Fixed Point Theory, 24(2023), No. 1, 221-232
DOI: 10.24193/fpt-ro.2023.1.11
http://www.math.ubbcluj.ro/~nodeacj/sfptcj.html

NONLINEAR ALTERNATIVES OF HYBRID TYPE FOR NONSELF VECTOR-VALUED MAPS AND APPLICATION

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Abstract. In this paper we obtain nonlinear alternatives of Leray-Schauder and Mönch type for nonself vector-valued operators, under hybrid conditions of Perov contraction and compactness. Thus, we give vector versions of the theorems of Krasnosel'skii, Avramescu, Burton-Kirk and Gao-Li-Zhang. An application is given to a boundary value problem for a system of second order differential equations in which some of the equations are implicit.
Key Words and Phrases: Nonlinear operator, nonself map, fixed point, Perov contraction, nonlinear boundary value problem.
2020 Mathematics Subject Classification: 47H10, 34B15.

1. Introduction

1.1. **Krasnosel'skii type results for self maps.** Any study of operator equations with hybrid conditions must begin with Krasnosel'skii's theorem for the sum of two operators.

Theorem 1.1 (Krasnosel'skii)**.** [12] *Let D be a closed bounded convex subset of a Banach space X, $A : D \to X$ a contraction and $B : D \to X$ a continuous mapping with $B(D)$ relatively compact. If*

$$A(x) + B(y) \in D \quad \text{for every} \quad x, y \in D, \tag{1.1}$$

then the map $N := A + B$ has at least one fixed point.

The hybrid character of Krasnosel'skii's theorem lies in the decomposition of the operator N as a sum of two maps A and B with different properties. Condition (1.1) shows that N is a self map of D as in the fixed point theorems of Banach and Schauder which Krasnosel'skii's theorem uses together in the proof.

An other possibility for a hybrid approach arises in case of systems, when the domain of N splits as a Cartesian product, say $X \times Y$, and correspondingly the operator N splits as a couple (N_1, N_2), where N_1, N_2 take their values in X and Y, respectively. A typical result in this direction is the following vector version of Krasnosel'skii's theorem, due to Avramescu [1], which we state here in a slightly modified form and whose proof is reproduced in [4].

Theorem 1.2 (Avramescu)**.** [1] *Let (D_1, d) be a complete metric space, D_2 a closed convex subset of a normed space Y and let $N_i : D_1 \times D_2 \to D_i$, $i = 1, 2$ be continuous mappings. Assume that the following conditions are satisfied:*

(a) *There is a constant $l \in [0, 1)$ such that*

$$d(N_1(x, y),\ N_1(\overline{x}, y)) \leq l d(x,\ \overline{x})$$

for all $x, \overline{x} \in D_1$ and $y \in D_2$;

(b) *$N_2(D_1 \times D_2)$ is a relatively compact subset of Y.*

Then there exists $(x, y) \in D_1 \times D_2$ with

$$N_1(x, y) = x, \quad N_2(x, y) = y.$$

In this regard, let us note the conclusion of the paper [18] according to which any theorem of continuous dependence of the fixed point on parameters can be associated with a fixed point existence result for operators of the form $N : X \times Y \to X \times Y$.

For the connection between Theorem 1.2 and Theorem 1.1, see Remark 1.2 in [4].

In the paper [6] it went further and added even more heterogeneity by mixing together the metrical topology of a complete metric space with the norm-topology and weak topology of a Banach space. Thus, results were obtained that combine the Banach-Perov, Schauder and Arino–Gautier-Penot fixed point theorems and the Banach-Perov theorem with the strong and weak-topology versions of Mönch's fixed point theorem. These results are dealing with operators of the form

$$N = (N_1, ..., N_n), \quad N_i : V \to V_i \ \text{ for } \ i = 1, ..., n,$$

on a Cartesian product space

$$V = V_1 \times ... \times V_n,$$

where a number p $(0 \leq p \leq n)$ of spaces, $V_1, ..., V_p$ are complete metric spaces endowed with the metrics d_i, $i = 1, ..., p$, a number q $(0 \leq q \leq n - p)$ are Banach spaces considered with their strong topologies, and the remaining ones are Banach spaces with their weak topologies. Let

$$X=\prod_{i=1}^{p}V_i,\quad Y=\prod_{i=p+1}^{p+q}V_i,\quad Z=\prod_{i=p+q+1}^{n}V_i$$

and

$$F=(N_1,..,N_p)\,,\quad G=(N_{p+1},...,N_{p+q})\,,\quad H=(N_{p+q+1},...,N_n)\,,$$

be defined respectively if $p>0$, $q>0$ and $n-(p+q)>0$. We note that if $p=0$ then the set X does not appear; analogously for the set Y when $q=0$, and for Z if $p+q=n$.

Denote by d the vector-valued metric $d:X\to\mathbb{R}_+^p$ given by

$$d(x,\overline{x})=(d_1(x_1,\overline{x}_1)\,,...,d_p(x_p,\overline{x}_p))^T\,,$$

where $x=(x_1,...,x_p)\,,\ \overline{x}=(\overline{x}_1,...,\overline{x}_p)\,,$ and use the notation

$$\begin{aligned}&d(F(x,y,z)\,,F(\overline{x},y,z))\\ =\quad&(d_1(N_1(x,y,z)\,,N_1(\overline{x},y,z))\,,...,d_p(N_p(x,y,z)\,,N_p(\overline{x},y,z)))^T\,,\end{aligned}$$

for $x,\overline{x}\in X$, $y\in Y$ and $z\in Z$.

Thus, we have a first generalization of Avramescu's theorem for self maps.

Theorem 1.3. [6] *Let $K_i\subset V_i$, $i=p+1,\ldots,n$, be convex such that K_i is compact for $i=p+1,...,p+q$, and weakly compact for $i=p+q+1,...,n$. Let*

$$K_Y=\prod_{i=p+1}^{p+q}K_i,\quad K_Z=\prod_{i=p+q+1}^{n}K_i,$$

and $N=(N_1,...,N_n):X\times K_Y\times K_Z\to V$ *a map with*

$$N_i:X\times K_Y\times K_Z\to V_i,\ i=1,...,p;\quad N_i:X\times K_Y\times K_Z\to K_i,\ i=p+1,...,n.$$

On the maps F,G,H one assumes that

(i) *for each point $x\in X$, $F(x,.,.)$ is sequentially continuous from $K_Y\times K_Z$ to X with respect to the strong topology on K_Y and weak topology on K_Z, and there exists a square matrix M of size p having nonnegative entries and with the spectral radius $\rho(M)<1$ such that*

$$d(F(x,y,z)\,,F(\overline{x},y,z))\le Md(x,\overline{x})\,,$$

for all $x,\overline{x}\in X$, $y\in K_Y$ and $z\in K_Z$.

(ii) *G and H are sequentially continuous from $X\times K_Y\times K_Z$ to K_Y and K_Z, respectively, with respect to the strong topology on K_Y and weak topology on K_Z.*

Then there exists $v^*=(x^*,y^*,z^*)\in X\times K_Y\times K_Z$ with $N(v^*)=v^*$.

A second generalization of Avramescu's theorem uses together the Banach-Perov fixed point theorem and the strong and weak versions of Mönch's fixed point theorem for self maps.

Theorem 1.4. [6] *Let $D_i \subset V_i$ be a closed convex set for $i = p+1, ..., n$,*

$$D_Y = \prod_{i=p+1}^{p+q} D_i, \quad D_Z = \prod_{i=p+q+1}^{n} D_i,$$

and let $N = (N_1, ..., N_n) : X \times D_Y \times D_Z \to V$ be a map with

$$N_i : X \times D_Y \times D_Z \to V_i, \quad i = 1, ..., p; \quad N_i : X \times D_Y \times D_Z \to D_i, \quad i = p+1, ..., n.$$

Assume that the maps F, G, H satisfy the following conditions:

(i) *for each point $x \in X$, $F(x, ., .)$ is sequentially continuous from $D_Y \times D_Z$ to X with respect the strong topology on D_Y and the weak topology on D_Z, and there exists a square matrix M of size p having nonnegative entries and spectral radius less than 1 such that*

$$d(F(x, y, z), F(\overline{x}, y, z)) \leq M d(x, \overline{x}),$$

for all $x, \overline{x} \in X$, $y \in D_Y$, and $z \in D_Z$;

(ii) *G and H are sequentially continuous from $X \times D_Y \times D_Z$ to D_Y and D_Z, respectively, with respect to the strong topology on D_Y and the weak topology on D_Z;*

(iii) *for some points $(x_{p+1}, ..., x_{p+q}) \in D_Y$ and $(x_{p+q+1}, ..., x_n) \in D_Z$, one has*

$$\begin{aligned}
&C_i \subset V_i \quad \text{countable for } i = 1, ..., p, \\
&C_i \subset D_i \quad \text{countable for } i = p+1, ..., n, \quad C = \textstyle\prod_{i=1}^{n} C_i, \\
&\overline{C_i} = \overline{\operatorname{conv}}(\{x_i\} \cup N_i(C)), \quad i = p+1, ..., p+q, \\
&\overline{C_i}^{w} = \overline{\operatorname{conv}}(\{x_i\} \cup N_i(C)), \quad i = p+q+1, ..., n, \\
&\text{implies} \\
&\overline{C_i} \quad \text{is strongly compact for } i = p+1, ..., p+q \quad \text{and} \\
&\overline{C_i}^{w} \quad \text{is weakly compact for } i = p+q+1, ..., n.
\end{aligned}$$

Then there exists $v^* = (x^*, y^*, z^*) \in X \times D_Y \times D_Z$ with $N(v^*) = v^*$.

1.2. **Krasnosel'skii type results for nonself maps.** Several extensions of Krasnosel'skii's theorem to nonself maps have been given. One is due to O'Regan and is based on the fact that the map $A + B$ in Krasnosel'skii's theorem is condensing and thus the Leray-Schauder continuation principle for condensing maps can be used. Stated as a continuation theorem, O'Regan's result reads as follows.

Theorem 1.5 (O'Regan)**.** [13] *Let U be an open set in a closed, convex set C of a Banach space $(X, |.|_X)$. Assume $0 \in U$ and $N : \overline{U} \to C$ is given by $N = A + B$, where $A : \overline{U} \to C$ is a ϕ-contraction, i.e., there exists a continuous nondecreasing function $\phi : [0, +\infty) \to [0, +\infty)$ satisfying $\phi(t) < t$ for $t > 0$, such that*

$$|A(x) - A(y)|_X \leq \varphi(|x - y|_X)$$

for all $x, y \in \overline{U}$, and $B : \overline{U} \to C$ is completely continuous. Then either,

(i) *N has a fixed point in $\overline{U}$, or*

(ii) *there is a point $u \in \partial U$ and $\lambda \in (0, 1)$ with $u = \lambda N(u)$.*

Using the original idea of the proof of Krasnosel'skii's theorem and Schaefer's fixed point theorem instead of Schauder's one, we have the following result due to Burton and Kirk, also presented as a continuation theorem.

Theorem 1.6 (Burton-Kirk)**.** [5] *Let X be a Banach space, $A, B : X \to X$, A a contraction and B completely continuous. Then either*
(i) $x = \lambda A(\frac{x}{\lambda}) + \lambda Bx$ *has a solution in X for $\lambda = 1$, or*
(ii) *the set of all such solutions, $0 < \lambda < 1$, is unbounded.*

A similar result is given in [9] by replacing the equation $x = \lambda A(x/\lambda) + \lambda Bx$ with $x = A(x) + \lambda Bx$.

Theorem 1.7 (Gao-Li-Zhang)**.** [9] *Let X be a Banach space, $A, B : X \to X$, A a contraction and B completely continuous. Then either*
(i) $x = A(x) + \lambda B(x)$ *has a solution in V for $\lambda = 1$, or*
(ii) *the set of all such solutions, $0 < \lambda < 1$, is unbounded.*

In the proofs, one uses the homotopy $\lambda (I - A)^{-1} B$ in case of Theorem 1.6, in contrast to Theorem 1.7 where the homotopy $(I - A)^{-1} \lambda B$ is used.

For other extensions of Krasnosel'skii's fixed point theorem we refer to [2], [3], [8], [10], [11], [15], [16], [14] and [19]. A variational version of Avramescu's theorem is given in [4].

The aim of this paper is to obtain nonlinear alternatives of Leray-Schauder and Mönch type for nonself vector-valued operators, under hybrid conditions of Perov contraction and compactness. Thus we shall extend the theorems of Krasnosel'skii, Avramescu, Burton-Kirk and Gao-Li-Zhang. An application is given to a boundary value problem for a system of second order differential equations in which some of the equations are implicit.

2. Nonlinear alternatives

2.1. **Avramescu type principle for nonself maps.** Consider a system of two operator equations

$$\begin{cases} N_1(x,y) = x \\ N_2(x,y) = y. \end{cases} \tag{2.1}$$

We have the following general topological principle in terms of fixed point index (see [7, Section 2.1]).

Theorem 2.1. *Let Y be a Banach space, $K \subset Y$ a retract of Y and $U \subset K$ open in K. Let Λ be a topological space and*

$$\begin{aligned} N_1 &: \Lambda \times \overline{U} \to \Lambda, \\ N_2 &: \Lambda \times \overline{U} \to K \end{aligned}$$

be two mappings such that the following conditions are satisfied:
(a) *For each $y \in \overline{U}$, there is a unique $x =: S(y) \in \Lambda$ with*

$$N_1(S(y), y) = S(y);$$

(b) *There is a compact map $H : \overline{U} \times [0,1] \to K$, $H_\lambda := H(\cdot, \lambda)$, with*

$$i(H_0,\ U,\ K) \neq 0 \quad \textit{and} \quad H_1 = N_2(S(.), .).$$

Then either
(i) *the system (2.1) has a solution $(x,y) \in \Lambda \times \overline{U}$, or*

(ii) *there is a point* $y \in \partial_K U$ *and* $\lambda \in (0,1)$ *with* $y = H(y, \lambda)$.

Proof. In virtue of (a), $(x, y) \in \Lambda \times \overline{U}$ is a solution of (2.1) if and only if $x = S(y)$ and y is a fixed point of H_1. Assume that (ii) does not hold. Hence H_λ is fixed point free on $\partial_K U$ for $\lambda \in (0,1)$. Obviously, condition $i(H_0, U, K) \neq 0$ makes necessary that H_0 is fixed point free on $\partial_K U$ too. If H_1 has a fixed point in $\partial_K U$, then (i) holds and we are finished. Otherwise, H is an admissible homotopy joining H_1 with H_0 and from the homotopy invariance of the fixed point index we should have $i(H_1, U, K) = i(H_0, U, K) \neq 0$, which guarantees that H_1 has a fixed point in U, thus again condition (i) is satisfied. □

Theorem 2.1 gives in particular hybrid results for nonself maps of Krasnosel'kii, Burton-Kirk and Gao-Li-Zhang types. Instead of the common contraction property, we consider its vector analogue, the Perov contraction (for this notion and related topics of vector analysis, we refer the reader to [17, Chapter 10]). To this aim, consider $(X_i, |.|_i)$, $i = 1, ..., n$ Banach spaces and $X = X_1 \times ... \times X_n$ the product space endowed with the norm

$$|x|_X = |x_1|_1 + ... + |x_n|_n .$$

All topological notions such as continuity, compactness, boundary of a set, related to the product space X will be considered with respect to the norm $|.|_X$. Also, on X we consider the vector-valued norm

$$\|x\| = (|x_1|_1, \ ..., \ |x_n|_n)^T .$$

With respect to the vector-valued metric $\|x - y\|$, the space X is a complete generalized metric space.

In such a product space X we have:

Corollary 2.2. *Let* $U \subset X$ *be open with* $0 \in U$, $A : X \to X$ *a Perov contraction and* $B : \overline{U} \to X$ *compact. Then either*

(i) *the map* $A + B$ *has a fixed point in* $\overline{U}$, *or*

(ii) *there is a point* $x \in \partial U$ *and* $\lambda \in (0,1)$ *with* $x = \lambda (A(x) + B(x))$.

Proof. Apply Theorem 2.1. Here

$$K = Y = \Lambda = X, \quad N_1(x, y) = A(x) + B(y), \ N_2(x, y) = x,$$

$S = (I - A)^{-1} B$ and $H(x, \lambda) = (I - \lambda A)^{-1} \lambda B$. □

A similar result is the vector version of Burton-Kirk theorem:

Corollary 2.3. *Let* $U \subset X$ *be open with* $0 \in U$, $A : X \to X$ *a Perov contraction and* $B : \overline{U} \to X$ *compact. Then either*

(i) *the map* $A + B$ *has a fixed point in* $\overline{U}$, *or*

(ii) *there is a point* $x \in \partial U$ *and* $\lambda \in (0,1)$ *with* $x = \lambda A\left(\frac{x}{\lambda}\right) + \lambda B(x)$.

Proof. Apply Theorem 2.1. Here

$$K = Y = \Lambda = X, \quad N_1(x, y) = A(x) + B(y), \ N_2(x, y) = x,$$

$S = (I - A)^{-1} B$ and $H(x, \lambda) = \lambda (I - A)^{-1} B$. □

Theorem 2.1 also yields the vector version of Gao-Li-Zhang:

Corollary 2.4. *Let $U \subset X$ be open with $0 \in U$, $A : X \to X$ a Perov contraction and $B : \overline{U} \to X$ compact. Then either*

(i) *the map $A + B$ has a fixed point in $\overline{U}$, or*

(ii) *there is a point $x \in \partial U$ and $\lambda \in (0, 1)$ with $x = A(x) + \lambda B(x)$.*

Proof. Apply Theorem 2.1. Here

$$K = Y = \Lambda = X, \quad N_1(x, y) = A(x) + B(y), \ N_2(x, y) = x,$$

$S = (I - A)^{-1} B$ and $H(x, \lambda) = (I - A)^{-1} \lambda B$. □

2.2. **Avramescu-Mönch theorem for nonself maps.** We finish this section by an Avramescu-Mönch type result for nonself maps. It involves a compactness condition which does not make possible the use of the fixed point index.

Theorem 2.5. *Let Λ be a complete generalized metric space endowed with the vector-valued metric d, Y a Banach space, $U \subset Y$ open with $0 \in U$, $N_1 : \Lambda \times \overline{U} \to \Lambda$ and $N_2 : \Lambda \times \overline{U} \to Y$ two maps such that*

(a) *$N_1(., y)$ is a Perov contraction with the same Lipschitz matrix for $y \in \overline{U}$;*

(b) *N_2 is continuous and a Mönch map, i.e., if $C = C_1 \times C_2 \subset \Lambda \times \overline{U}$ is countable and $C_2 \subset \overline{conv}(\{0\} \cup N_2(C))$, then $\overline{C}_2$ is compact.*

Then either

(i) *the system (2.1) has a solution $(x, y) \in \Lambda \times \overline{U}$, or*

(ii) *there is a point $(x, y) \in \Lambda \times \partial U$ and $\lambda \in (0, 1)$ with*

$$\begin{cases} N_1(x, y) = x \\ \lambda N_2(x, y) = y. \end{cases} \tag{2.2}$$

Proof. Solving (2.1) reduces to the fixed point equation $y = N_2(S(y), y)$ in $\overline{U}$, where $S : \overline{U} \to \Lambda$ is defined by $N_1(S(y), y) = S(y)$ and whose well-definition and continuity are guarantees by (a). Next we use Mönch's fixed point theorem for nonself maps ([7, Theorem 18.1]). To this end we first need to check Mönch's compactness condition for the operator $T := N_2(S(.), .)$. If $C \subset \overline{U}$ is countable and $C \subset \overline{\text{conv}}(\{0\} \cup T(C))$, then $C \subset \overline{\text{conv}}(\{0\} \cup N_2(S(C), C))$, whence in view of (b) implies that $\overline{C}_2$ is compact. Next observe that if a point $(x, y) \in \Lambda \times \partial U$ and $\lambda \in (0, 1)$ satisfy (2.2) then $x = S(y)$ and $\lambda N_2(S(y), y) = y$, that is $\lambda T(y) = y$, which means that the Leray-Schauder boundary condition does not hold. Thus Mönch's fixed point theorem applies and gives the result. □

3. Application

We present an application of the vector version of Burton-Kirk theorem, Corollary 2.3, to the following boundary value problem for a system of n equations

$$\begin{aligned} -u_i'' &= f_i(t, V_i u) + g_i(t, V_0 u), \quad \text{a.e. } t \in (0, 1) \\ u_i(0) &= u_i(1) = 0, \quad i = 1, 2, \cdots, n, \end{aligned} \tag{3.1}$$

where V_0u and V_iu denote the vectors

$$V_0u = (u, u') = (u_1, \cdots, u_n, u_1', \cdots, u_n'), \qquad V_iu = (V_0u,\ u_i'') = (u, u', u_i'').$$

Also the mappings f_i and g_i are assumed to satisfy the Carathéodory conditions and some additional conditions given below.

Note that the equations are implicit due to the dependence on u_i'' of the terms $f_i(t, V_iu)$.

Denoting $Lu = -u''$ and letting $x_i = Lu_i$, the system is equivalent to

$$\begin{aligned} x_i &= f_i\left(t, V_iL^{-1}x\right) + g_i\left(t, V_0L^{-1}x\right), \quad \text{a.e. } t \in (0,1) \\ x_i &\in L^2(0,1), \quad i = 1,2,...,n. \end{aligned}$$

Let

$$\begin{aligned} A_i(x) &:= f_i\left(t, V_iL^{-1}x\right) \\ B_i(x) &:= g_i\left(t, V_0L^{-1}x\right). \end{aligned}$$

Thus, our problem appears as a fixed point equation

$$x = A(x) + B(x), \qquad x \in L^2(0,1;\mathbb{R}^n).$$

We shall apply the vector version of Burton-Kirk theorem in the space $L^2(0,1;\mathbb{R}^n)$. Before checking the conditions of this theorem, let us recall:

0) Some basic results on Sobolev spaces in one dimension.

One denotes by $H_0^1(0,1)$ the space of all absolutely continuous functions on $[0,1]$ vanishing at 0 and 1 and whose derivative belongs to $L^2(0,1)$. This is a Hilbert space under the scalar product and norm

$$\langle x, y\rangle_{H_0^1} = \int_0^1 x'y'dt, \qquad |x|_{H_0^1} = |x'|_{L^2} = \left(\int_0^1 x'^2 dt\right)^{1/2}.$$

By $H^{-1}(0,1)$ one denotes the dual of $H_0^1(0,1)$ and the following embeddings hold: $H_0^1(0,1) \subset L^2(0,1) \subset H^{-1}(0,1)$. According to Poincaré's inequality one has

$$\begin{aligned} |x|_{L^2} &\leq \frac{1}{\pi}|x|_{H_0^1} \quad \left(x \in H_0^1(0,1)\right), \\ |x|_{H^{-1}} &\leq \frac{1}{\pi}|x|_{L^2} \quad \left(x \in L^2(0,1)\right). \end{aligned}$$

The operator $Lx = -x''$ is an isometry between $H_0^1(0,1)$ and $H^{-1}(0,1)$, so $\left|L^{-1}x\right|_{H_0^1} = |x|_{H^{-1}} \ \left(x \in H^{-1}(0,1)\right)$.

Also note that the number $\mu_1 = \pi^2$ is the first eigenvalue of the operator $-x''$ under the Dirichlet boundary condition $x(0) = x(1) = 0$.

1) We now guarantee that A is a **Perov contraction** on $L^2(0,1;\mathbb{R}^n)$. To this aim assume that f_i are Lipschitz continuous, more exactly

$$|f_i(t,x,y,z) - f_i(t,\overline{x},\overline{y},\overline{z})| \leq \sum_{j=1}^{n}\left(a_{ij}|x_j - \overline{x}_j| + b_{ij}\left|y_j - \overline{y}_j\right|\right) + c_i|z - \overline{z}|, \quad (3.2)$$

for all $x, \overline{x}, y, \overline{y} \in \mathbb{R}^n$; $z, \overline{z} \in \mathbb{R}$ and a.e. $t \in (0,1)$.

Then for any $x, y \in L^2(0,1;\mathbb{R}^n)$, one has

$$\begin{aligned} & |A_i(x) - A_i(y)|_{L^2} \\ \leq\ & \sum_{j=1}^n \left(a_{ij}\left|L^{-1}(x_j - y_j)\right|_{L^2} + b_{ij}\left|L^{-1}(x_j - y_j)'\right|_{L^2}\right) + c_i |x_i - y_i|_{L^2} \\ \leq\ & \sum_{j=1}^n \left(\frac{a_{ij}}{\pi^2} + \frac{b_{ij}}{\pi}\right)|x_j - y_j|_{L^2} + c_i |x_i - y_i|_{L^2}, \end{aligned} \tag{3.3}$$

where we used

$$\left|L^{-1}(x_j - y_j)\right|_{L^2} \leq \frac{1}{\pi}\left|L^{-1}(x_j - y_j)\right|_{H_0^1} = \frac{1}{\pi}|x_j - y_j|_{H^{-1}} \leq \frac{1}{\pi^2}|x_j - y_j|_{L^2}.$$

Therefore A is a Perov contraction if

$$\rho(M) < 1, \tag{3.4}$$

where $\rho(M)$ is the spectral radius of the matrix $M = [m_{ij}]_{1 \leq i,j \leq n}$ whose entries are

$$m_{ij} = \frac{a_{ij}}{\pi^2} + \frac{b_{ij}}{\pi} \text{ for } j \neq i, \quad m_{ii} = \frac{a_{ii}}{\pi^2} + \frac{b_{ii}}{\pi} + c_i.$$

2) A priori bounds for the solutions of the equations

$$x = \lambda A\left(\frac{1}{\lambda}x\right) + \lambda B(x), \quad \lambda \in (0,1). \tag{3.5}$$

Step 1: Bounds for $|x_i|_{H^{-1}}$.

We have

$$x_i = \lambda A_i\left(t, \frac{1}{\lambda}V_i L^{-1}x\right) + \lambda B_i\left(t, V_0 L^{-1}x\right). \tag{3.6}$$

Now we introduce a sign condition on g_i, namely

$$x_i g_i(t, x, y) \leq 0 \quad \text{for every } x, y \in \mathbb{R}^n \text{ and a.e. } t \in (0,1). \tag{3.7}$$

Now we multiply by $L^{-1}x_i$ in (3.6) and integrate over $[0,1]$ and we observe that $L^{-1}x_i g_i\left(t, V_0 L^{-1}x\right) \leq 0$, we obtain

$$\begin{aligned} & \left\langle x_i,\ L^{-1}x_i\right\rangle_{L^2} \\ =\ & \left|L^{-1}x_i\right|_{H_0^1}^2 = |x_i|_{H^{-1}}^2 \leq \lambda \int_0^1 L^{-1}x_i A_i\left(t, \frac{1}{\lambda}V_i L^{-1}x\right) dt \\ \leq\ & \int_0^1 \left|L^{-1}x_i\right| \left(\sum_{j=1}^n \left(a_{ij}\left|L^{-1}x_j\right| + b_{ij}\left|\left(L^{-1}x_j\right)'\right|\right) + c_i|x_i| + |f_i(t,0)|\right) dt \\ \leq\ & \sum_{j=1}^n \left(a_{ij}\left|L^{-1}x_j\right|_{L^2} + b_{ij}\left|\left(L^{-1}x_j\right)'\right|_{L^2}\right)\left|L^{-1}x_i\right|_{L^2} + \phi_i\left|L^{-1}x_i\right|_{L^2} \\ & + c_i \int_0^1 \left|L^{-1}x_i\right| |x_i|\, dt, \end{aligned}$$

where $\phi_i = |f_i(\cdot,0)|_{L^2}$. We have

$$\left|\left(L^{-1}x_j\right)'\right|_{L^2} = \left|L^{-1}x_j\right|_{H_0^1}, \qquad \left|L^{-1}x_j\right|_{L^2} \le \frac{1}{\pi}\left|L^{-1}x_j\right|_{H_0^1} = \frac{1}{\pi}\left|x_j\right|_{H^{-1}}.$$

Also

$$\int_0^1 \left|L^{-1}x_i\right| |x_i|\, dt = \left\langle x_i, \sigma L^{-1}x_i\right\rangle_{L^2},$$

where $\sigma(t)$ gives the sign of the function $x_i(t)\left(L^{-1}x_i\right)(t)$. Furthermore

$$\left\langle x_i, \sigma L^{-1}x_i\right\rangle_{L^2} = |x_i|_{H^{-1}}\left|\sigma L^{-1}x_i\right|_{H_0^1} = |x_i|_{H^{-1}}\left|L^{-1}x_i\right|_{H_0^1} = |x_i|_{H^{-1}}^2.$$

Then

$$|x_i|_{H^{-1}}^2 \le \sum_{j=1}^n \left(\frac{a_{ij}}{\pi^2} + \frac{b_{ij}}{\pi}\right)|x_i|_{H^{-1}}|x_j|_{H^{-1}} + c_i |x_i|_{H^{-1}}^2 + \frac{\phi_i}{\pi}|x_i|_{H^{-1}},$$

whence

$$|x_i|_{H^{-1}} \le \sum_{j=1}^n \left(\frac{a_{ij}}{\pi^2} + \frac{b_{ij}}{\pi}\right)|x_j|_{H^{-1}} + c_i |x_i|_{H^{-1}} + \frac{\phi_i}{\pi}.$$

These can be put under the matrix form

$$\|x\|_{H^{-1}} \le M\|x\|_{H^{-1}} + \overline{\phi},$$

with columns $\|x\|_{H^{-1}} = \left[|x_i|_{H^{-1}}\right]_{1\le i\le n}^T$ and $\overline{\phi} = \frac{1}{\pi}\left[\phi_i\right]_{1\le i\le n}^T$. Since the spectral radius of M is less than one, we may multiply by $(I-M)^{-1}$ and obtain

$$\|x\|_{H^{-1}} \le (I-M)^{-1}\overline{\phi},$$

which proves that $|x_i|_{H^{-1}}$ are bounded.

Step 2: $|B_i(x)|_{L^2} \le \overline{C}_i$. To this aim we impose a second condition to g_i, namely the growth condition

$$|g_i(t,x,y)| \le \sum_{j=1}^n \left(\alpha_{ij}|x_j|^p + \beta_{ij}|y_j|\right) + \gamma_i(t) \qquad (x,y\in\mathbb{R}^n), \tag{3.8}$$

where α_{ij}, β_{ij} are nonnegative, $p \ge 1$ and $\gamma_i \in L^2(0,1;\mathbb{R}_+)$. Notice the large generality of this growth condition with respect to x_j since no restriction on the exponent p is required.

Then we have

$$|B_i(x)|_{L^2} \le \sum_{j=1}^n \left(\alpha_{ij}\left|L^{-1}x_j\right|_{L^{2p}}^p + \beta_{ij}\left|\left(L^{-1}x_j\right)'\right|_{L^2}\right) + |\gamma_i|_{L^2}.$$

Furthermore, since $H_0^1(0,1) \subset L^{2p}(0,1)$ continuously, there is a constant η_1 such that

$$\begin{aligned}\left|L^{-1}x_j\right|_{L^{2p}}^p &\le \eta_1\left|L^{-1}x_j\right|_{H_0^1}^p = \eta_1|x_j|_{H^{-1}}^p,\\ \left|\left(L^{-1}x_j\right)'\right|_{L^2} &= \left|L^{-1}x_j\right|_{H_0^1} = |x_j|_{H^{-1}}.\end{aligned}$$

Hence

$$\left|B_i(x)\right|_{L^2} \leq \sum_{j=1}^{n}\left(\overline{\alpha}_{ij}\left|x_j\right|_{H^{-1}}^{p} + \beta_{ij}\left|x_j\right|_{H^{-1}}\right) + \left|\gamma_i\right|_{L^2}$$

which in view of the result from Step 1 is bounded.

Step 3: $\left|x_i\right|_{L^2} \leq C_i$. Indeed, using (3.3) we have

$$\begin{aligned} \left|x_i\right|_{L^2} &\leq \lambda\left|A_i\left(\frac{1}{\lambda}x\right)\right|_{L^2} + \lambda\left|B_i(x)\right|_{L^2} \\ &\leq \sum_{j=1}^{n} m_{ij}\left|x_j\right|_{L^2} + \phi_i + \overline{C}_i, \end{aligned}$$

or, under the matrix form

$$\|x\|_{L^2} \leq M\,\|x\|_{L^2} + \widetilde{\phi},$$

where $\|x\|$, $\widetilde{\phi}$ are the column vectors $\|x\| = \left[\left|x_i\right|_{L^2}\right]^T_{1\leq i\leq n}$ and $\widetilde{\phi} = \left[\phi_i + \overline{C}_i\right]^T_{1\leq i\leq n}$. Then

$$\|x\|_{L^2} \leq (I-M)^{-1}\,\widetilde{\phi}$$

which proves our claim.

3) Complete continuity of B. The linear operator L^{-1} is compact from $L^2(0,1)$ to $C^1[0,1]$, while due to the growth property (3.8), the Nemytskii's operator associated to g_i is continuous and bounded (maps bounded sets into bounded sets) from C^1 to L^2. Consequently, as the composition of the previous two operators, B_i is completely continuous from $L^2(0,1;\mathbb{R}^n)$ to $L^2(0,1)$.

4) Application of Corollary 2.3. The set of all solutions of the equations (3.5) being bounded it can be included in an open ball U of $L^2(0,1;\mathbb{R}^n)$ centered at the origin and of a sufficiently large radius. Then all the assumptions of Corollary 2.3 are fulfilled.

Thus we can state following conclusion result.

Theorem 3.1. *If* f_i *and* g_i *satisfy the Carathéodory conditions and conditions* (3.2), (3.4), (3.7) *and* (3.8) *hold, then problem* (3.1) *has at least one solution.*

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Received: October 1st, 2021; Accepted: March 20, 2022.