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A NOTE ON FIXED POINT THEORY FOR MULTIVALUED MAPPINGS

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Abstract. In this paper, we establish two fixed point theorems of Himmelberg's type for multivalued maps. Using abstract measures of weak noncompactness, these results are applied to derive some fixed point theorems of Sadovskii's type.

Key Words and Phrases: Fixed point theorem, multivalued maps, upper semicontinuous multivalued map, weakly sequentially upper semicontinuous multivalued map, measure of weak noncompactness, condensing multivalued map.
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1. Introduction and preliminary

The purpose of this note is to study the existence of fixed points in Banach spaces for upper semicontinuous multivalued maps and weakly sequentially upper semicontinuous multivalued maps. This work is motivated by Himmelberg's theorem (Theorem 1.1) and the condition (A) which was introduced in [15]. Indeed, if instead of assuming in Theorem 1.1 that $F(M)$ is relatively compact, we suppose that $F(M)$ is relatively weakly compact and each selection of F satisfies condition (A), then F has a fixed point (cf. Theorem 2.1). Moreover, in the case where F is a weakly sequentially upper semicontinuous multivalued maps, assuming that $F(M)$ is relatively weakly compact and using O'Regan's theorem (Theorem 1.2), we establish a new fixed point theorem of Himmelberg's type without supposing condition (A). In Section 3 we use our results to establish fixed point theorem of Sadovskii's type for multivalued maps. We underline that, in the last years, the fixed point theory under weak topology for single or multivalued maps experienced new developments and aroused a lot of interest (see, for example, [1, 2, 5, 8, 9, 10, 15, 16, 17, 18] or [14, Chapter 4]).

Now we introduce notations and definitions which are required in the paper. Let X be Banach space and let $\mathsf{B}(X)$ and $\mathsf{P}_{cl,cv}$ be the subsets defined by

$$\mathsf{B}(X) = \Big\{ M \subset X : M \text{ is nonempty and bounded} \Big\},$$

$$\mathsf{P}_{cl,cv}(X) = \Big\{ M \subset X : M \text{ is nonempty, convex and closed} \Big\}.$$

We shall now give the notion of a measure of weak noncompactness on a Banach space.

Definition 1.1. A map $\mu : \mathsf{B}(X) \to [0, +\infty[$ is said to be a measure of weak noncompactness on X if it satisfies the following conditions:

(1) The family $\ker \mu := \big\{ M \in \mathsf{B}(X) : \mu(M) = 0 \big\}$ is non-empty and $\ker \mu$ is contained in the set of relatively weakly compact subsets of X.
(2) Monotonicity: $M_1 \subset M_2 \Rightarrow \mu(M_1) \leq \mu(M_2)$.
(3) Invariance under passage to the closed convex hull: $\mu(\overline{co}(M)) = \mu(M)$ where $\overline{co}$ denotes the closed convex hull of M.
(4) Maximum Property: $\mu(M_1 \cup M_2) = \max\big(\mu(M_1), \mu(M_2)\big)$, for all $M_1, M_1 \in \mathsf{B}(X)$.
(5) Homogeneity: $\mu(\lambda M) = \lambda \mu(M)\ \forall \lambda \in \mathbb{R}^+$.
(6) Fullness: $\mu(M) = 0$ if and only if M is a relatively weakly compact set.

The family $\ker \mu$ given in first assertion is called the kernel of the measure μ. It should be noticed that the containements $M \subseteq \overline{M^w} \subseteq \overline{co}(M)$ together with the item (3) of Definition 1.1 implies

(7) $\mu(\overline{M^w}) = \mu(M)$.

Note that $\mu(\cdot)$ is a full measure of weak noncompactness having the maximum property is non-singular, that is:

(8) $\mu(M \cup \{x\}) = \mu(M)$, for all $M \in \mathsf{B}(X)$ and $x \in X$.

Proposition 1.1. *Let X be a Banach space and μ a measure of weak noncompactness on X. If $(M_n)_{n\in\mathbb{N}}$ is a decreasing sequence (in the sense of inclusion) of weakly closed subsets of $\mathsf{B}(X)$ such that $\lim\limits_{n\to+\infty} \mu(M_n) = 0$, then $M_\infty = \bigcap_{n=1}^{\infty} M_n$ is a nonempty weakly compact subset of X.*

Proof. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence of points of X such that for all $n \in \mathbb{N}$, $x_n \in M_n$. Let $(S_n)_{n\in\mathbb{N}}$ be a sequence of subsets where $S_n = \{x_k : k \geq n\}$. It is clear that $(S_n)_{n\in\mathbb{N}}$ is decreasing and $S_n \subseteq M_n$ for all $n \in \mathbb{N}$. Since $\mu(\cdot)$ is nonsingular, for all $n \in \mathbb{N}$, $\mu(S_0) = \mu(S_n) \leq \mu(M_n)$. Because $\lim\limits_{n\to+\infty} \mu(M_n) = 0$, we have $\mu(S_0) = 0$ and therefore the set $\{x_n,\, n \in \mathbb{N}\}$ is relatively weakly compact. Let x be the weak limit of a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$. It is clear that for any $n \in \mathbb{N}$, $x \in M_n$ and therefore $x \in M_\infty$ which proves that $M_\infty \neq \emptyset$. Moreover, for all $n \in \mathbb{N}$, $\mu(M_\infty) \leq \mu(K_n)$ and then $\mu(M_\infty) = 0$ because $\lim\limits_{n\to+\infty} \mu(M_n) = 0$. Hence, M_∞ is weakly compact. □

We note that the first example of measure of weak noncompactness was introduced by F. S. De Blasi [6]. For an axiomatic definition, we refer, for example, to [3].

Let (X, d) and (Y, d) be two metric spaces and let $F : X \to \mathsf{P}_{cl,cv}(Y)$ be a multi-valued map.

• F is called upper semi-continuous (u.s.c. for short) provided that, for every open set U of Y, the set $F^{-1}(U)$ is open in X, where $F^{-1}(U) = \{x \in X : F(x) \subset U\}$.

• F is called weakly upper semicontinuous (w.u.s.c. for short) if F is u.s.c. with respect to the weak topologies of X and Y.

• F is called weakly sequentially upper semicontinuous (w.s.u.s.c. for short) if for any weakly closed set V of Y, $F^{-1}(V)$ is weakly sequentially closed.

Let us now recall the following definition (see, for example, [11]).

Definition 1.2. Let X and Y be two metric spaces and $F : X \to \mathsf{P}_{cl,cv}(Y)$ be a multivalued map. A single valued map $f : X \to Y$ is called a selection of F if for every $x \in X$, $f(x) \in F(x)$.

We recall also the following two results established in [12, Theorem 2] and [18, Theorem 4.1] respectively.

Theorem 1.1. *Let M be a nonempty, convex and closed subset of a locally convex Hausdorff space X. Let $F : M \to \mathsf{P}_{cl,cv}(M)$ be a u.s.c. multivalued map such that $F(M)$ is relatively compact. Then F has a fixed point.*

Theorem 1.2. *Let X be a Banach space and let M be a weakly compact subset of X. If $F : M \to \mathsf{P}_{cl,cv}(M)$ is w.s.u.s.c. multivalued map, then F is a w.u.s.c. multivalued map.*

2. Two theorems of Himmelberg's type

Let M be a nonempty closed convex subset of a Banach space X and let $f : M \to M$ be a single-valued map. In the remainder of this work, we need the following hypothesis introduced in [15, p. 260].

(A) For each weakly convergent sequence $(x_n)_{n\in\mathbb{N}}$ of M, the sequence $(f(x_n))_{n\in\mathbb{N}}$ has a strongly convergent subsequence.

Now we are ready to state our first result:

Theorem 2.1. *Let M be a nonempty closed, convex subset of a Banach space X and let $F : M \to \mathsf{P}_{cl,cv}(M)$ be a u.s.c. multivalued map. Suppose that all selections of F satisfy condition* (A) *and $F(M)$ is relatively weakly compact. Then there exists $x \in M$ such that $x \in F(x)$.*

Proof. Let $C = \overline{co}(F(M))$. According to Krein-Ŝmulian's theorem (see [7, p. 434]), C is a weakly compact convex subset of X. Moreover, it is not difficult to check that $F(C) \subseteq F(M) \subset C$. Now, we shall show that $F(C)$ is relatively compact. To this end, let $(x_n)_{n\in\mathbb{N}}$ be a sequence of points in C. Since C is weakly compact, there exists a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ such that $x_{n_k} \rightharpoonup x$, ($x \in C$ because C is weakly closed), as $n \to +\infty$. Let $(y_{n_k})_{k\in\mathbb{N}}$ be a sequence in $F(C)$ such that, for each $k \in \mathbb{N}$,

$y_{n_k} \in F(x_{n_k})$. Hence, there exists a selection f of F such that $y_{n_k} = f(x_{n_k})$. Since f satisfies the hypothesis (A), we infer that $y_{n_k} = f(x_{n_k}) \in C$ has a strongly convergent subsequence in C . Hence $F(C)$ is relatively compact. Now, applying Theorem 1.1, we conclude that there exists $z \in C$ such that $z \in F(z)$. □

Theorem 2.2. *Let X be a Banach space, M a nonempty closed, convex subset of X. Let $F : M \to \mathsf{P}_{cl,cv}(M)$ be a w.s.u.s.c. multivalued map and $F(M)$ is relatively weakly compact. Then there exists $x \in M$ such that $x \in F(x)$.*

Proof. Put $K = \overline{co}(F(M))$. As in the previous theorem, K is a weakly compact convex subset of X and satisfies $F(M) \subseteq K \subset M$. Since $F(K) \subseteq K$ and K is weakly compact, we infer that $F(K)$ is relatively weakly compact. Now, we note that $F : K \to \mathsf{P}_{cl,cv}(K)$ is a w.s.u.s.c. multi-valued map. By Theorem 1.2, F is a w.u.s.c. multi-valued map. Since X equipped with its weak topology $\sigma(X, X^*)$ is a Hausdorff locally convex space, applying Theorem 1.1, we deduce that F has a fixed point $z \in K \subseteq M$ which ends the proof. □

3. Theorems of Sadovskii's type

In this section we shall derive some fixed point theorems of Sadovskii's type for multi-valued maps. The next two results deal with mappings satisfying the condition (A).

Definition 3.1. Let M be a nonempty closed, convex subset of a Banach space X and μ a measure of weak noncompactness on X. Denote by $F : M \to 2^X$ a multi-valued mapping. We say that F is μ-condensing if

$$\mu(F(D)) < \mu(D), \quad \text{for each } D \in \mathsf{B}(X) \text{ such that } \mu(D) > 0.$$

We now use Theorem 2.1 to obtain a fixed point result of Sadovskii's Type.

Theorem 3.1. *Let M be a nonempty bounded, closed, convex subset of a Banach space X and let $F : M \to \mathsf{P}_{cl,cv}(M)$ be a u.s.c. multivalued map and $\mu(\cdot)$ a measure of weak noncompactness on X. Suppose that all selections of F satisfy condition (A). If F is μ-condensing, then there exists $x \in M$ such that $x \in F(x)$.*

Proof. Let $y \in M$. We define the family Π by

$$\Pi := \{D \subseteq M : D \text{ is closed, convex, } y \in D \text{ and } F : D \to \mathsf{P}_{cl,cv}(D)\}.$$

We denote

$$K = \bigcap_{D \in \Pi} D \qquad \text{and} \qquad K^* = \overline{co}(F(K) \cup \{y\}).$$

It is clear that K is closed convex containing y. Moreover for all $D \in \Pi$, we have $F(K) \subseteq F(D) \subseteq D$ and so $F(K) \subseteq \bigcap_{D \in \Pi} D = K$. If $y \in D$, we see that $F(K) \cup \{y\} \subseteq K$ and so, taking into account the fact that K is closed and convex, we deduce that

$$K^* = \overline{co}(F(K) \cup \{y\}) \subseteq \overline{co}(K) = K.$$

This implies that $K^* \subseteq K$ and therefore $F(K^*) \subseteq F(K) \subseteq K^*$. Hence we conclude that $K^* \in \Pi$ and $K \subseteq K^*$ which proves that $K = K^*$. It suffices to show that K is relatively weakly compact. If it is not the case, then $\mu(K) > 0$. Moreover, using the properties of $\mu(\cdot)$ we can write

$$\mu(K) = \mu\big(\overline{co}(F(K) \cup \{y\})\big) \leq \mu(F(K)) < \mu(K)$$

which is a contraction. Hence $\mu(K) = 0$ and so K is relatively weakly compact. Now using the fact that $F(K) \subset K$ we infer that $F(K)$ is relatively weakly compact and therefore $F : K \to \mathsf{P}_{cl,cv}(K)$ is u.s.c. mapping. To conclude the proof, it suffices to apply Theorem 2.1. □

It should be noticed that the result of Theorem 3.1 is also valid for μ-k-contractive mapping for some k belonging to $[0, 1)$.

The proof of the previous result uses the boundedness of M. In the case where M is an unbounded convex subset of X, we have the following result.

Theorem 3.2. *Let M be a nonempty closed convex unbounded subset of X and $\mu(\cdot)$ a measure of weak noncompactness on X. Let $F : M \to \mathsf{P}_{cl,cv}(M)$ be a u.s.c. μ-condensing multivalued map. If $F(M)$ is bounded and all selections of F satisfy condition* (A)*, then there exists $z \in M$ such that $z \in F(z)$.*

To prove this theorem we shall make use of the following result established in [13, p. 636].

Lemma 3.1. *Let X be a topological space and let $F : X \to \mathsf{P}_{cl,cv}(X)$ be a u.s.c. multivalued map with closed graph. If there is a nonempty subset A of X such that $\overline{A}$ is compact and $F(A) \subseteq A$, then there exists a nonempty, closed and compact subset M of X such that $M \subset F(M)$.*

Proof of Theorem 3.2. Let $\zeta \in M$ and $A = \{F^n(\zeta), n = 0, 1, 2, \cdots\}$ where $F^0(\zeta) = \zeta$. It is clear that $A = F(A) \cup \{\zeta\}$. Moreover, the use of properties (1) and (4) of Definition 1.1, implies $\mu(F(A) \cup \{\zeta\}) = \max\Big(\mu(F(A)), \mu(\{\zeta\})\Big) = \mu(F(A))$ because $\{\zeta\}$ is trivially weakly compact. Hence we deduce that $\mu(F(A)) = \mu(A)$. Since F is μ-condensing, we get $\mu(A) = 0$ and so A is relatively weakly compact. By the hypothesis that all selections of F satisfy the condition (A), we deduce that $F(A)$ is relatively compact (see the proof of Theorem 2.1). Since $F(F(A)) \subset F(A)$, we derive by Lemma 3.1 that one can choose a compact subset A_0 contained in $F(A)$ such that $A_0 \subseteq \overline{co}(F(A_0))$. We put

$$I := \{C \text{ such that } A_0 \subseteq C, \overline{co}(C) = C, F(C) \subseteq C\}.$$

It is clear that $I \neq 0$ because $M \in I$. If Π is a chain of a partially ordered subsets in $(X, \subseteq)$, then $\bigcap_{C \in \Pi} C$ is a lower bound of Π. Hence, by Zorn's lemma (see for example, [4, p. 2] or [7, p. 6]), one sees that I has a minimal element S. Because S is closed and convex, we infer that $F(S) \subseteq S$ and so $\overline{co}(F(S)) \subseteq S$. Accordingly,

$$F(\overline{co}(F(S))) \subseteq F(S) \subseteq \overline{co}(F(S)).$$

Since $A_0 \subseteq \overline{co}(F(S))$, we deduce that $\overline{co}(F(S))$ is a element of I. Using the fact that S is a minimal element of I, we conclude that $S = \overline{co}(F(S))$. Hence $\mu(S) = \mu(\overline{co}(F(S))) = \mu(F(S))$. Since F is μ-condensing, we get $\mu(S) = \mu(F(S)) = 0$ and therefore $F(S)$ is relatively weakly compact. Because each selection of F satisfies condition (A), the use of Theorem 2.1 ends the proof. □

We close this section by establishing the next fixed point result for w.s.u.s.c. μ-condensing multi-valued maps.

Theorem 3.3. *Let X be a Banach space and M a nonempty bounded, closed, convex subset of X. Assume that $\mu(\cdot)$ is a measure of weak noncompactness on X. If $F : M \to \mathsf{P}_{cl,cv}(M)$ is a w.s.u.s.c. μ-condensing multivalued map, then there exists $x \in M$ such that $x \in F(x)$.*

Proof. Let $k_0 \in M$. We consider the set

$$\mathcal{K} := \{A \subset M \,\text{such that}\, F(A) \subseteq A,\ k_0 \in A \text{ and } \ A \text{ is closed, convex}\}.$$

So, using the same proof as Theorem 3.1, we show that the set

$$B := \bigcap_{A\in\mathcal{K}} A = co(F(B) \cup \{k_0\})$$

belongs to $\mathcal{K}$. Next, using the properties of μ and F, one sees that

$$\mu(B) = \mu(\overline{co}\big\{(F(B) \cup \{k_0\}\big\}) = \mu(F(B)) < \mu(B)$$

which is a contradiction. Hence $\mu(B) = 0$ and so B is relatively weakly compact. On the other hand, since $B \in \mathcal{K}$, we have $F(B) \subset B$ and therefore $F(B)$ is relatively weakly compact. Moreover, the restriction of F to B, that is $F : B \to \mathsf{P}_{cl,cv}(B)$, is a w.s.u.s.c. multivalued map. Now the use of Theorem 2.2 shows that there exists $z \in B$ such that $z \in F(z)$. □

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