some $p\in\mathbb{N}$. Assume that one of the following assertions holds:*

(I) *There exists $x_0\in X$ such that $\delta(x_0,Tx_0)\le 0$.*

(II) *There exists $x_0\in X$ such that*

$$\varphi(x_0,Tx_0)\le \inf_{k\in\mathbb{N}}\psi(T^kx_0,T^{k+1}x_0), \tag{3.1}$$

and for all $x,y\in X$, we have

$$\delta(x,y)>0 \implies \psi(T^{m(x,y)}x,T^{m(x,y)}y)<\varphi(x,y). \tag{3.2}$$

Then T has a fixed point.

Proof. Let $x_0\in X$ satisfying (I), by definition of δ, we have

$$(T^px_0,T^{p+1}x_0)\notin T^p(Gr(T))\setminus\Delta,$$

then $(T^p x_0, T^{p+1}x_0) \in \Delta$, which implies that $T^p x_0$ is a fixed point of T. In case where (II) holds, assume that $\delta(x_0, Tx_0) > 0$. Using (3.2), we get

$$\psi(T^{m(x_0,Tx_0)}x_0, T^{m(x_0,Tx_0)+1}x_0) < \varphi(x_0, Tx_0).$$

Then, from (3.1), we deduce that

$$\psi(T^{m(x_0,Tx_0)}x_0, T^{m(x_0,Tx_0)+1}x_0) < \inf_{k\in\mathbb{N}} \psi(T^k x_0, T^{k+1}x_0).$$

which is a contradiction. Consequently, $\delta(x_0, Tx_0) \leq 0$ and we conclude by (I). □

We next provide two corollaries for m-δ-(ψ, φ)-contractive mappings defined on topological spaces, which have precompact images or a nonempty ω-limit set. Recall that a subset Y of a topological space X is said to be precompact if its closure is compact.

Corollary 3.2. *Let X be a topological space and $T\colon X \to X$ be a continuous mapping such that $T(X)$ is precompact. If ψ is lower semi-continuous and T is m-δ-(ψ, φ)-contractive, then T has a fixed point.*

Proof. Let $x \in X$, since $T(X)$ is precompact, $\omega_T(x)$ is compact. Let $\{y_n\}$ be a sequence in $\omega_T(x)$ such that

$$\inf_{y\in\omega_T(x)} \psi(y, Ty) = \lim_{n\to\infty} \psi(y_n, Ty_n).$$

By the compactness of $\omega_T(x)$, without loosing the generality, we may assume that the sequence $\{y_n\}$ converge to some $x_0 \in \omega_T(x)$. Using the semi-continuity of ψ and the continuity of T, we see that

$$\psi(x_0, Tx_0) \leq \inf_{y\in\omega_T(x)} \psi(y, Ty).$$

As $T^k x_0 \in \omega_T(x)$ for all k, then

$$\psi(x_0, Tx_0) \leq \inf_{k\in\mathbb{N}} \psi(T^k x_0, T^{k+1}x_0).$$

If $\delta(x_0, Tx_0) \leq 0$, then (I) holds. Otherwise, assume that $\delta(x_0, Tx_0) > 0$. By Definition 2.15-(i), we deduce that

$$\varphi(x_0, Tx_0) \leq \inf_{k\in\mathbb{N}} \psi(T^k x_0, T^{k+1}x_0),$$

that is (3.1) holds and from Definition 2.15-(ii), we have (3.2), then the result follows from Theorem 3.1. □

Remark 3.3. *As first consequence of this corollary, any continuous mapping T, satisfying one of the contractions of the previous table has a fixed point.*

In the next corollary, we relax the condition of compactness.

Corollary 3.4. *Let X be a topological space and $T\colon X \to X$ be a continuous mapping such that there exists $x_0 \in X$ satisfying $\omega_T(x_0) \neq \emptyset$. If ψ is continuous and T is δ-(ψ, φ)-contractive, then T has a fixed point.*

Proof. Let $x_0 \in X$ such that $\omega_T(x_0) \neq \emptyset$. If there exists k such that $\delta(T^k x_0, T^{k+1}x_0) \leq 0$, then we conclude that (I) holds. Otherwise, for all $k \geq 0$ we have $\delta(T^k x_0, T^{k+1}x_0) > 0$. Using Definition 2.15, we deduce that

$$\psi(T^{k+1}x_0, T^{k+2}x_0) < \varphi(T^k x_0, T^{k+1}x_0) \leq \psi(T^k x_0, T^{k+1}x_0).$$

Consequently, the sequence $\{\psi(T^k x_0, T^{k+1}x_0)\}$ is decreasing. Since this sequence is bounded below then it is convergent. Hence, for any $y \in \omega_T(x_0)$, by continuity of ψ and T, we have

$$\psi(y, Ty) = \inf_{k\in\mathbb{N}} \psi(T^k x_0, T^{k+1}x_0).$$

In particular, we have $\psi(y, Ty) = \psi(T^k y, T^{k+1}y)$ for all $k \geq 0$, since by continuity of T, we get $T^k y \in \omega_T(x_0)$. Hence,

$$\psi(y, Ty) = \inf_{k\in\mathbb{N}} \psi(T^k y, T^{k+1}y).$$

If $\delta(y, Ty) \leq 0$, then (I) holds. Otherwise, assume that $\delta(y, Ty) > 0$. Then, using Definition 2.15-(i), we obtain

$$\varphi(y, Ty) \leq \inf_{k\in\mathbb{N}} \psi(T^k y, T^{k+1}y),$$

that is, (3.1) holds. In addition, (3.2) follows from Definition 2.15-(ii). We conclude then by Theorem 3.1-(II). □

4. Some consequences

In this section, we present a series of results for mappings defined on topological spaces with precompact images, as direct consequence of Corollary 3.2. These results remain valid even if we replace the pre-compactness of $T(X)$ and semi-continuity of ρ (defined in the corollaries below) by the hypotheses:
(*) There exists $x_0 \in X$ such that $\omega_T(x_0) \neq \emptyset$, ρ is continuous and $m(x, y) = 1$ for all $x, y \in X$.

In fact, the furnished proofs are exactly the same under the hypotheses (*), using Corollary 3.2, therefore omitted. Let us start by introducing a concept needed later to show the uniqueness of the fixed point.

Definition 4.1. Let X be a nonempty set and $T\colon X \to X$ be a mapping. We say that δ is T-perfect if there exists $p \in \mathbb{N}$ such that $\delta \in S_T^p(X)$ and

$$\mathrm{Fix}(T)^2 \setminus \Delta \subseteq \{(x, y) : \delta(x, y) > 0\}.$$

Remark 4.2. *All mappings δ given in Example 2.13 are T-perfect.*

The following corollary is an extension of fixed point theorems of Nymetskii [10], Edelstein [5], Bailey [1], Liepiņš [9], Suzuki [15], Popescu [12].

Corollary 4.3. *Let X be a topological space and $T\colon X \to X$ be a continuous mapping such that $T(X)$ is precompact. Let $\rho : X \times X \to \mathbb{R}_+$ be a lower semi-continuous function. Assume that δ is T-perfect and there exists a function $m : X \times X \to \mathbb{N}$ such that:*

$$\delta(x, y) > 0 \implies \rho(T^{m(x,y)}x, T^{m(x,y)}y) < \rho(x, y).$$

Then T has a unique fixed point.

Proof. The result of existence follows from Corollary 3.2 by taking $\psi = \varphi = \rho$. Assume now that T has two distinct fixed points x, y, then by the perfectness of δ, we get $\delta(x, y) > 0$, and by using the contractive condition, we infer a contradiction.□

We present next an extension of Wardowski [16] fixed point theorem.

Corollary 4.4. *Let X be a topological space and $T: X \to X$ be a continuous mapping such that $T(X)$ is precompact. Let $\rho : X \times X \to \mathbb{R}$ be a lower semi-continuous function. Assume that $\delta \in S_T^p(X)$ for some $p \in \mathbb{N}$ and there exists a function $m : X \times X \to \mathbb{N}$ such that:*

$$\delta(x, y) > 0 \implies \tau + \rho(T^{m(x,y)}x, T^{m(x,y)}y) < \rho(x, y),$$

where $\tau \geq 0$. Then T has a fixed point. In addition, if δ is T-perfect, then T has a unique fixed point.

Proof. Using the monotony of the exponential function, we obtain that the contractive condition is equivalent to

$$\delta(x, y) > 0 \implies c \exp \rho(T^{m(x,y)}x, T^{m(x,y)}y) < \exp \rho(x, y),$$

where $c = \exp \tau$. Hence, for $\psi(x, y) = c \exp \rho(x, y)$ and $\varphi(x, y) = \exp \rho(x, y)$, the mapping T becomes an m-δ-(ψ, φ)-contractive mapping and satisfies the properties of Corollary 3.2. Assume next that δ is T-perfect. If x, y are two distinct fixed points, then $\delta(x, y) > 0$, and by the contractive condition we deduce a contradiction. □

Under the additional hypothesis of continuity of T, the two following corollaries extend Reich's [14, Theorem 3] and Skof's [13, Theorem 5.8] in different ways.

Corollary 4.5. *Let X be a topological space and $T: X \to X$ be a continuous mapping such that $T(X)$ is precompact. Let $\rho : X \times X \to \mathbb{R}_+$ be a lower semi-continuous function such that $\rho(x, x) = 0$ for all $x \in X$. Assume $\delta \in S_T^p(X)$ for some $p \in \mathbb{N}$ and there exists a function $m : X \times X \to \mathbb{N}$ such that:*

$$\delta(x, y) > 0 \implies \rho(T^{m(x,y)}x, T^{m(x,y)}y) < M_1(x, y) \tag{4.1}$$

where

$$M_1(x, y) = a\rho(x, y) + b\rho(Tx, y) + c\rho(T^{m(x,y)-1}y, T^{m(x,y)}y) + d\rho(x, Tx),$$

with $a, b, c, d \in \mathbb{R}_+$ and $0 \leq a + c + d \leq 1$. Then T has a fixed point. In particular, if δ is T-perfect, then T has a unique the fixed point.

Proof. Consider the functions $\psi, \varphi : X \times X \to \mathbb{R}_+$ given by

$$\psi(x, y) = \rho(x, y) \quad \text{and} \quad \varphi(x, y) = M_1(x, y).$$

Then to conclude that T has a fixed point, it is sufficient to show that T is m-δ-(ψ, φ)-contractive. Observe that Definition 2.15-(ii) follows from from (4.1), so it remains to prove that ψ and φ satisfy Definition 2.15-(i). For this purpose, let $x \in X$ such that $\delta(x, Tx) > 0$, then for $m = m(x, Tx)$, we have

$$\rho(T^m x, T^{m+1}x) < a\rho(x, Tx) + b\rho(Tx, Tx) + c\rho(T^m x, T^{m+1}x) + d\rho(x, Tx). \tag{4.2}$$

So, it follows from (4.1) that

$$\rho(T^m x, T^{m+1}x) < \frac{a+d}{1-c}\rho(x, Tx). \tag{4.3}$$

Combining (4.2), (4.3) and using the fact that $a+c+d \leq 1$, we get

$$\varphi(x, Tx) < \frac{a+d}{1-c}\rho(x, Tx) \leq \psi(x, Tx).$$

Finally, assume that δ is T-perfect and x, y be two distinct fixed points. Then by assumption on δ, we get

$$\rho(x, y) < (a+b)\rho(x, y) \leq \rho(x, y),$$

which is a contradiction. □

Corollary 4.6. *Let X be a topological space and $T\colon X \to X$ be a continuous mapping such that $T(X)$ is precompact. Let $\rho : X \times X \to \mathbb{R}_+$ be a lower semi-continuous function such that $\rho(x,x) = 0$ for all $x \in X$. Assume $\delta \in S_T^p(X)$ for some $p \in \mathbb{N}$ and there exists a function $m : X \times X \to \mathbb{N}$ such that:*

$$\delta(x, y) > 0 \implies \rho(T^{m(x,y)}x, T^{m(x,y)}y) < M_2(x, y), \tag{4.4}$$

where

$$M_2(x, y) = \max\left\{\rho(x, y),\, \alpha\, \rho(T^k x, T^{k-1}y),\, \rho(T^{m(x,y)-1}y, T^{m(x,y)}y),\, \rho(x, Tx)\right\},$$

where $k \in \mathbb{N}$ and $\alpha \in \mathbb{R}$. Then T has a fixed point. In particular, if δ is T-perfect, then T has a unique fixed point.

Proof. Consider the functions $\psi, \varphi : X \times X \to \mathbb{R}_+$ given by

$$\psi(x, y) = \rho(x, y) \quad \text{and} \quad \varphi(x, y) = M_2(x, y).$$

In order to conclude that T has a fixed point, it suffice to show that T is m-δ-(ψ, φ)-contractive. From (4.4), we obtain that Definition 2.15-(ii) holds, so we shall prove that ψ and φ satisfy Definition 2.15-(i). Let $x \in X$ such that $\delta(x, Tx) > 0$ then for $m = m(x, Tx)$, we obtain

$$\rho(T^m x, T^{m+1}x) < \max\left\{\rho(x, Tx),\, \rho(Tx, Tx),\, \rho(T^m x, T^{m+1}x),\, \rho(x, Tx)\right\}. \tag{4.5}$$

Thus, we deduce that $\rho(T^m x, T^{m+1}x) < \rho(x, Tx)$ and consequently $\varphi(x, Tx) \leq \psi(x, Tx)$. Finally, the uniqueness is obtained in a similar way as in the previous corollary. □

The following result extends [3, Theorems 1 & 2] of Ćirić.

Corollary 4.7. *Let X be a topological space and $T\colon X \to X$ be a continuous mapping such that $T(X)$ is precompact. Let $\rho : X \times X \to \mathbb{R}_+$ be a lower semi-continuous function such that $\rho(x,x) = 0$ for all $x \in X$. Assume $\delta \in S_T^p(X)$ for some $p \in \mathbb{N}$ and there exists a function $m : X \times X \to \mathbb{N}$ such that:*

$$\delta(x, y) > 0 \implies \rho(T^{m(x,y)}x, T^{m(x,y)}y) < M_3(x, y), \tag{4.6}$$

where

$$M_3(x,y) = \max\Big\{\rho(x,y), \min\{\rho(x,Tx),\rho(y,Ty)\} + \mu\min\{\rho(x,Ty),\rho(y,Tx)\}\Big\} + \lambda\min\{\rho(x,Ty),\rho(y,Tx)\},$$

where $k \in \mathbb{N}$ and $\lambda, \mu \in \mathbb{R}_+$. Then T has a fixed point. In particular, if δ is T-perfect, $\lambda = 0$ and $\mu \leq 1$, then T has a unique fixed point.

Proof. Consider the functions $\psi, \varphi : X \times X \to \mathbb{R}_+$ given by

$$\psi(x,y) = \rho(x,y) \quad \text{and} \quad \varphi(x,y) = M_3(x,y).$$

As previously, from (4.6), it is clear that Definition 2.15-(ii) is satisfied. Now, observe that for all $x \in X$ we have $\varphi(x,Tx) = \psi(x,Tx)$, then Definition 2.15-(i) holds. To see the uniqueness result, assume that x, y are two distinct fixed points, then from (4.6), we have

$$\rho(x,y) < \rho(x,y)\big(\max\{1,\mu\} + \lambda\big).$$

If $\lambda = 0$ and $\mu \leq 1$, we obtain a contradiction. □

The following corollary extend the fixed point theorems of Jaggi [7] and Popa [11] for mappings satisfying rational contractive condition.

Corollary 4.8. *Let X be a topological space and $T\colon X \to X$ be a continuous mapping such that $T(X)$ is precompact. Let $\rho : X \times X \to \mathbb{R}_+$ be a lower semi-continuous function such that $\rho(x,y) \neq 0$ for all $x \neq y$. Assume $\delta \in S_T^p(X)$ for some $p \in \mathbb{N}$ and there exists a function $m : X \times X \to \mathbb{N}$ such that:*

$$\delta(x,y) > 0 \implies \rho(T^{m(x,y)}x, T^{m(x,y)}y) < M_4(x,y), \tag{4.7}$$

where

$$M_4(x,y) = a\frac{\rho(x,Tx)\,\rho(T^{m(x,y)-1}y, T^{m(x,y)}y)}{\rho(x,y)} + b\rho(x,y),$$

with $a, b \in \mathbb{R}_+$ and $a + b \leq 1$. Then T has a fixed point. In particular, if δ is T-perfect, then T has a unique fixed point.

Proof. Consider the functions $\psi, \varphi : X \times X \to \mathbb{R}_+$ given by

$$\psi(x,y) = \rho(x,y) \quad \text{and} \quad \varphi(x,y) = M_4(x,y).$$

As previously, we have to show Definition 2.15-(i). Let $x \in X$ such that $\delta(x,Tx) > 0$ then for $m = m(x,Tx)$, we have

$$\rho(T^m x, T^{m+1}x) < M_3(x,Tx) = a\rho(T^m x, T^{m+1}x) + b\rho(x,Tx). \tag{4.8}$$

Thus, by using (4.7), we deduce

$$\rho(T^m x, T^{m+1}x) < \frac{b}{1-a}\rho(x,Tx). \tag{4.9}$$

Combining (4.8), (4.9) and using that $a + b \leq 1$, we get

$$\varphi(x,Tx) \leq \rho(x,Tx) = \psi(x,Tx).$$

Finally, the uniqueness is obtained in a similar way as in the previous corollary. □

Next, we present an extension of Dass-Gupta's [4] fixed point theorem.

Corollary 4.9. *Let X be a topological space and $T\colon X \to X$ be a continuous mapping such that $T(X)$ is precompact. Let $\rho : X \times X \to \mathbb{R}_+$ be a lower semi-continuous function. Assume $\delta \in S_T^p(X)$ for some $p \in \mathbb{N}$ and there exists a function $m : X \times X \to \mathbb{N}$ such that:*

$$\delta(x,y) > 0 \implies \rho(T^{m(x,y)}x, T^{m(x,y)}y) < M_5(x,y),$$

where

$$M_5(x,y) = a\frac{(1+\rho(x,Tx))\,\rho(T^{m(x,y)-1}y, T^{m(x,y)}y)}{1+\rho(x,y)} + b\rho(x,y),$$

with $a, b \in \mathbb{R}_+$ and $a+b \le 1$. Then T has a fixed point. In particular, if δ is T-perfect, then T has a unique fixed point.

Proof. The proof is similar to the previous one, where $\psi(x,y) = \rho(x,y)$ and $\varphi(x,y) = M_5(x,y)$. □

The following result may be viewed as an extension of Hardy-Rogers' [6] theorem (see also Remark 4.12-(R_3)).

Corollary 4.10. *Let (X,d) be a metric space and $T\colon X \to X$ be a continuous mapping such that $T(X)$ is precompact. Assume $\delta \in S_T^p(X)$ for some $p \in \mathbb{N}$ and there exist two functions $m, r : X \times X \to \mathbb{N}$ such that:*

$$\delta(x,y) > 0 \implies d(T^{m(x,y)}x, T^{m(x,y)}y) < M_6(x,y),$$

where for $n = m(x,y)$ and $k = r(x,y)$,

$$\begin{aligned} M_6(x,y) &= ad(T^{n-1}x, T^{n-1}y) + bd(T^k x, T^{k-1}y) + cd(T^{n-1}y, T^n y) \\ &+ ed(T^{n-1}x, T^n x) + fd(T^{n-1}x, T^n y), \end{aligned}$$

with $a, b, c, d, e \in \mathbb{R}_+$ and $a + c + e + 2f \le 1$. Then T has a fixed point. In particular, if δ is T-perfect and $a + b + f \le 1$, then T has a unique fixed point.

Proof. Consider the functions $\psi, \varphi : X \times X \to \mathbb{R}_+$ given by

$$\psi(x,y) = d(T^{m(x,y)-1}x, T^{m(x,y)-1}y) \quad \text{and} \quad \varphi(x,y) = M_6(x,y).$$

By this consideration T is a δ-(ψ,φ)-contractive. As previously, we have to show Definition 2.15-(i). Let $x \in X$ such that $\delta(x,Tx) > 0$ then for $n = m(x,Tx)$, we obtain

$$d(T^n x, T^{n+1}x) < \frac{a+e}{1-c} d(T^{n-1}x, T^n x) + \frac{f}{1-c} d(T^{n-1}x, T^{n+1}x). \tag{4.10}$$

Since by triangle inequality, we have

$$d(T^{n-1}x, T^{n+1}x) - d(T^{n-1}x, T^n x) \le d(T^n x, T^{n+1}x),$$

then

$$d(T^{n-1}x, T^{n+1}x) < \frac{a+e+1-c}{1-c-f} d(T^{n-1}x, T^n x). \tag{4.11}$$

Substituting (4.11) in (4.10) we get

$$\varphi(x,Tx) < \frac{a+e+f}{1-c-f} d(T^{n-1}x, T^n x) \le \psi(x,Tx).$$

Finally, assume that x, y are two distinct fixed points and δ is T-perfect, using the contractive condition and the hypothesis $a+b+f \leq 1$, we get

$$d(x,y) < (a+b+f)d(x,y) \leq d(x,y),$$

which is a contradiction. □

We end this series of corollaries by presenting a result that extends Theorem 2.1 of Karapınar [8], under a supplementary condition of continuity of T.

Corollary 4.11. *Let (X,d) be a metric space and $T : X \to X$ be a continuous mapping such that $T(X)$ is precompact. Assume $\delta \in S_T^p(X)$ for some $p \in \mathbb{N}$ and there exist two functions $m, r : X \times X \to \mathbb{N}$ such that:*

$$\delta(x,y) > 0 \implies d(T^{m(x,y)}x, T^{m(x,y)}y) < M_7(x,y),$$

where for $n = m(x,y)$ and $k = r(x,y)$,

$$\begin{aligned} M_7(x,y) \quad = \quad & \max\left\{d(T^{n-1}x, T^{n-1}y), \alpha d(T^k x, T^{k-1}y), d(T^{n-1}y, T^n y),\right. \\ & \left. d(T^{n-1}x, T^n x), \frac{1}{2}d(T^{n-1}x, T^n y)\right\}, \end{aligned}$$

with $\alpha \in \mathbb{R}$. Then T has a fixed point. In particular, if δ is T-perfect, then T has a unique fixed point.

Proof. Consider the functions $\psi, \varphi : X \times X \to \mathbb{R}_+$ given by

$$\psi(x,y) = d(T^{m(x,y)-1}x, T^{m(x,y)-1}y) \quad \text{and} \quad \varphi(x,y) = M_7(x,y).$$

Then T is δ-(ψ,φ)-contractive. As previously, we have to show Definition 2.15-(i). Let $x \in X$ such that $\delta(x,Tx) > 0$ then for $n = m(x,Tx)$, we obtain

$$d(T^n x, T^{n+1}x) < \max\left\{d(T^{n-1}x, T^n x), d(T^n x, T^{n+1}x), \frac{1}{2}d(T^{n-1}x, T^{n+1}x)\right\}.$$

Thus using the triangle inequality, we obtain

$$M_7(x,Tx) = d(T^{n-1}x, T^n x).$$

We deduce then

$$\varphi(x,Tx) = d(T^{n-1}x, T^n x) = \psi(x,Tx).$$

Finally, assume that x, y are two distinct fixed points and δ is T-perfect, then by the contractive condition, we infer a contradiction. □

Remark 4.12.

- (R_1) Compared to Theorem 2.1 of Wardowski [16], Corollary 4.4 is valid for lower semi-continuous mappings ρ defined on topological spaces such that $T(X)$ is precompact, or (*) is satisfied. However, it does not require additional constraints on ρ.
- (R_2) The Corollary 4.8 in the context of hypotheses (*), generalize Popa [11, Theorem 2] without imposing neither the separation of the space nor the condition (ii) of Theorem 2.10.

(R_3) In Corollary 4.10, if we assume $m(x,y)=k$ for all x,y and $\delta(x,y)>0$ iff $\delta(y,x)>0$, then M_5 becomes a symmetric function. So, by using the same argument as in the proof of [6, Theorem 1], we may replace the condition $a+c+e+2f\leq 1$ by $a+b+c+e+f\leq 1$.

Question 4.13. In the compact metric setting, the Suzuki Theorem 3 [15] and Karapınar Theorem 2.1 [8] deals with non continuous mappings. It is of interest to know under what conditions on δ, ψ and φ we may relax the continuity of T for non metric spaces.

5. An application to an initial value problem

The objective of this section is to investigate the existence of solutions to the following initial value problem:

$$\begin{cases} x'(t)=f(t,x(t)), & 0\leq t<a,\\ x(0)=0. \end{cases} \tag{5.1}$$

Let $E=C([0,a],\mathbb{R}_+)$ be the space of continuous functions on $[0,a]$ endowed with its uniform distance, and let $L^1[0,a]$ the set of integrable functions on $[0,a]$. Denote $L=L^1[0,a]\cap C([0,a),\mathbb{R}_+)$.

Definition 5.1. Let $\eta : L\times L\to L$ be a function and $\xi : [0,a]\times L\times L\to\mathbb{R}_+$ defined by

$$\xi(t,x,y)=\int_0^t \eta(x,y)(s)ds.$$

A continuous mapping $f:[0,a)\times\mathbb{R}\to\mathbb{R}_+$ is said to be ξ-Lipschitz, if for all $x_1,x_2\in E$ such that $x_1\neq x_2$, there exists a continuous function $u_{x_1,x_2}:[0,a)\to\mathbb{R}_+$ satisfying:

$$\xi\big(t,f(\cdot,x_1),f(\cdot,x_2)\big)\leq u_{x_1,x_2}(t)\xi\big(t,x_1,x_2\big),\ \text{for all } t\in[0,a),$$

with $f(\cdot,x):[0,a)\to\mathbb{R}_+,\ t\mapsto f(t,x(t))$ is integrable, for all $x\in E$.

We give next some sufficient conditions ensuring the existence of a unique solution to (5.1) in E.

Theorem 5.2. *Assume that ξ is lower semi-continuous function and $f:[0,a)\times\mathbb{R}\to\mathbb{R}_+$ be a function. Suppose that:*

(i) *f is ξ-Lipschitz function and for all $x_1,x_2\in E$ such that $x_1\neq x_2$, we have*

$$\int_0^a u_{x_1,x_2}(s)ds<1.$$

(ii) *For all $x_1,x_2\in E$ such that $x_1\neq x_2$ and for all $t\in[0,a]$, we have*

$$\xi(t,y_1,y_2)\leq\int_0^t \xi(s,x_1,x_2)ds,$$

where y_i is an antiderivative of x_i such that $y_i(0)=0$, $i=1,2$.

(iii) *There exists a positive integrable function v on $[0,a]$ such that*

$$\int_{t_1}^{t_2} f(s,x(s))ds \leq \int_{t_1}^{t_2} v(s)ds.$$

for all $x \in E$ and $t_1, t_2 \in [0,a]$ with $t_1 < t_2$.

(iv) *The mapping $T\colon X \to E$ defined by*

$$Tx(t) := \int_0^t f(s,x(s))ds, \text{ for all } t \in [0,a),$$

is continuous, where $X := \left\{x \in E,\ x(t) \leq \int_0^t v(s)ds \text{ for } t \in [0,a)\right\}$.

Then the Cauchy problem (5.1) *has a unique solution in E.*

Proof. The proof follows from Corollary 4.3 and is divided into four steps.

Step 1. We start by establishing that the mapping $T\colon X \to X$ is well-defined and continuous. Observe that by hypothesis on v, the function $F : [0,a] \to \mathbb{R}_+$ defined by

$$F(t) = \int_0^t v(s)ds.$$

is continuous on $[0,a]$. Now, by definition of f, for all $x \in X$, the image Tx is positive. Using the integrability condition on f in (iii), we see that Tx is also continuous and $T(X) \subseteq X$.

Step 2. The closure of $T(X)$ is compact. Indeed, for all $t_1, t_2 \in [0,a]$, we have

$$\begin{aligned} \left|Tx(t_1) - Tx(t_2)\right| &= \left|\int_{t_1}^{t_2} f(s,x(s))ds\right| \\ &\leq \left|\int_{t_1}^{t_2} v(s)ds\right| \\ &\leq \left|F(t_1) - F(t_2)\right|. \end{aligned}$$

Thus, using Arzelà-Ascoli theorem, it follows that $T(X)$ is precompact.

Step 3. There exists a lower semi-continuous function ρ. By construction, the function $t \mapsto \xi(t,x_1,x_2)$ is continuous, so the function $\rho : X \times X \to \mathbb{R}_+$ given by

$$\rho(x_1,x_2) = \sup_{t\in[0,a]} \xi(t,x_1,x_2),$$

is well defined. Let see that ρ is lower semi-continuous. Take $b \in [0,a]$ such that

$$\rho(x_1,x_2) = \xi(b,x_1,x_2),$$

Then, for $\{x_{1n}\}$ and $\{x_{2n}\}$ two sequences which converge respectively to x_1 and x_2, we have

$$\rho(x_{1n},x_{2n}) = \sup_{t\in[0,a]} \xi(t,x_{1n},x_{2n}) \geq \xi(b,x_{1n},x_{2n}).$$

Using the fact that ξ is lower semi-continuous we obtain

$$\liminf \rho(x_{1n},x_{2n}) \geq \liminf \xi(b,x_{1n},x_{2n}) \geq \xi(b,x_1,x_2) = \rho(x_1,x_2),$$

which proves that ρ is lower semi-continuous.

Step 4. The contractive condition of Corollary 4.3 is satisfied. Let $x_1, x_2 \in X$ such that $\delta_0(x_1, x_2) > 0$ (see Remark 2.12 for the definition of δ_0). Since, by definition of ξ, we have

$$\xi\big(0, Tx_1, Tx_2\big) = 0,$$

then there exists a $t_0 \in (0, a]$ such that

$$\rho(Tx_1, Tx_2) = \xi(t_0, Tx_1, Tx_2).$$

Using (i) and (ii), we obtain

$$\begin{aligned} \rho(Tx_1, Tx_2) &= \xi(t_0, Tx_1, Tx_2) \\ &\leq \int_0^{t_0} \xi\big(s, f(\cdot, x_1), f(\cdot, x_2)\big) ds \\ &\leq \int_0^{t_0} u_{x_1,x_2}(s)\xi(s, x_1, x_2) ds \\ &\leq \rho(x_1, x_2) \int_0^{t_0} u_{x_1,x_2}(s) ds \\ &< \rho(x_1, x_2). \end{aligned}$$

□

Define the set of nonnegative functions:

$$\mathcal{A} = \left\{ \ell : [0,1) \to \mathbb{R}_+ \;\middle|\; \begin{array}{l} \ell \text{ increasing} \\ \lim_{t\to 1} \ell(t) = +\infty \\ \int_0^1 \ell(t)dt < 1 \end{array} \right\}.$$

Note that $\mathcal{A}$ is nonempty, since for all positive reals $a_1 \geq 1$ and $b_1 < 1$, there exists a constant $c > 0$ such that $\ell(t) := c(1 - t^{a_1})^{-b_1} \in \mathcal{A}$.

Theorem 5.3. *Let $h, g : [0,1) \to \mathbb{R}_+$ and $\alpha : [0,1) \to (0,+\infty)$ be continuous functions such that g, h are integrable and $\frac{h}{\alpha} \in \mathcal{A}$. Then the following initial value problem:*

$$\begin{cases} x'(t) = g(t) + h(t)\, \dfrac{x(t)}{\alpha(t) + x(t)}, & t \in [0,1), \\ x(0) = 0, \end{cases} \tag{5.2}$$

has a unique solution in $E = C([0,1], \mathbb{R}_+)$.

Proof. Consider,

$$f(t, x(t)) = g(t) + h(t)\, \frac{x(t)}{\alpha(t) + x(t)} \text{ and } v(t) = g(t) + h(t), \text{ for all } t \in [0,1).$$

Clearly f is continuous and $f(\cdot, x)$ is integrable for all $x \in C([0,1), \mathbb{R}_+)$.

Let $\eta : L \times L \to L$ be the mapping given by

$$\eta(x, y) = x - y + |x - y|\,.$$

Then the function ξ given by

$$\xi(t, x_1, x_2) = \int_0^t (x_1(s) - x_2(s)) ds + \int_0^t |x_1(s) - x_2(s)|\, ds,$$

is continuous. Next, we shall prove that conditions (i)–(iv) of Theorem 5.2 are fulfilled.

(i) For all $x_1, x_2 \in E$ such that $x_1 \neq x_2$, then we have:

$$\begin{aligned}
&\xi\big(t, f(\cdot, x_1), f(\cdot, x_2)\big)\\
&= \int_0^t f(s, x_1(s)) - f(s, x_2(s))ds + \int_0^t |f(s, x_1(s)) - f(s, x_2(s))|\, ds\\
&= \int_0^t h(s) \left(\frac{x_1(s)}{\alpha(s) + x_1(s)} - \frac{x_2(s)}{\alpha(s) + x_2(s)} + \left| \frac{x_1(s)}{\alpha(s) + x_1(s)} - \frac{x_2(s)}{\alpha(s) + x_2(s)} \right| \right)\\
&\leq \frac{h(t)}{\alpha(t)} \int_0^t x_1(s) - x_2(s) + \big|x_1(s) - x_2(s)\big| ds\\
&\leq \frac{h(t)}{\alpha(t)} \xi(t, x_1, x_2).
\end{aligned}$$

Thus, for $u_{x_1,x_2}(t) = \dfrac{h(t)}{\alpha(t)}$, we have $\displaystyle\int_0^1 u_{x_1,x_2}(s)ds < 1$ for all $t \in [0, 1)$.

(ii) For all $x_1, x_2 \in E$ and their respective antiderivative y_1, y_2, we have

$$\begin{aligned}
\xi(t, y_1, y_2) &= \int_0^t y_1(s) - y_2(s)ds + \int_0^t |y_1(s) - y_2(s)|\, ds\\
&= \int_0^t \int_0^s x_1(r) - x_2(r)dr + \int_0^t \left| \int_0^s x_1(r) - x_2(r)dr \right| ds\\
&\leq \int_0^t \left(\int_0^s x_1(r) - x_2(r) + \big|x_1(r) - x_2(r)\big| dr \right) ds\\
&= \int_0^t \xi(s, x_1, x_2)ds.
\end{aligned}$$

(iii) As $\big|f(t, x(t))\big| \leq g(t) + h(t)$, for all $x \in E$ and $t \in [0, 1)$, then since g and h are integrable, it follows that

$$\begin{aligned}
\int_{t_1}^{t_2} f(s, x(s))ds &= \int_{t_1}^{t_2} g(s) + h(s)\, \frac{x(s)}{\alpha(s) + x(s)} ds\\
&\leq \int_{t_1}^{t_2} v(s)ds.
\end{aligned}$$

for all $x \in E$ and $t_1, t_2 \in [0, a]$ with $t_1 < t_2$.

(iv) For $x, y \in X$, we have

$$\begin{aligned}
\|Tx - Ty\| &= \sup_{t\in[0,1]} \left| \int_0^t f(s, x(s)) - f(s, y(s))ds \right|\\
&\leq \sup_{t\in[0,1]} \int_0^t |f(s, x(s)) - f(s, y(s))|\, ds\\
&\leq \sup_{t\in[0,1]} \int_0^t h(s)\big|x(s) - y(s)\big| ds\\
&\leq \|x - y\| \int_0^1 h(s)ds,
\end{aligned}$$

this show that T is continuous.
Therefore, the existence of a unique solution in E of the initial value problem (5.2), follows from Theorem 5.2. $\square$

Example 5.4. The following initial value problem

$$\begin{cases} x'(t) = t(1-t)^{-a} + (1-t)^{-b}\,\dfrac{x(t)}{2(1-t)^{-c} + x(t)}, & t \in [0,1), \\ x(0) = 0, \end{cases}$$

has a unique solution in $C([0,1], \mathbb{R}_+)$, where $a, b, c \in (0,1)$ and $b > c$. Indeed, for $g(t) = t(1-t)^{-a}$, $h(t) = (1-t)^{-b}$ and $\alpha(t) = 2(1-t)^{-c}$, we have $h,\, g : [0,1) \to \mathbb{R}_+$ and $\alpha : [0,1) \to (0,+\infty)$ are continuous functions such that g, h are integrable and $\frac{h}{\alpha} \in \mathcal{A}$. Hence, we conclude by Theorem 5.3.

References

[1] D. Bailey, *Some theorems on contractive mappings*, Journal of the London Mathematical Society, **1**(1966), no. 1, 101-106.

[2] S. Banach, *Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales*, Fundamenta Mathematicae, **3**(1922), no.1, 133-181.

[3] L. Ćirić, *Coincidence and fixed points for maps on topological spaces*, Topology and its Applications, **154**(2007), no. 17, 3100-3106.

[4] B.K. Dass, S. Gupta, *An extension of Banach contraction principle through rational expression*, Indian Journal of Pure and Applied Mathematics, **6**(1975), no. 12, 1455-1458.

[5] M. Edelstein, *On fixed and periodic points under contractive mappings*, Journal of the London Mathematical Society, **1**(1962), no. 1, 74-79.

[6] G.E. Hardy, T. Rogers, *A generalization of a fixed point theorem of Reich*, Canadian Mathematical Bulletin, **16**(1973), no. 2, 201-206.

[7] D. Jaggi, *Some unique fixed point theorems*, Indian Journal of Pure and Applied Mathematics, **8**(1977), no. 2, 223-230.

[8] E. Karapınar, *Edelstein type fixed point theorems*, Annals of Functional Analysis, **2**(2011), no. 1, 51-58.

[9] A. Liepiņš, *Edelstein's fixed point theorem in topological spaces*, Numerical Functional Analysis and Optimization, **2**(1980), no. 5, 387-396.

[10] V.V. Nemytskii, *Fixed-point method in analysis*, Uspekhi Matematicheskikh Nauk. Ross. Akad. Nauk, (1936), no. 1, 141-174.

[11] V. Popa, *Some unique fixed point theorems in Hausdorff spaces*, Indian Journal of Pure and Applied Mathematics, **14**(1983), no. 6, 713-717.

[12] O. Popescu, *Some generalizations of Suzuki and Edelstein type theorems*, Fixed Point Theory and Applications, **2013**(2013), no. 1, 1-11.

[13] P.D. Proinov, *Fixed point theorems for generalized contractive mappings in metric spaces*, Journal of Fixed Point Theory and Applications, **22**(2020), no. 1, 21.

[14] S. Reich, *Some remarks concerning contraction mappings*, Canadian Mathematical Bulletin, **14**(1971), no. 1, 121-124.

[15] T. Suzuki, *A new type of fixed point theorem in metric spaces*, Nonlinear Analysis. Theory, Methods & Applications, vol. 71, **11**(2009), 5313-5317.

[16] D. Wardowski, *Fixed points of a new type of contractive mappings in complete metric spaces*, Fixed Point Theory and Applications, **2012**(2012), no. 1, 94.

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