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ITERATIVE ALGORITHMS FOR A FINITE FAMILY OF EQUILIBRIUM PROBLEMS AND FIXED POINT PROBLEM IN AN HADAMARD SPACE

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Abstract. The main purpose of this paper is to introduce a viscosity-type proximal point algorithm for approximating a common solution of a finite family of equilibrium problems and fixed point problem for a certain class of nonspreading-type mappings recently introduced by Phuengrattana [Applied General Topology 18(2017), 117-129]. We further establish a strong convergence of our proposed algorithm to a common solution of a finite family of equilibrium problems which is also a fixed point of this class of mappings and a unique solution of some variational inequality problems in an Hadamard space. We also analyse the asymptotic behaviour of the sequence generated by a viscosity-type algorithm and extend the analysis to approximate a common solution of a finite family of equilibrium problems in an Hadamard space. Furthermore, we applied our results to solve some optimization problems in Hadamard spaces.
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1. Introduction

Let C be a nonempty subset of a metric space X and $T : C \to C$ be a nonlinear mapping. A point $x \in C$ is called a fixed point of T if $Tx = x$. We denote the set of fixed points of T by $F(T)$. The mapping T is said to be

(i) *a contraction*, if there exists $k \in (0, 1)$ such that

$$d(Tx, Ty) \leq kd(x, y) \ \forall x, y \in C,$$

if $k = 1$, then T is called *nonexpansive*;

(ii) *quasinonexpansive*, if $F(T) \neq \emptyset$ and

$$d(p, Tx) \leq d(p, x) \ \forall p \in F(T), \ x \in C;$$

(iii) *nonspreading* (see [30]) if

$$2d^2(Tx, Ty) \leq d^2(Tx, y) + d^2(Ty, x) \ \forall x, y \in C.$$

The approximation of fixed points of nonlinear mappings is known to be one of the most flourishing areas of research in mathematics that has received a lot of attention in recent time, due to its wide applications in solving many mathematical problems, (see [3, 33, 36, 37, 49] and the references therein). For instance, the approximation of fixed points of nonspreading mappings are known to be very useful in solving mean ergodic problems (see for example, [30]). Also, approximating fixed points of certain nonspreading mappings is equivalent to finding zero points of monotone operators and minimizers of proper convex and lower semi-continuous mappings (see [14, 30] and the references contained therein). Thus, there is rapid increase in the study of this class of mappings and its generalizations by numerious authors. For example, Naraghirad [34] introduced and studied a generalization of the class of nonspreading mappings in a real Banach space, called the class of asymptotically nonspreading mappings, which he defined as follows: Let C be a nonempty closed and convex subset of a real Banach space E. A mapping $T : C \to C$ is called asymptotically nonspreading if

$$||T^n x - T^n y||^2 \leq ||x - y||^2 + 2\langle x - T^n x, J(y - T^n y)\rangle \ \forall x, y \in C \text{ and } n \in \mathbb{N}, \quad (1.1)$$

where J is the duality mapping on C. One can easily verify that in a real Hilbert space, (1.1) is equivalent to

$$2||T^n x - T^n y||^2 \leq ||T^n x - y||^2 + ||T^n y - x||^2 \ \forall x, y \in C \text{ and } n \in \mathbb{N}. \quad (1.2)$$

Naraghirad [34] proved some weak and strong convergence theorems for approximating fixed points of asymptotically nonspreading mappings in a real Banach space. Phuengrattana [42] continue along this line and introduced a new class of nonspreading-type mappings in a convex metric space, which is more general than the class of asymptotically nonspreading mappings. He called this class of mappings, the class of generalized asymptotically nonspreading mappings and defined it as follows: A mapping $T : C \to C$ is called generalized asymptotically nonspreading, if there exist two functions $f, g : C \to [0, \gamma], \ \gamma < 1$ such that

$$d^2(T^n x, T^n y) \leq f(x)d^2(T^n x, y) + g(x)d^2(T^n y, x) \ \forall x, y \in C, \ n \in \mathbb{N},$$

and

$$0 < f(x) + g(x) \leq 1 \ \forall x \in C.$$

Furthermore, he proved a Δ-convergence of the Mann-type iteration to a fixed point of this class of mappings in an Hadamard space. It is easy to see that, if $f(x) = \frac{1}{2} = g(x)$ for all $x \in C$ in the above definition, then T reduces to an asymptotically nonspreading mapping. This shows that the class of generalized asymptotically nonspreading mappings includes the class of asymptotically nonspreading mappings. To show that this inclusion is actually proper, Phuengrattana [42] gave the following example:

Example 1.1. [42] Let $T : [0, \infty) \to [0, \infty)$ be defined by

$$Tx = \begin{cases} 0.9, & \text{if } x \geq 1, \\ 0, & \text{if } x \in [0, 1). \end{cases}$$

Then, T is not an asymptotically nonspreading mapping. To see this, take $x = 1.2$ and $y = 0.7$. However, T is a generalized asymptotically nonspreading mapping.
We observe that the mapping defined in Example 1 is a constant mapping (in each of the sub-intervals). It will be more desirable and interesting to consider an example of a generalized asymptotically nonspreading mapping which is not a constant mapping and also not an asymptotically nonspreading mapping. Unfortunately, to the best of our knowledge, such example cannot be found in the literature. To this end, we show that such example actually exists.
Example 1.2. Let $T : [0, \infty) \to [0, \infty)$ be defined by

$$Tx = \begin{cases} \frac{1}{x + \frac{1}{10}}, & \text{if } x \geq 1, \\ 0, & \text{if } x \in [0, 1). \end{cases}$$

Then, T is not an asymptotically nonspreading mapping. In fact, if we take $x = 1$ and $y = 0.5$, then

$$2|Tx - Ty|^2 = 1.65 > 0.17 + 1 = |Tx - y|^2 + |Ty - x|^2.$$

However, T is a generalized asymptotically nonspreading mapping. To see this, let $f, g : [0, \infty) \to [0, 0.9]$ be defined by

$$f(x) = \begin{cases} 0, & \text{if } x \geq 1, \\ 0.9, & \text{if } x \in [0, 1) \end{cases} \qquad \text{and} \qquad g(x) = \begin{cases} \frac{1}{(x + \frac{1}{10})^2}, & \text{if } x \geq 1, \\ 0, & \text{if } x \in [0, 1). \end{cases}$$

Case 1. If $x \geq 1$ and $y \in [0, 1)$, then $f(x) = 0$ and $g(x) = \frac{1}{(x + \frac{1}{10})^2}$. For $n = 1$, we have that $Tx = \frac{1}{x + \frac{1}{10}}$ and $Ty = 0$. Thus, we obtain that

$$|Tx - Ty|^2 = \frac{1}{(x + \frac{1}{10})^2} \leq 0 + g(x)x^2 = f(x)|Tx - y|^2 + g(x)|Ty - x|^2.$$

Now, observe that $Tx = \frac{1}{x + \frac{1}{10}} \in [0, 1)$, thus for $n \geq 2$, we have that $T^n x = 0 = T^n y$, and

$$|T^n x - T^n y|^2 = 0 \leq f(x)|T^n x - y|^2 + g(x)|T^n y - x|^2.$$

Case 2. If $x \in [0, 1)$ and $y \geq 1$, then $f(x) = 0.9$ and $g(x) = 0$. For $n = 1$, we have that $Tx = 0$ and $Ty = \frac{1}{y + \frac{1}{10}}$. Thus, we obtain that

$$|Tx - Ty|^2 = \frac{1}{(y + \frac{1}{10})^2} < f(x)y^2 + 0 = f(x)|Tx - y|^2 + g(x)|Ty - x|^2.$$

For $n \geq 2$, we have that $T^n y = 0 = T^n x$, and

$$|T^n x - T^n y|^2 = 0 \leq f(x)|T^n x - y|^2 + g(x)|T^n y - x|^2.$$

Case 3. If $x \geq 1$ and $y \geq 1$, then $f(x) = 0$ and $g(x) = \frac{1}{(x+\frac{1}{10})^2}$. For $n = 1$, we have that $Tx = \frac{1}{x+\frac{1}{10}}$ and $Ty = \frac{1}{y+\frac{1}{10}}$. Thus, we obtain that

$$\begin{aligned} |Tx - Ty|^2 &= \frac{(x-y)^2}{(x+\frac{1}{10})^2(y+\frac{1}{10})^2} \\ &< \frac{(1-xy-\frac{x}{10})^2}{(x+\frac{1}{10})^2(y+\frac{1}{10})^2} \\ &= \frac{1}{(x+\frac{1}{10})^2}\Big|x - \frac{1}{y+\frac{1}{10}}\Big|^2 \\ &= f(x)|Tx-y|^2 + g(x)|Ty-x|^2. \end{aligned}$$

Now, for $n \geq 2$, $T^n x = 0 = T^n y$ and $|T^n x - T^n y|^2 = 0$. Hence, the conclusion follows.
Case 4. If $x, y \in [0, 1)$, we have that

$$|T^n x - T^n y|^2 = 0 \leq f(x)|T^n x - y|^2 + g(x)|T^n y - x|^2.$$

Therefore, we conclude that T is a generalized nonspreading mapping.
Equilibrium Problem (EP) is another important area of research in mathematics that has attracted the interest of many researchers. The EP is defined as:

$$\text{Find } x^* \in C \text{ such that } f(x^*, y) \geq 0, \ \forall y \in C. \tag{1.3}$$

The point x^* for which (1.3) is satisfied is called an equilibrium point of f. The solution set of problem (1.3) is denoted by EP(C, f). The EP can be considered to be of central importance in optimization theory since it includes many other optimization and mathematical problems as special cases; namely, minimization problems, variational inequality problems, complementarity problems, fixed point problems, convex feasibility problems, among others (see for example [20, 25, 31, 38, 39, 40, 47]). Thus, numerious authors have extensively studied EPs in Hilbert, Banach and topological vector spaces (see [1, 2, 9, 10, 13, 22, 41, 46, 47]), as well as in Hadamard manifolds (see [12, 35]). The study of the EP was recently studied in Hadamard spaces by Kumam and Chaipunya [31]. First, they established the existence of an equilibrium point of a bifunction satisfying some convexity, continuity and coercivity assumptions, and they also established some fundamental properties of the resolvent of the bifunction. Furthermore, they proved that the PPA Δ-converges to an equilibrium point of a monotone bifunction in an Hadamard space. More precisely, they proved the following theorem.
Theorem 1.3. Let C be a nonempty closed and convex subset of an Hadamard space X and $f : C \times C \to \mathbb{R}$ be monotone, Δ-upper semicontinuous in the first variable such that $D(J_\lambda^f) \supset C$ for all $\lambda > 0$ (where $D(J_\lambda^F)$ means the domain of J_λ^f). Suppose that $EP(C, f) \neq \emptyset$ and for an initial guess $x_0 \in C$, the sequence $\{x_n\} \subset C$ is generated by

$$x_n := J_{\lambda_n}^f(x_{n-1}), \ n \in \mathbb{N},$$

where $\{\lambda_n\}$ is a sequence of positive real numbers bounded away from 0. Then, $\{x_n\}$ Δ-converges to an element of $EP(C, f)$.
It is worthy to note that other authors have also studied EPs in Hadamard spaces (see for example [24]).

We also note that the results of Kumam and Chaipunya [31] are natural generalizations of corresponding results in Hilbert spaces. Furthermore, in general, Hadamard spaces are more suitable frameworks for the study of optimization problems and other related mathematical problems, since many recent results in these spaces have already found applications in diverse fields than they do in Hilbert spaces. For instance, the minimizers of energy functional (which is an example of a convex and lower semi-continuous functional in an Hadamard space) called hamonic maps, are very useful in geometry and analysis (see [5]). Also, the gradient flow theorem in Hadamard space was used to attack a conjecture of Donaldson on the asymptotic behavior of the Calabi flow in Kähler geometry (see [6]). Moreover, the theory of optimization has successfully been applied to find minimizers of submodular functions on modular lattices (see [6]). Furthermore, the study of optimization problems has also been successfully applied in Hadamard spaces, for computing medians and means, which are very important in computational phylogenetics, difussion tensor imaging, censensus algorithms and modeling of airway systems in human lungs and blood vessels (see [4, 18, 19] for details). Thus, it is not out of place to expect that EPs will prove very useful in Hadamard spaces. Hence, the generalization by Kumam and Chaipunya [31] and [24] are necessary and very important.
Based on this, we shall continue along this line and introduce a viscosity-type proximal point algorithm (since viscosity-type algorithms generally have higher rate of convergence that the Halpern-types, see [45]), for approximating a common solution of a finite family of EPs which is also a fixed point of a generalized asymptotically nonspreading mappings and a unique solution of some variational inequality problems in an Hadamard space. We shall also analyse the asymptotic behaviour of the sequence generated by a viscosity-type algorithm and extend the analysis to approximate a common solution of a finite family of equilibrium problems in an Hadamard space. It is also important to note that, in all our convergence analysis, we obtained strong convergence results which are more desirable than Δ-convergence results in Hadamard spaces. Furthermore, we applied our results to solve some optimization problems in Hadamard spaces.

2. Preliminaries

In this section, we recall some basic and useful results that will be needed in establishing our main results. We categorize our study into brief-detailed subsections.

2.1 Geometry of Hadamard spaces

Definition 2.1. Let (X, d) be a metric space, $x, y \in X$ and $I = [0, d(x, y)]$ be an interval. A curve c (or simply a geodesic path) joining x to y is an isometry $c : I \to X$ such that $c(0) = x$, $c(d(x, y)) = y$ and $d(c(t), c(t') = |t - t'|$ for all $t, t' \in I$. The image of a geodesic path is called the geodesic segment, which is denoted by $[x, y]$ whenever it is unique.

Definition 2.2. [17] A metric space (X, d) is called a geodesic space if every two points of X are joined by a geodesic, and X is said to be uniquely geodesic if every two points of X are joined by exactly one geodesic. A subset C of X is said to be convex if C includes every geodesic segments joining two of its points. Let $x, y \in X$ and $t \in [0, 1]$, we write $tx \oplus (1 - t)y$ for the unique point z in the geodesic segment joining from x to y such that

$$d(x,z) = (1-t)d(x,y) \text{ and } d(z,y) = td(x,y). \tag{2.1}$$

A geodesic triangle $\Delta(x_1, x_2, x_3)$ in a geodesic metric space (X,d) consists of three vertices (points in X) with unparameterized geodesic segment between each pair of vertices. For any geodesic triangle there is comparison (Alexandrov) triangle $\bar{\Delta} \subset \mathbb{R}^2$ such that $d(x_i, x_j) = d_{\mathbb{R}^2}(\bar{x}_i, \bar{x}_j)$ for $i, j \in \{1,2,3\}$. Let Δ be a geodesic triangle in X and $\bar{\Delta}$ be a comparison triangle for $\bar{\Delta}$, then Δ is said to satisfy the CAT(0) inequality if for all points $x, y \in \Delta$ and $\bar{x}, \bar{y} \in \bar{\Delta}$,

$$d(x,y) \leq d_{\mathbb{R}^2}(\bar{x}, \bar{y}). \tag{2.2}$$

Let x, y, z be points in X and y_0 be the midpoint of the segment $[y,z]$, then the CAT(0) inequality implies

$$d^2(x, y_0) \leq \frac{1}{2}d^2(x,y) + \frac{1}{2}d^2(x,z) - \frac{1}{4}d(y,z). \tag{2.3}$$

Definition 2.3. A geodesic space X is said to be a CAT(0) space if all geodesic triangles satisfy inequality (2.3). A complete CAT(0) space is called an Hadamard space.

Definition 2.4. [8] Let X be a CAT(0) space and let the pair $(a,b) \in X \times X$ which is denoted by $\overrightarrow{ab}$, be called a vector. A quasilinearization mapping

$$\langle \cdot, \cdot \rangle : (X \times X) \times (X \times X) \to \mathbb{R}$$

is defined by

$$\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle = \frac{1}{2}\left(d^2(a,d) + d^2(b,c) - d^2(a,c) - d^2(b,d)\right) \ \forall a,b,c,d \in X.$$

One can easily verify that

$$\langle \overrightarrow{ab}, \overrightarrow{ab} \rangle = d^2(a,b), \ \langle \overrightarrow{ba}, \ \overrightarrow{cd} \rangle = -\langle \overrightarrow{ab}, \ \overrightarrow{cd} \rangle, \ \langle \overrightarrow{ab}, \ \overrightarrow{cd} \rangle = \langle \overrightarrow{ae}, \ \overrightarrow{cd} \rangle + \langle \overrightarrow{eb}, \ \overrightarrow{cd} \rangle$$

and $\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle = \langle \overrightarrow{cd}, \overrightarrow{ab} \rangle$ for all $a,b,c,d,e \in X$. A geodesic space X is said to satisfy the Cauchy-Swartz inequality if $\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle \leq d(a,b)d(c,d) \ \forall a,b,c,d \in X$. It has been established in [8] that a geodesically connected metric space is a CAT(0) space if and only if it satisfies the Cauchy-Schwartz inequality. Examples of CAT(0) spaces includes: Euclidean spaces $\mathbb{R}^n$, Hilbert spaces, simply connected Riemannian manifolds of nonpositive sectional curvature [44], $\mathbb{R}$-trees, Hilbert ball [21], among others. We also note that CAT(0) spaces are uniquely geodesic spaces.

We end this subsection with the following important lemmas which characterizes CAT(0) spaces.

Lemma 2.5. Let X be a CAT(0) space, $x, y, z \in X$ and $t, s \in [0,1]$. Then

(i) $d(tx \oplus (1-t)y, z) \leq td(x,z) + (1-t)d(y,z)$ (see[17]).
(ii) $d^2(tx \oplus (1-t)y, z) \leq td^2(x,z) + (1-t)d^2(y,z) - t(1-t)d^2(x,y)$ (see [17]).
(iii) $d^2(tx \oplus (1-t)y, z) \leq t^2d^2(x,z) + (1-t)^2d^2(y,z) + 2t(1-t)\langle \overrightarrow{xz}, \overrightarrow{yz} \rangle$ (see [15]).
(iv) $d(tw \oplus (1-t)x, ty \oplus (1-t)z) \leq td(w,y) + (1-t)d(x,z)$ (see [11]).

2.2 The notion of Δ-convergence

Definition 2.6. Let $\{x_n\}$ be a bounded sequence in a CAT(0) space X. Then, the asymptotic center $A(\{x_n\})$ of $\{x_n\}$ is defined by

$$A(\{x_n\}) = \{\bar{v} \in X : \limsup_{n\to\infty} d(\bar{v}, x_n) = \inf_{v\in X} \limsup_{n\to\infty} d(v, x_n)\}.$$

It is generally known that in an Hadamard space, $A(\{x_n\})$ consists of exactly one point. A sequence $\{x_n\}$ in X is said to be Δ-convergent to a point $\bar{v} \in X$ if $A(\{x_{n_k}\}) = \{\bar{v}\}$ for every subsequence $\{x_{n_k}\}$ of $\{x_n\}$. In this case, we write $\Delta\text{-}\lim_{n\to\infty} x_n = \bar{v}$ (see [16]). The notion of Δ-convergence in metric spaces is known as analogue of the classical notion of weak convergence in Banach spaces (see [29]).

The following lemmas are very important as regards to Δ-convergence in Hadamard spaces.

Lemma 2.7. [17] Every bounded sequence in an Hadamard space always have a $\triangle$-convergent subsequence.

Lemma 2.8. [32] Let X be an Hadamard space. Then, every bounded sequence in X has a unique asymptotic center.

Lemma 2.9. [27] Let X be an Hadamard space, $\{x_n\}$ be a sequence in X and $v \in X$. Then $\{x_n\}\triangle-$converges to v if and only if $\limsup_{n\to\infty}\langle \overrightarrow{vx_n}, \overrightarrow{vy}\rangle \leq 0$ for all $y \in C$.

Lemma 2.10. [43, Opial's Lemma] Let X be an Hadamard space and $\{x_n\}$ be a sequence in X. If there exists a nonempty subset F in which

(i) $\lim_{n\to\infty} d(x_n, z)$ exists for every $z \in F$, and

(ii) if $\{x_{n_k}\}$ is a subsequence of $\{x_n\}$ which is Δ-convergent to x, then $x \in F$.

Then, there is a $p \in F$ such that $\{x_n\}$ is Δ-convergent to p in X.

Lemma 2.11. [42] Let C be a nonempty, closed and convex subset of a complete CAT(0) space X and $T : C \to C$ be a generalized asymptotically nonspreading mapping. Let $\{x_n\}$ be a bounded sequence in C such that $\{x_n\}$ Δ-converges to v and $\lim_{n\to\infty} d(x_n, Tx_n) = 0$. Then, $Tv = v$.

2.3 Existence of solution of equilibrium problems and resolvent operators

Theorem 2.12. [31, Theorem 4.1] Let C be a nonempty closed and convex subset of an Hadamard space X and $f : C \times C \to \mathbb{R}$ be a bifunction satisfying the following:

(A1) $f(x, x) \geq 0$ for each $x \in C$,

(A2) for every $x \in C$, the set $\{y \in C : f(x, y) < 0\}$ is convex,

(A3) for every $y \in C$, the function $x \mapsto f(x, y)$ is upper semicontinuous,

(A4) there exists a compact subset $L \subset C$ containing a point $y_0 \in L$ such that $f(x, y_0) < 0$ whenever $x \in C\backslash L$.

Then, problem (1.3) has a solution.

In [31], the authors introduce the resolvent of the bifunction f associated with the EP (1.3). They defined a perturbed bifunction $\bar{f}_{\bar{x}} : C \times C \to \mathbb{R}$ ($\bar{x} \in X$) of f by

$$\bar{f}_{\bar{x}}(x, y) := f(x, y) - \langle \overrightarrow{\bar{x}x}, \overrightarrow{xy}\rangle, \ \forall x, y \in C. \tag{2.4}$$

The perturbed bifunction $\bar{f}$ has a unique equilibrium, called the resolvent operator $J^f : X \to 2^C$ of the bifunction f (see [31]), which is defined by

$$J^f(x) := EP(C, \bar{f}_x) = \{w \in C : f(w,y) - \langle \overrightarrow{wx}, \overrightarrow{wy} \rangle \geq 0,\ y \in C\},\ x \in X. \qquad (2.5)$$

It was established in [31] that J^f is well defined. In the next subsection, we shall study some of the basic properties of this resolvent operator.

2.4. Fundamental properties of resolvent operators

We begin this subsection with the following definitions which will be needed in the sequel.

Definition 2.13. Let X be a CAT(0) space and C be a nonempty closed and convex subset of X. A function $f : C \times C \to \mathbb{R}$ is called monotone if $f(x,y) + f(y,x) \leq 0$ for all $x, y \in C$.

Definition 2.14. Let X be a CAT(0) space. A function $f : D(f) \subseteq X \to (-\infty, +\infty]$ is said to be convex if

$$f(tx \oplus (1-t)y) \leq tf(x) + (1-t)f(y)\ \forall x, y \in X,\ t \in (0,1).$$

f is called proper, if $D(f) := \{x \in X : f(x) < +\infty\} \neq \emptyset$. The function $f : D(f) \to (-\infty, \infty]$ is lower semi-continuous at a point $x \in D(f)$ if $f(x) \leq \liminf_{n\to\infty} f(x_n)$, for each sequence $\{x_n\}$ in $D(f)$ such that $\lim_{n\to\infty} x_n = x$; f is said to be lower semi-continuous on $D(f)$ if it is lower semi-continuous at any point in $D(f)$.

Lemma 2.15. [31, Proposition 5.4] Let C be a nonempty closed and convex subset of an Hadamard space X and f be a monotone bifunction, with $D(J^f) \neq \emptyset$. Then, the following properties hold.

(i) J^f is single-valued.
(ii) If $D(J^f) \supset C$, then J^f is nonexpansive restricted to C.
(iii) If $D(J^f) \supset C$, then $F(J^f) = EP(f, C)$.

Theorem 2.16. [31, Theorem 5.2] Let X be an Hadamard space and C be a nonempty closed and convex subset of X. Suppose that f has the following properties

(i) $f(x,x) = 0$ for all $x \in C$,
(ii) f is monotone,
(iii) for each $x \in C$, $y \mapsto f(x,y)$ is convex and lower semicontinuous,
(iv) for each $x \in C$, $f(x,y) \geq \limsup_{t\downarrow 0} f((1-t)x \oplus tz, y)$ for all $x, z \in C$.

Then $D(J^f) = X$ and J^f single-valued.

Remark 2.17. By (2.5), we have that the resolvent J^f_λ of the bifunction f and of order $\lambda > 0$, is given as

$$J^f_\lambda(x) := EP(C, \bar{f}_x) = \{w \in C : f(w,y) + \frac{1}{\lambda}\langle \overrightarrow{xw}, \overrightarrow{wy} \rangle \geq 0,\ y \in C\},\ x \in X, \qquad (2.6)$$

where $\bar{f}$ is defined in this case as

$$\bar{f}_{\bar{x}}(x,y) := f(x,y) + \frac{1}{\lambda}\langle \overrightarrow{\bar{x}\,x}, \overrightarrow{xy} \rangle,\ \forall x, y \in C,\ \bar{x} \in X. \qquad (2.7)$$

Lemma 2.18. [24] Let C be a nonempty, closed and convex subset of an Hadamard space X and $f : C \times C \to \mathbb{R}$ be a monotone bifunction such that $C \subset D(J_\lambda^f)$ for $\lambda > 0$. Then, the following hold:

(i) J_λ^f is firmly nonexpansive restricted to C. That is,

$$d^2(J_\lambda^f x, J_\lambda^f y) \leq \langle \overrightarrow{J_\lambda^f x J_\lambda^f y}, \overrightarrow{xy} \rangle \ \forall x, y \in C, \ \lambda > 0.$$

(ii) If $F(J_\lambda^f) \neq \emptyset$, then

$$d^2(J_\lambda^f x, x) \leq d^2(x, v) - d^2(J_\lambda^f x, v) \ \forall x \in C, \ v \in F(J_\lambda^f).$$

(iii) If $0 < \lambda \leq \mu$, then $d(J_\mu^f x, J_\lambda^f x) \leq \sqrt{1 - \frac{\lambda}{\mu}} d(x, J_\mu^f x)$, which implies that $d(x, J_\lambda^f x) \leq 2d(x, J_\mu^f x) \ \forall x \in C$.

Remark 2.19. (See also [24]) If the bifunction f satisfies assumptions (i)-(iv) of Theorem 2.16, then by Theorem 2.16, $D(J_\lambda^f) = X$ for any $\lambda > 0$ and hence, the conclusions of Lemma 2.18 hold in the whole space X.

The following lemma will be very useful in establishing our strong convergence theorem.

Lemma 2.20. (Xu, [48]) Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying the following relation:

$$a_{n+1} \leq (1 - \alpha_n) a_n + \alpha_n \sigma_n + \gamma_n, n \geq 0,$$

where

(i) $\{\alpha_n\} \subset [0, 1]$, $\sum \alpha_n = \infty$;

(ii) $\limsup \ \sigma_n \leq 0$; (iii) $\gamma_n \geq 0$; $(n \geq 0)$, $\sum \gamma_n < \infty$.

Then, $a_n \to 0$ as $n \to \infty$.

3. Strong convergence analysis

We now present our strong convergence theorems.

3.1 Viscosity-type proximal point algorithm

Theorem 3.1. Let C be a nonempty closed and convex subset of an Hadamard space X and $f_i : C \times C \to \mathbb{R}$, $i = 1, 2, \ldots, N$ be a finite family of bifunctions satisfying assumptions (i)-(iv) of Theorem 2.16. Let $T : C \to C$ be a uniformly L-Lipschitzian generalized asymptotically nonspreading mapping which is also asymptotically regular, and g be a contraction mapping on C with coefficient $\gamma \in (0, 1)$. Suppose that $\Gamma := \cap_{i=1}^N EP(f_i, C) \cap F(T) \neq \emptyset$ and for arbitrary $x_1 \in C$, the sequence $\{x_n\}$ is generated by

$$\begin{cases} y_n = J_{\lambda_n}^{f_N} \circ J_{\lambda_n}^{f_{(N-1)}} \circ \cdots \circ J_{\lambda_n}^{f_2} \circ J_{\lambda_n}^{f_1} x_n, \\ x_n = \alpha_n g(y_n) \oplus (1 - \alpha_n) T^n y_n, \ n \geq 1, \end{cases} \tag{3.1}$$

where $0 < \lambda_n \leq \lambda \ \forall n \geq 1$ and $\{\alpha_n\}$ is in $(0, 1)$ satisfying the following conditions:

(i) $\lim\limits_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^\infty \alpha_n = \infty$,

(ii) $L < (1 - \alpha_n \gamma)/(1 - \alpha_n)$.

Then, $\{x_n\}$ converges strongly to $w \in \Gamma$ which solves the variational inequality

$$\langle \overrightarrow{wg(w)}, \overrightarrow{uw} \rangle \geq 0, \quad \forall u \in \Gamma. \tag{3.2}$$

Proof. By Lemma 2.15, we obtain that, for any $v \in \Gamma$, $v = J_{\lambda_n}^{f_i} v$ and $J_{\lambda_n}^{f_i}$ is nonexpansive for each $i = 1, 2, \dots, N$. Also, by Remark 2.19, we have that $D(J_{\lambda_n}^{f_i}) = X$ for each $i = 1, 2, \dots, N$.
We now divide our proof into steps.
Step 1. We show that (3.1) is well defined. Now, define the mapping $T_n^g : C \to C$ by

$$T_n^g x = \alpha_n g(y) \oplus (1 - \alpha_n) T^n y,$$

where $y = J_{\lambda_n}^{f_N} \circ J_{\lambda_n}^{f_{(N-1)}} \circ \cdots \circ J_{\lambda_n}^{f_2} \circ J_{\lambda_n}^{f_1} x$ for all $n \geq 1$. Then, by Lemma 2.5 (iv), we obtain that

$$\begin{aligned} d(T_n^g x_1, T_n^g x_2) &\leq \alpha_n d(g(y_1), g(y_2)) + (1 - \alpha_n) d(T^n y_1, T^n y_2) \\ &\leq \gamma \alpha_n d(y_1, y_2) + (1 - \alpha_n) L d(y_1, y_2) \\ &\leq (\gamma \alpha_n + (1 - \alpha_n) L)\, d(x_1, x_2). \end{aligned}$$

By condition (ii), we have that $0 < (\gamma \alpha_n + (1 - \alpha_n) L) < 1$. Hence, T_n^g is a contraction for each $n \geq 1$. Therefore, by Banach contraction mapping principle, there exists a unique fixed point x_n of T_n^g for each $n \geq 1$. Thus, (3.1) is well defined.
Step 2. We show that $\{x_n\}$ is bounded. Let $v \in \Gamma$, since T is generalized asymptotically nonspreading, we obtain that

$$(1 - g(v)) d^2(v, T^n y_n) \leq f(v) d^2(v, y_n).$$

Since $0 < f(v) + g(v) \leq 1$, we obtain that

$$d(v, T^n y_n) \leq d(v, y_n). \tag{3.3}$$

Thus, by (3.1) and Lemma 2.5 (i), we obtain

$$\begin{aligned} d(x_n, v) &= d(\alpha_n g(y_n) \oplus (1 - \alpha_n) T^n y_n, v) \\ &\leq \alpha_n d(g(y_n), v) + (1 - \alpha_n) d(T^n y_n, v) \\ &\leq \alpha_n \gamma d(y_n, v) + \alpha_n d(g(v), v) + (1 - \alpha_n) d(y_n, v) \\ &\leq \alpha_n \gamma d(x_n, v) + \alpha_n d(g(v), v) + (1 - \alpha_n) d(x_n, v) \\ &= \Big(1 - \alpha_n (1 - \gamma)\Big) d(x_n, v) + \alpha_n d(g(v), v), \end{aligned} \tag{3.4}$$

which implies that

$$d(x_n, v) \leq \frac{d(g(v), v)}{1 - \gamma}.$$

Thus, $\{x_n\}$ is bounded. Consequently, $\{y_n\}$ $\{T^n y_n\}$ and $\{g(y_n)\}$ are all bounded.
Step 3. We show that $\lim\limits_{n \to \infty} d(J_{\lambda^{(i)}} x_n, x_n) = 0 = \lim\limits_{n \to \infty} d(y_n, T y_n),\ i = 1, 2, \dots, N.$
Now, by (3.1), we get

$$\begin{aligned} d(x_n, T^n y_n) &= d(\alpha_n g(y_n) \oplus (1 - \alpha_n) T^n y_n, T^n y_n) \\ &\leq \alpha_n d(g(y_n), T^n y_n) \to 0, \text{ as } n \to \infty. \end{aligned} \tag{3.5}$$

Also, we obtain from Lemma 2.5 (ii) and (3.3) that

$$\begin{aligned} d^2(x_n, v) &= d^2(\alpha_n g(y_n) \oplus (1-\alpha_n)T^n y_n, v) \\ &\leq \alpha_n d^2(g(y_n), v) + (1-\alpha_n) d^2(T^n y_n, v) \\ &\leq \alpha_n d^2(g(y_n), v) + (1-\alpha_n) d^2(y_n, v). \end{aligned} \tag{3.6}$$

Set $u_n^{(i+1)} = J_{\lambda_n}^{f_i} u_n^{(i)}$, for each $i = 1, 2, \ldots, N$, where $u_n^{(1)} = x_n$, for all $n \geq 1$. Then, $u_n^{(2)} = J_{\lambda_n}^{f_1}(x_n)$, $u_n^{(3)} = J_{\lambda_n}^{f_2} \circ J_{\lambda_n}^{f_1}(x_n)$, $\ldots$, $u_n^{(N+1)} = y_n$. Then by Lemma 2.18 (ii), we obtain for each $i = 1, 2, \ldots, N$ that

$$d^2(u_n^{(i+1)}, v) \leq d^2(u_n^{(i)}, v) - d^2(u_n^{(i)}, u_n^{(i+1)}). \tag{3.7}$$

For $i = N$, we obtain from (3.6) and (3.7) that

$$\begin{aligned} d^2(x_n, v) &\leq \alpha_n d^2(g(y_n), v) + (1-\alpha_n) d^2(u_n^{(N+1)}, v) \\ &\leq \alpha_n d^2(g(y_n), v) + (1-\alpha_n) d^2(u_n^{(N)}, v) - (1-\alpha_n) d^2(u_n^{(N)}, u_n^{(N+1)}) \\ &\leq \alpha_n d^2(g(y_n), v) + (1-\alpha_n) d^2(x_n, v) - (1-\alpha_n) d^2(u_n^{(N)}, u_n^{(N+1)}) \\ &\leq \alpha_n (d^2(g(y_n), v) - d^2(x_n, v)) + d^2(x_n, v) - (1-\alpha_n) d^2(u_n^{(N)}, u_n^{(N+1)}), \end{aligned}$$

which implies by condition (i) that

$$\lim_{n\to\infty} d^2(u_n^{(N)}, u_n^{(N+1)}) = 0. \tag{3.8}$$

In a similar way, we can get that

$$\lim_{n\to\infty} d^2(u_n^{(N-1)}, u_n^{(N)}) = 0. \tag{3.9}$$

Thus, if we continue in the same manner, we can show that

$$\lim_{n\to\infty} d(u_n^{(i)}, u_n^{(i+1)}) = 0, \ i = 1, 2, \ldots, N. \tag{3.10}$$

From (3.10), and applying triangle inequality, we obtain for each $i = 1, 2, \ldots, N$, that

$$\lim_{n\to\infty} d(x_n, u_n^{(i+1)}) = 0. \tag{3.11}$$

Thus, for $i = N$, we have

$$\lim_{n\to\infty} d(x_n, y_n) = 0. \tag{3.12}$$

Since $0 < \lambda_n \leq \lambda$ for all $n \geq 1$, we obtain by Lemma 2.18 (iii) and (3.10) that

$$d\left(u_n^{(i)}, J_\lambda^{f_i} u_n^{(i)}\right) \leq 2d\left(u_n^{(i)}, J_{\lambda_n}^{f_i} u_n^{(i)}\right) \to 0, \text{ as } n \to \infty, \ i = 1, 2, \ldots, N. \tag{3.13}$$

Again, since $J_\lambda^{f_i}$ is nonexpansive for each i, we obtain from (3.10) and (3.11) that

$$\begin{aligned} d(J_\lambda^{f_i} x_n, J_\lambda^{f_i} u_n^{(i)}) &\leq d(J_\lambda^{f_i} x_n, J_\lambda^{f_i} u_n^{(i+1)}) + d(J_\lambda^{f_i} u_n^{(i+1)}, J_\lambda^{f_i} u_n^{(i)}) \\ &\leq d(x_n, u_n^{(i+1)}) + d(u_n^{(i+1)}, u_n^{(i)}) \to 0, \text{ as } n \to \infty. \end{aligned} \tag{3.14}$$

From (3.10) to (3.14), we obtain for each $i = 1, 2, \ldots, N$ that

$$\begin{aligned} d\left(J_\lambda^{f_i} x_n, x_n\right) &\leq d\left(J_\lambda^{f_i} x_n, J_\lambda^{f_i} u_n^{(i)}\right) + d\left(J_\lambda^{f_i} u_n^{(i)}, u_n^{(i)}\right) \\ &\quad + d\left(u_n^{(i)}, u_n^{(i+1)}\right) + d\left(u_n^{(i+1)}, x_n\right) \to 0 \text{ as } n \to \infty. \end{aligned} \tag{3.15}$$

Furthermore, we obtain from (3.5) and (3.12) that

$$\lim_{n\to\infty} d(y_n, T^n y_n) = 0. \tag{3.16}$$

Since T is asymptotically regular, we obtain that

$$\begin{aligned} d(y_n, Ty_n) &\le d(y_n, T^n y_n) + d(T^n y_n, T^{n+1} y_n) + d(T^{n+1} y_n, Ty_n) \\ &\le (1+L) d(y_n, T^n y_n) + d(T^{n+1} y_n, T^n y_n) \to 0, \text{ as } n \to \infty. \end{aligned} \tag{3.17}$$

Since $\{x_n\}$ is bounded and X is an Hadamard space, we obtain from Lemma 2.7 that there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ which $\triangle$-converges to w. It then follows from (3.12) that there exists a subsequence $\{y_{n_k}\}$ of $\{y_n\}$ which $\triangle$-converges to w. Thus, from (3.17) and Lemma 2.11, we obtain that $w \in F(T)$. Also, since $J_\lambda^{f_i}$ is nonexpansive for each i and every nonexpansive mapping is demiclosed, we obtain from (3.15) that $w \in F(J_\lambda^{f_i})$. Hence, $w \in \Gamma$.

Step 4. We now show that $\{x_n\}$ converges strongly to w. Since $\{y_{n_k}\}$ $\triangle-$converges to $w \in \Gamma$, we obtain by Lemma 2.9 that

$$\lim_{k\to\infty} \langle \overrightarrow{g(w)w}, \overrightarrow{y_{n_k}w} \rangle \le 0. \tag{3.18}$$

Also, by Lemma 2.5 (iii) and (3.1), we have

$$\begin{aligned} d^2(x_n, w) &= d^2(\alpha_n g(y_n) \oplus (1-\alpha_n) T^n y_n, w) \\ &\le \alpha_n^2 d^2(g(y_n), w) + (1-\alpha_n) d^2(T^n y_n, w) \\ &\quad + 2\alpha_n(1-\alpha_n) \langle \overrightarrow{g(y_n)w}, \overrightarrow{T^n y_n w} \rangle \\ &\le \alpha_n^2 d^2(g(y_n), w) + (1-\alpha_n) d^2(y_n, w) \\ &\quad + 2\alpha_n(1-\alpha_n) [\langle \overrightarrow{g(y_n)w}, \overrightarrow{T^n y_n y_n} \rangle + \langle \overrightarrow{g(y_n)g(w)}, \overrightarrow{y_n w} \rangle \\ &\quad + \langle \overrightarrow{g(w)w}, \overrightarrow{y_n w} \rangle] \\ &\le \alpha_n^2 d^2(g(y_n), w) + (1-\alpha_n) d^2(y_n, w) \\ &\quad + 2\alpha_n(1-\alpha_n) [\langle \overrightarrow{g(y_n)w}, \overrightarrow{T^n y_n y_n} \rangle + \gamma d^2(y_n, w) + \langle \overrightarrow{g(w)w}, \overrightarrow{y_n w} \rangle] \\ &\le \Big[(1-\alpha_n) + 2\gamma\alpha_n(1-\alpha_n)\Big] d^2(x_n, w) \\ &\quad + \alpha_n \Big[\alpha_n d^2(g(y_n), w) + 2(1-\alpha_n) d(T^n y_n, y_n)\Big] d(g(y_n), w) \\ &\quad + 2\alpha_n(1-\alpha_n) \langle \overrightarrow{g(w)w}, \overrightarrow{y_n w} \rangle, \end{aligned} \tag{3.19}$$

which implies

$$\begin{aligned} d^2(x_n, w) &\le \frac{[\alpha_n d^2(g(y_n), w) + 2(1-\alpha_n) d(T^n y_n, y_n)] d(g(y_n), w)}{[1 - 2\gamma(1-\alpha_n)]} \\ &\quad + \frac{2(1-\alpha_n) \langle \overrightarrow{g(w)w}, \overrightarrow{y_n w} \rangle}{[1 - 2\gamma(1-\alpha_n)]}. \end{aligned} \tag{3.20}$$

Thus, by condition (i), (3.16) and (3.18), we obtain

$$\lim_{k\to\infty} d^2(x_{n_k}, w) = 0.$$

Therefore, $\lim_{k\to\infty} x_{n_k} = w$.

Step 5. Lastly, we show that w is a solution of (3.2). From Lemma 2.5 (ii) and (3.1), we obtain for all $u \in \Gamma$ that

$$\begin{aligned} d^2(x_m,u) &\le \alpha_m d^2(g(y_m),u) + (1-\alpha_m)d^2(T^m y_m,u) \\ &\quad -\alpha_m(1-\alpha_m)d^2(g(y_m),T^m y_m) \\ &\le \alpha_m d^2(g(y_m),u) + (1-\alpha_m)d(x_m,u) \\ &\quad -\alpha_m(1-\alpha_m)d^2(g(y_m),T^m y_m), \end{aligned}$$

which implies that

$$d^2(x_m,u) \le d^2(g(y_m),u) - (1-\alpha_m)d^2(g(y_m),T^m y_m).$$

Thus, taking limit as $m \to \infty$, we obtain

$$d^2(w,u) \le d^2(g(w),u) - d^2(g(w),w).$$

Hence,

$$\langle \overrightarrow{wg(w)}, \overrightarrow{uw} \rangle = \frac{1}{2}\Big(d^2(g(w),u) - d^2(w,u) - d^2(g(w),w)\Big) \ge 0, \ \forall u \in \Gamma.$$

Therefore, we have that w solves the variational inequality (3.2).

Now, assume that $\{x_{n_k}\}$ $\triangle-$converges to u. Then, by the same argument, we also obtain that $u \in \Gamma$ solves the variational inequality (3.2). That is,

$$\langle \overrightarrow{ug(u)}, \overrightarrow{uw} \rangle \le 0. \quad \text{Also} \quad \langle \overrightarrow{wg(w)}, \overrightarrow{wu} \rangle \le 0.$$

Thus, we obtain that

$$\begin{aligned} (1-\gamma)d^2(w,u) &= d^2(w,u) - \gamma d^2(u,w) \\ &\le \langle \overrightarrow{wu}, \overrightarrow{wu} \rangle - d(g(u)g(w))d(u,w) \\ &\le \langle \overrightarrow{wu}, \overrightarrow{wu} \rangle - \langle \overrightarrow{g(u)g(w)}, \overrightarrow{uw} \rangle \\ &= -\langle \overrightarrow{uw}, \overrightarrow{wu} \rangle + \langle \overrightarrow{g(u)g(w)}, \overrightarrow{wu} \rangle \\ &\quad +\langle \overrightarrow{wg(u)}, \overrightarrow{wu} \rangle - \langle \overrightarrow{wg(u)}, \overrightarrow{wu} \rangle \\ &= \langle \overrightarrow{ug(u)}, \overrightarrow{uw} \rangle + \langle \overrightarrow{wg(w)}, \overrightarrow{wu} \rangle \le 0, \end{aligned}$$

which implies that $d(w,u) = 0$. Hence, $w = u$. Therefore, $\{x_n\}$ converges strongly to w, which is a solution of the variational inequality (3.2).

By setting $g(x) = z$ for arbitrary but fixed $z \in C$ and for all $x \in C$ in Theorem 3.1, we obtain the following corollary whose algorithm is of Halpern-type.

Corollary 3.2. Let C be a nonempty closed and convex subset of an Hadamard space X and $f_i : C \times C \to \mathbb{R}, \ i = 1,2,\ldots,N$ be a finite family of bifunctions satisfying assumptions (i)-(iv) of Theorem 2.16. Let $T : C \to C$ be a uniformly L-Lipschitzian generalized asymptotically nonspreading mapping which is also asymptotically regular. Suppose that $\Gamma := \cap_{i=1}^N EP(f_i,C) \cap F(T) \neq \emptyset$ and for arbitrary $z, x_1 \in C$, the sequence $\{x_n\}$ is generated by

$$\begin{cases} y_n = J_{\lambda_n}^{f_N} \circ J_{\lambda_n}^{f_{(N-1)}} \circ \cdots \circ J_{\lambda_n}^{f_2} \circ J_{\lambda_n}^{f_1} x_n, \\ x_n = \alpha_n z \oplus (1-\alpha_n)T^n y_n, \ n \ge 1, \end{cases} \tag{3.21}$$

where $0 < \lambda_n \leq \lambda \ \forall n \geq 1$ and $\{\alpha_n\}$ is in $(0,1)$ satisfying the following conditions:

(i) $\lim\limits_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(ii) $L < (1-\alpha_n\gamma)/(1-\alpha_n)$.

Then, $\{x_n\}$ converges strongly to $w \in \Gamma$ which is nearest to z.

By setting $N = 1$ in Theorem 3.1, we obtain the following corollary.

Corollary 3.3. Let C be a nonempty closed and convex subset of an Hadamard space X and $f : C \times C \to \mathbb{R}$ be a bifunctions satisfying assumptions (i)-(iv) of Theorem 2.16. Let $T : C \to C$ be a uniformly L-Lipschitzian generalized asymptotically nonspreading mapping which is also asymptotically regular, and g be a contraction mapping on C with coefficient $\gamma \in (0,1)$. Suppose that $\Gamma := EP(f,C) \cap F(T) \neq \emptyset$ and for arbitrary $x_1 \in C$, the sequence $\{x_n\}$ is generated by

$$\begin{cases} y_n = J^f_{\lambda_n} x_n, \\ x_n = \alpha_n g(y_n) \oplus (1-\alpha_n) T^n y_n, \ n \geq 1, \end{cases} \tag{3.22}$$

where $0 < \lambda_n \leq \lambda \ \forall n \geq 1$ and $\{\alpha_n\}$ is in $(0,1)$ satisfying the following conditions:

(i) $\lim\limits_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(ii) $L < (1-\alpha_n\gamma)/(1-\alpha_n)$.

Then, $\{x_n\}$ converges strongly to $w \in \Gamma$ which solves the variational inequality

$$\langle \overrightarrow{wg(w)}, \overrightarrow{uw} \rangle \geq 0, \quad \forall u \in \Gamma. \tag{3.23}$$

Corollary 3.4. Let C be a nonempty closed and convex subset of an Hadamard space X and $T : C \to C$ be a uniformly L-Lipschitzian generalized asymptotically nonspreading mapping which is also asymptotically regular. Let g be a contraction mapping on C with coefficient $\gamma \in (0,1)$. Suppose that $F(T) \neq \emptyset$ and for arbitrary $x_1 \in C$, the sequence $\{x_n\}$ is generated by

$$x_n = \alpha_n g(x_n) \oplus (1-\alpha_n) T^n x_n, \ n \geq 1, \tag{3.24}$$

where $\{\alpha_n\}$ is in $(0,1)$ satisfying the following conditions:

(i) $\lim\limits_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(ii) $L < (1-\alpha_n\gamma)/(1-\alpha_n)$.

Then, $\{x_n\}$ converges strongly to $w \in F(T)$ which solves the variational inequality

$$\langle \overrightarrow{wg(w)}, \overrightarrow{uw} \rangle \geq 0, \quad \forall u \in F(T). \tag{3.25}$$

3.2. The asymptotic behavior of viscosity-type proximal point algorithm

In this subsection, we study the asymptotic behavior of the sequence given by the following viscosity-type PPA and extend the study to approximate a common solution of finite family of equilibrium problems. For $x_1 \in C$, define the sequence $\{x_n\} \subset C$ by

$$x_{n+1} = \alpha_n g(x_n) \oplus (1-\alpha_n) J^f_{\lambda_n} x_n, \tag{3.26}$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$, $\{\lambda_n\}$ is in $(0,\infty)$, g is a contraction on C and f is a bifunction from $C \times C$ into $\mathbb{R}$.

We begin by first establishing the following lemmas which we will be needing for our strong convergence analysis.

Lemma 3.5. Let C be a nonempty closed and convex subset of an Hadamard space X and $f : C \times C \to \mathbb{R}$ be a bifunction satisfying assumptions (i)-(iv) of Theorem 2.16. Then, for $\lambda, \mu > 0$ and $x, y \in C$, we have the following inequalities:

$$d^2(J_\lambda^f x, J_\mu^f y) \leq 2\lambda f(J_\lambda^f x, J_\mu^f y) + d^2(x, J_\mu^f y) - d^2(x, J_\lambda^f x) \tag{3.27}$$

and

$$(\lambda+\mu) d^2(J_\lambda^f x, J_\mu^f y) + \mu d^2(J_\lambda^f x, x) + \lambda d^2(J_\mu^f y, y) \leq \lambda d^2(J_\lambda^f x, y) + \mu d^2(J_\lambda^f y, x). \tag{3.28}$$

Proof. We first prove (3.27). Let $\lambda,\ \mu > 0$ and $x, y \in C$. Then, by the definition of the resolvent, we obtain that

$$f(J_\lambda^f x, z) + \frac{1}{\lambda}\langle \overrightarrow{xJ_\lambda^f x}, \overrightarrow{J_\lambda^f xz}\rangle \geq 0 \ \forall\ z \in C,$$

which implies that

$$\begin{aligned} 0 &\leq 2\lambda f(J_\lambda^f x, z) + 2\langle \overrightarrow{xJ_\lambda^f x}, \overrightarrow{J_\lambda^f xz}\rangle \\ &= 2\lambda f(J_\lambda^f x, z) + d^2(x, z) - d^2(x, J_\lambda^f) - d^2(J_\lambda^f x, z) \\ &\leq 2\lambda f(J_\lambda^f x, z) + d^2(x, z) - d^2(x, J_\lambda^f x). \end{aligned} \tag{3.29}$$

Now, set $z = tJ_\mu^f y \oplus (1-t)J_\lambda^f x$ for all $t \in (0,1)$ in (3.29). Since f satisfies conditions (i) and (iii) of Theorem 2.16, we obtain that

$$\begin{aligned} d^2(x, J_\lambda^f x) &\leq 2\lambda\Big(tf(J_\lambda^f x, J_\mu^f y) + (1-t)f(J_\lambda^f x, J_\lambda^f x)\Big) \\ &\quad + td^2(x, J_\mu^f y) + (1-t)d^2(x, J_\lambda^f x) - t(1-t)d^2(J_\mu^f y, J_\lambda^f x) \\ &= 2\lambda t f(J_\lambda^f x, J_\mu^f y) + td^2(x, J_\mu^f y) + (1-t)d^2(x, J_\lambda^f x) \\ &\quad - t(1-t)d^2(J_\mu^f y, J_\lambda^f x), \end{aligned} \tag{3.30}$$

which implies that

$$d^2(x, J_\lambda^f x) \leq 2\lambda f(J_\lambda^f x, J_\mu^f y) + d^2(x, J_\mu^f y) - (1-t)d^2(J_\mu^f y, J_\lambda^f x). \tag{3.31}$$

Thus, taking limit as $t \to 0$, we obtain

$$d^2(J_\lambda^f x, J_\mu^f y) \leq 2\lambda f(J_\lambda^f x, J_\mu^f y) + d^2(x, J_\mu^f y) - d^2(x, J_\lambda^f x). \tag{3.32}$$

Next, we prove (3.28). From (3.32), we obtain that

$$\mu d^2(J_\lambda^f x, J_\mu^f y) \leq 2\lambda\mu f(J_\lambda^f x, J_\mu^f y) + \mu d^2(x, J_\mu^f y) - \mu d^2(x, J_\lambda^f x).$$

Similarly, we have

$$\lambda d^2(J_\mu^f y, J_\lambda^f x) \leq 2\mu\lambda f(J_\mu^f y, J_\lambda^f x) + \lambda d^2(y, J_\lambda^f x) - \lambda d^2(y, J_\mu^f y).$$

Adding both inequalities and using condition (ii) of Theorem 2.16, we get

$$(\lambda+\mu) d^2(J_\lambda^f x, J_\mu^f y) + \mu d^2(x, J_\lambda^f x) + \lambda d^2(y, J_\mu^f y) \leq \mu d^2(x, J_\mu^f y) + \lambda d^2(y, J_\lambda^f x).$$

Lemma 3.6. Let C be a nonempty closed and convex subset of an Hadamard space X and $f : C \times C \to \mathbb{R}$ be a bifunction satisfying assumptions (i)-(iv) of Theorem 2.16. Let $\{\lambda_n\}$ be a sequence in $(0,\infty)$ and $\bar{v}$ be an element of C. Suppose that $\lim\limits_{n\to\infty} \lambda_n = \infty$ and $A(\{J^f_{\lambda_n} x_n\}) = \{\bar{v}\}$ for some bounded sequence $\{x_n\}$ in X, then $\bar{v} \in EP(f,C)$.

Proof. From (3.28), we obtain that

$$(\lambda_n+1)d^2(J^f_{\lambda_n} x_n, J^f\bar{v})+d^2(J^f_{\lambda_n} x_n, x_n)+\lambda_n d^2(J^f\bar{v},\bar{v}) \leq d^2(J^f\bar{v},x_n)+\lambda_n d^2(J^f_{\lambda_n} x_n,\bar{v}),$$

which implies that

$$d^2(J^f_{\lambda_n} x_n, J^f\bar{v}) \leq \frac{1}{\lambda_n} d^2(J^f\bar{v},x_n) + d^2(J^f_{\lambda_n} x_n,\bar{v})^2.$$

Since $\lim\limits_{n\to\infty} \lambda_n = \infty, \{x_n\}$ is bounded and $A(\{J^f_{\lambda_n} x_n\}) = \{\bar{v}\}$, we obtain that

$$\begin{aligned}\limsup_{n\to\infty} d(J^f_{\lambda_n} x_n, J^f\bar{v}) &\leq \limsup_{n\to\infty} d(J^f_{\lambda_n} x_n,\bar{v}) \\ &= \inf_{y\in X} \limsup_{n\to\infty} d(J^f_{\lambda_n} x_n, y),\end{aligned}$$

which by Lemma 2.8 and Lemma 2.15 (iii) implies that $\bar{v} \in F(J^f) = EP(f,C)$.

Theorem 3.7. Let C be a nonempty closed and convex subset of an Hadamard space X and $f : C\times C \to \mathbb{R}$ be a bifunction satisfying assumptions (i)-(iv) of Theorem 2.16. Let g be a contraction on C with coefficient $\gamma \in (0,1)$ and $\{x_n\}$ be a sequence defined by (3.26), where $\{\alpha_n\}$ is a sequence in $(0,1)$ and $\{\lambda_n\}$ is a sequence in $(0,\infty)$ such that $\lim\limits_{n\to\infty} \lambda_n = \infty$. Then, we have the following:

(i) The sequence $\{J^f_{\lambda_n} x_n\}$ is bounded if and only if $EP(f,C)$ is nonempty

(ii) If $\lim\limits_{n\to\infty} \alpha_n = 0$, $\sum\limits_{n=1}^{\infty} \alpha_n = \infty$ and $EP(f,C) \neq \emptyset$, then $\{x_n\}$ and $\{J^f_{\lambda_n} x_n\}$ converge strongly to an element of $EP(f,C)$.

Proof. (i) Suppose that $\{J^f_{\lambda_n} x_n\}$ is bounded. Then by Lemma 2.8, there exists $\bar{v} \in X$ such that $A(\{J^f_{\lambda_n} x_n\}) = \{\bar{v}\}$. Since $\alpha_n, \gamma \in (0,1)$, we obtain from (3.26) that

$$\begin{aligned} d(x_{n+1},\bar{v}) &\leq \alpha_n d(g(x_n),\bar{v}) + (1-\alpha_n) d(J^f_{\lambda_n} x_n,\bar{v}) \\ &\leq \alpha_n\gamma d(x_n,\bar{v}) + \alpha_n d(g(\bar{v}),\bar{v}) + (1-\alpha_n) d(J^f_{\lambda_n} x_n,\bar{v}) \\ &\leq d(x_n,\bar{v}) + \alpha_n d(g(\bar{v}),\bar{v}) + d(J^f_{\lambda_n} x_n,\bar{v}) \\ &\leq \alpha_{n-1}\gamma d(x_{n-1},\bar{v}) + \alpha_{n-1} d(g(\bar{v}),\bar{v}) + (1-\alpha_{n-1}) d(J^f_{\lambda_{n-1}} x_{n-1},\bar{v}) \\ &\quad +\alpha_n d(g(\bar{v}),\bar{v}) + d(J^f_{\lambda_n} x_n,\bar{v}) \\ &\leq d(x_{n-1},\bar{v}) + \alpha_{n-1} d(g(\bar{v}),\bar{v}) + d(J^f_{\lambda_{n-1}} x_{n-1},\bar{v}) \\ &\quad +\alpha_n d(g(\bar{v}),\bar{v}) + d(J^f_{\lambda_n} x_n,\bar{v}). \end{aligned}$$

Thus, by induction and the fact that $\{J^f_{\lambda_n} x_n\}$ is bounded for all $n \geq 1$, we get that $\{x_n\}$ is bounded. Also, since $\lim\limits_{n\to\infty} \lambda_n = \infty$ and $A(\{J^f_{\lambda_n} x_n\}) = \{\bar{v}\}$, we obtain by Lemma 3.6 that $\bar{v} \in EP(f,C)$. Hence, $EP(f,C)$ is nonempty.

Conversely, let $EP(f,C)$ be nonempty. Then, there exists a point say $\bar{v} \in C$ such that $\bar{v} \in EP(f,C)$. Thus by (3.26), we obtain that

$$\begin{aligned} d(x_{n+1},\bar{v}) &\leq \alpha_n d(g(x_n),\bar{v}) + (1-\alpha_n) d(J^f_{\lambda_n} x_n, \bar{v}) \\ &\leq \alpha_n \gamma d(x_n,\bar{v}) + \alpha_n d(g(\bar{v}),\bar{v}) + (1-\alpha_n) d(J^f_{\lambda_n} x_n, \bar{v}) \\ &\leq (1-\alpha_n(1-\gamma)) d(x_n,\bar{v}) + \alpha_n d(g(\bar{v}),\bar{v}) \\ &\leq \max\{d(x_n,\bar{v}), \frac{d(g(\bar{v}),\bar{v})}{1-\gamma}\} \\ &\vdots \\ &\leq \max\{d(x_1,\bar{v}), \frac{d(g(\bar{v}),\bar{v})}{1-\gamma}\}. \end{aligned}$$

Therefore, $\{x_n\}$ is bounded. Consequently, $\{J^f_{\lambda_n} x_n\}$ is also bounded.
(ii) Since $EP(f,C)$ is nonempty, we obtain from part (i) that $\{x_n\}$ and $\{J^f_{\lambda_n} x_n\}$ are bounded. Now, let $v_n = J^f_{\lambda_n} x_n$ for all $n \geq 1$ and $\bar{v} \in EP(f,C)$, then we obtain from Lemma 2.5 (iii) that

$$\begin{aligned} d^2(x_{n+1},\bar{v}) &\leq (1-\alpha_n)^2 d^2(v_n,\bar{v}) + 2\alpha_n(1-\alpha_n)\langle \overrightarrow{g(x_n)\bar{v}}, \overrightarrow{v_n\bar{v}} \rangle + \alpha_n^2 d^2(g(x_n),\bar{v}) \\ &\leq (1-\alpha_n)^2 d^2(x_n,\bar{v}) + 2\alpha_n(1-\alpha_n)\langle \overrightarrow{g(x_n)\bar{v}}, \overrightarrow{v_n\bar{v}} \rangle + \alpha_n^2 d^2(g(x_n),\bar{v}) \\ &\leq (1-\alpha_n)^2 d^2(x_n,\bar{v}) + 2\alpha_n(1-\alpha_n)\left(\langle \overrightarrow{g(x_n)g(\bar{v})}, \overrightarrow{v_n\bar{v}} \rangle + \langle \overrightarrow{g(\bar{v})\bar{v}}, \overrightarrow{v_n\bar{v}} \rangle\right) \\ &+ \alpha_n^2 d^2(g(x_n),\bar{v}) \\ &\leq (1-\alpha_n)^2 d^2(x_n,\bar{v}) + 2\alpha_n(1-\alpha_n)\left(\gamma d^2(x_n,\bar{v}) + \langle \overrightarrow{g(\bar{v})\bar{v}}, \overrightarrow{v_n\bar{v}} \rangle\right) \\ &+ \alpha_n^2 d^2(g(x_n),\bar{v}) \\ &\leq (1-2\alpha_n(1-\gamma)) d^2(x_n,\bar{v}) + 2\alpha_n^2(1-\gamma) d^2(x_n,\bar{v}) \\ &+ 2\alpha_n(1-\alpha_n)\langle \overrightarrow{g(\bar{v})\bar{v}}, \overrightarrow{v_n\bar{v}} \rangle + \alpha_n^2 d^2(g(x_n),\bar{v}) \\ &= (1-2\alpha_n(1-\gamma)) d^2(x_n,\bar{v}) + 2\alpha_n(1-\gamma)\delta_n, \end{aligned} \tag{3.33}$$

where

$$\delta_n = \frac{(1-\alpha_n)}{(1-\gamma)}\langle \overrightarrow{g(\bar{v})\bar{v}}, \overrightarrow{v_n\bar{v}} \rangle + \alpha_n\left(d^2(x_n,\bar{v}) + \frac{1}{2(1-\gamma)} d^2(g(x_n),\bar{v})\right) \tag{3.34}$$

for all $\bar{v} \in EP(f,C)$.
Furthermore, since $\{v_n\}$ is bounded, we obtain from Lemma 2 that there exists a subsequence $\{v_{n_k}\}$ of $\{v_n\}$ that Δ-converges to some $\hat{v} \in C$. Thus, by Lemma 2, we obtain that $A(\{v_{n_k}\}) = \{\hat{v}\}$. Moreover, $\lim_{k\to\infty} \lambda_{n_k} = \infty$ and $\{x_{n_k}\}$ is bounded. Hence, by Lemma 3, we obtain that $\hat{v} \in EP(f,C)$.
Next, we show that $\{x_n\}$ converges strongly to an element of $EP(f,C)$. Since the subsequence $\{v_{n_k}\}$ of $\{v_n\}$ Δ-converges to $\hat{v} \in EP(f,C)$, we obtain from Lemma 2.10 that there exists $\bar{z} \in EP(f,C)$ such that $\{v_n\}$ Δ-converges to $\bar{z}$. Thus, by Lemma

2.9, we obtain that

$$\limsup_{n\to\infty}\langle\overrightarrow{g(\bar{z})\bar{z}},\overrightarrow{v_n\bar{z}}\rangle \leq 0, \tag{3.35}$$

which by setting $\bar{v}=\bar{z}$ in (3.34), implies that $\limsup_{n\to\infty}\delta_n \leq 0$. Therefore, applying Lemma 2 to (3.33), gives that $\{x_n\}$ converges strongly to $\bar{z}\in EP(f,C)$. It then follows that $\{J^f_{\lambda_n}x_n\}$ also converges strongly to $\bar{z}\in EP(f,C)$.

We are now going to apply Theorem 3.7 to approximate a common solution of finite family of equilibrium problems. We begin with the following lemma whose proof is similar to the proof of [26, Theorem 3.14].

Lemma 3.8. Let C be a nonempty closed and convex subset of an Hadamard space X and $f_j : C\times C\to\mathbb{R},\ j=1,2,\dots,m$ be a finite family of bifunctions satisfying assumptions (i)-(iv) of Theorem 2.16. Then, for $\lambda>0$, we have

$$F\left(\prod_{j=1}^m J_\lambda^{f_j}\right)=\bigcap_{j=1}^m F\left(J_\lambda^{f_j}\right),$$

where

$$\prod_{j=1}^m J_\lambda^{f_j}=J_\lambda^{f_1}\circ J_\lambda^{f_2}\circ\cdots\circ J_\lambda^{f_{m-1}}\circ J_\lambda^{f_m}.$$

Proof. Clearly,

$$\bigcap_{j=1}^m F\left(J_\lambda^{f_j}\right)\subseteq F\left(\prod_{j=1}^m J_\lambda^{f_j}\right).$$

Thus, we only have to show that

$$F\left(\prod_{j=1}^m J_\lambda^{f_j}\right)\subseteq\bigcap_{j=1}^m F\left(J_\lambda^{f_j}\right).$$

For this, let $x\in F\left(\prod_{j=1}^m J_\lambda^{f_j}\right)$ and $y\in\bigcap_{j=1}^m F\left(J_\lambda^{f_j}\right)$, we obtain that

$$\begin{aligned} d^2(x,y) &= d^2\left(\prod_{j=1}^m J_\lambda^{f_j}x,\prod_{j=1}^m J_\lambda^{f_j}y\right)\\ &\leq d^2\left(\prod_{j=2}^m J_\lambda^{f_j}x,y\right). \end{aligned}\tag{3.36}$$

Furthermore, we obtain by Lemma 2.18 (ii) and (3.36) that

$$\begin{aligned} d^2\left(\prod_{j=2}^{m} J_\lambda^{f_j}x, \prod_{j=1}^{m} J_\lambda^{f_j}x\right) &\leq d^2\left(\prod_{j=2}^{m} J_\lambda^{f_j}x, y\right) - d^2\left(\prod_{j=1}^{m} J_\lambda^{f_j}x, y\right) \\ &\vdots \\ &\leq d^2(x,y) - d^2\left(\prod_{j=1}^{m} J_\lambda^{f_j}x, y\right) \\ &= d^2\left(\prod_{j=1}^{m} J_\lambda^{f_j}x, y\right) - d^2\left(\prod_{j=1}^{m} J_\lambda^{f_j}x, y\right), \end{aligned}$$

which implies

$$\prod_{j=1}^{m} J_\lambda^{f_j}x = \prod_{j=2}^{m} J_\lambda^{f_j}x. \tag{3.37}$$

Similarly, we obtain that

$$\prod_{j=2}^{m} J_\lambda^{f_j}x = \prod_{j=3}^{m} J_\lambda^{f_j}x. \tag{3.38}$$

Continuing in this manner, we can show that

$$\prod_{j=3}^{m} J_\lambda^{f_j}x = \prod_{j=4}^{m} J_\lambda^{f_j}x = \cdots = \prod_{j=m-1}^{m} J_\lambda^{f_j}x = J_\lambda^{f_m}x = x. \tag{3.39}$$

From (3.39), we have

$$x = J_\lambda^{f_m}x. \tag{3.40}$$

From (3.39) and (3.40), we obtain

$$x = \prod_{j=m-1}^{m} J_\lambda^{f_j}x = J_\lambda^{f_{m-1}}\left(J_\lambda^{f_m}x\right) = J_\lambda^{f_{m-1}}x. \tag{3.41}$$

Continuing in this manner, we obtain from (3.37)-(3.41) that

$$x = J_\lambda^{f_{m-2}}x = \cdots = J_\lambda^{f_2}x = J_\lambda^{f_1}x, \tag{3.42}$$

which together with (3.40) and (3.41) gives the desired conclusion.

Theorem 3.9. Let C be a nonempty closed and convex subset of an Hadamard space X and $f_j : C \times C \to \mathbb{R},\ j = 1, 2, \ldots, m$ be a finite family of bifunctions satisfying assumptions (i)-(iv) of Theorem 2.16. Let g be a contraction mapping on C with coefficient $\gamma \in (0,1)$. Suppose that for arbitrary $x_1 \in C$, the sequence $\{x_n\}$ is generated by

$$x_{n+1} = \alpha_n g(x_n) \oplus (1-\alpha_n) \prod_{j=1}^{m} J_{\lambda_n}^{f_j}x_n,\ n \geq 1, \tag{3.43}$$

where

$$\prod_{j=1}^{m} J_{\lambda_n}^{f_j} = J_{\lambda_n}^{f_1} \circ J_{\lambda_n}^{f_2} \circ \cdots \circ J_{\lambda_n}^{f_{m-1}} \circ J_{\lambda_n}^{f_m},$$

$\{\alpha_n\}$ is a sequence in $(0,1)$ and $\{\lambda_n\}$ is a sequence in $(0,\infty)$ such that $\lim_{n\to\infty} \lambda_n = \infty$. If $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\Gamma := \cap_{i=1}^{N} EP(f_i, C) \neq \emptyset$, then the sequence $\{x_n\}$ converges strongly to an element of Γ.

Proof. By Theorem 3.7 (ii) and Lemma 2.15 (iii), we obtain that $\{x_n\}$ converges strongly to an element of $F\left(\prod_{j=1}^{m} J_{\lambda}^{f_j}\right)$. Therefore, we conclude by Lemma 3.8 and Lemma 2.15 (iii) that $\{x_n\}$ converges strongly to an element of Γ.

4. Application to optimization problems

We now give some applications of our results to optimization problems. We shall assume for the rest of this paper that, X is an Hadamard space and C is a nonempty closed and convex subset of X.

4.1. Minimization problem

Let $h : X \to \mathbb{R}$ be a proper convex and lower semi-continuous function. Now, define the bifunction $f_h : C \times C \to \mathbb{R}$ by

$$f_h(x,y) = h(y) - h(x), \ \forall x, y \in C.$$

Then, f_h satisfies assumptions (i)-(iv) of Theorem 2.16 (see [31]). Moreover, $\text{EP}(f_h, C) = \arg\min_C h$, $J^{f_h} = \text{prox}^h$ and $D(\text{prox}^h) = X$ (see [31]). Consider the following finite family of minimization problems:

$$\text{Find } x \in C \text{ such that } h_j(x) \leq h_j(y), \ \forall y \in C, \ j = 1, 2 \ldots, m. \tag{4.1}$$

Thus, by setting $J_{\lambda_n}^{f_j} = \text{prox}_{\lambda_n}^{h_j}$ in Algorithm (3.43), we can apply Theorem 3.9 to approximate solutions of problem (4.1).

4.2. Variational inequality problem

Let $T : C \to C$ be a nonexpansive mapping. Consider the bifunction $f_T : C \times C \to \mathbb{R}$ defined by $f_T(x,y) = \langle \overrightarrow{Txx}, \overrightarrow{xy} \rangle$. Then, f_T satisfies assumptions (i)-(iv) of Theorem 2.16, and $J^{f_T} = J^T$ (see [7, 28]). Now, consider the following finite family of variational inequality problems:

$$\text{Find } x \in C \text{ such that } \langle \overrightarrow{T_j x x}, \overrightarrow{xy} \rangle \geq 0, \ \forall y \in C, \ j = 1, 2 \ldots, m. \tag{4.2}$$

Thus, by setting $J_{\lambda_n}^{f_j} = J_{\lambda_n}^{T_i}$ in Algorithm (3.43), we can apply Theorem 3.9 to approximate solutions of problem (4.2).

4.3. Convex feasibility problem

Let $C_j, j = 1, 2, \ldots, m$ be a finite family of nonempty closed and convex subsets of C such that $\cap_{j=1}^{m} C_j \neq \emptyset$. Then, the convex feasibility problem is defined as:

$$\text{Find } x \in C \text{ such that } x \in \cap_{j=1}^{N} C_j. \tag{4.3}$$

Furthermore, the indicator function $\delta_C : X \to \mathbb{R}$ defined by

$$\delta_C(x) = \begin{cases} 0, & \text{if } x \in C, \\ +\infty, & \text{otherwise} \end{cases}$$

is known to be proper convex and lower semi-continuous. Thus, by letting $\delta_C = h$ and following similar argument as in Subsection 4.1, we obtain that f_{δ_C} satisfies assumptions (i)-(iv) of Theorem 2.16, and $J^{f_{\delta_C}} = \text{prox}^{\delta_C} = P_C$. Therefore, by setting $J^{f_j} = P_{C_j},\ j = 1, 2, \ldots, m$ in Algorithm (3.43), we can apply Theorem 3.9 to approximate solutions of (4.3).

Remark 4.1. The motivation for using viscosity-type algorithms in our main theorems instead of Halpern-type algorithms that also converges strongly (as seen in Corollary 3.2), is due to the fact that viscosity-type algorithms have higher rate of convergence than Halpern-types. Moreover, it has been established in [45] that Halpern-type convergence theorems imply viscosity convergence theorems. Furthermore, one other advantage of adopting the viscosity-type algorithm for our strong convergence analysis is that it also converges strongly to a unique solution of some variational inequalities which cannot be achieved if the Halpern-type algorithm is used, as seen in Corrollary 3.

Remark 4.2. Our main theorems improve and extend the main theorems of Phuengrattana [42], Kumam and Chaipunya [31] in the following ways:

(i) In [42, Theorem 3.12], the author proved a Δ-convergence of the Mann-type iteration to a fixed point of a generalized asymptotically nonspreading mapping while in Theorem 3.1 of this paper, we prove a strong convergence of a viscosity-type algorithm to a fixed point of a generalized asymptotically nonspreading mapping which is also a common solution of a finite family of equilibrium problems and a unique solution of some variational inequality problems. Furthermore, the non-constant example given in this paper (see Example 1.2) is in general more desirable and applicable than the constant example considered in [42] (see Example 1.1).

(ii) In [31, Theorem 7.3], the authors proved a Δ-convergence of the PPA to a solution of an equilibrium problem (see Theorem 2.12 while in Theorem 3.1 of this paper, we prove a strong convergence of a viscosity-type algorithm to a common solution of a finite family of equilibrium problems which is also a fixed point of a generalized asymptotically nonspreading mapping and a unique solution of a variational inequality problem.

(iii) We also studied the asymptotic behavior of the sequence generated by a viscosity-type algorithm and extend this study to approximate a common solution of finite family of equilibrium problems.

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